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Chern–Simons Solitons and a Nonlinear Elliptic Equation

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Abstract

We prove an existence theorem for the following quasilinear elliptic equation

$$(1-e^u)\Delta u = |\nabla u|^2 e^u - \lambda(1-e^u)^2 e^u + 4\pi \sum_{j=1}^{N} \delta_{p_j}$$

over the full plane subject to the boundary condition that $u \to 0$ as $|x| \to \infty$, where $\lambda > 0$ is a physical parameter and δ_p is the Dirac distribution concentrated at the point p. The solutions of the equation are vortex-like multi-solitons arising in a unified relativistic self-dual Chern–Simons theory.

1 Introduction

It is well known that relativistic self-dual Chern–Simons models [7, 8, 9, 10, 13, 15] appear in quantum field theory as approximations of the physically important anyon models which have applications in high-temperature superconductivity and quantized Hall effect. The Chern–Simons solitons behave like dually (electrically and magnetically) charged particles [1, 12, 16, 21, 22] which are absent in the classical $(2+1)$-dimensional Yang–Mills theory. Self-duality [2] singles out a unique situation in which multi-solitons exist to saturate various quantized energy levels as in the Abelian Higgs model [11]. However, unless the model is nonrelativistic, the nonlinear governing equations are always nonintegrable and one has to pursue their solutions by functional analysis [4, 5, 17, 18, 19]. Since these solutions are absolute energy minimizers, they are relevant in quantum theory in sense of perturbative constructions.

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Here we study a more general, unified, Chern–Simons model [3] for which the existence problem was previously solved [20] only in the category of radially symmetric solutions in the framework of [6] but the problem of existence of multi-solitons has been left open due to the nonlinearity involved in the governing elliptic equation. The aim of this paper is to solve this problem: we will prove, by a globally convergent (constructive) method, the existence of multi-solitons in the general Chern–Simons model originally proposed in [3].

The rest of the paper is outlined as follows. In the next section we introduce the Chern–Simons equations to be studied and state the results for the existence of multi-sloitons. In Section 3 we reduce the Chern–Simons equations into two equivalent nonlinear elliptic equations, quasilinear and semilinear, respectively, and state the results for the existence of solutions for these PDE's. In Section 4 we provide proofs. In Section 5 we return to the Chern–Simons equations again and calculate the energy of a multi-soliton solution. In Section 6 we show that the solutions of the Chern–Simons equations obtained earlier may be used to get multi-soliton solutions of the general self-dual Abelian Higgs equations, also found in [3].

2 Multi-solitons

We use ϕ to denote a complex scalar field and and $A = (A_1, A_2)$ a vector field, both defined over the full plane, $\mathbb{R}^2$. The relativistic self-dual Chern–Simons equations to be solved are

$$
\begin{aligned}
D_1\phi &= \mathrm{i}D_2\phi, &(2.1)\\
(1-|\phi|^2)F_{12} &= \mathrm{i}(D_1\phi\overline{D_2\phi} - \overline{D_1\phi}D_2\phi) + \frac{1}{2}\lambda(1-|\phi|^2)^2|\phi|^2, &(2.2)
\end{aligned}
$$

where $D_j\phi = \partial_j\phi + \mathrm{i}A_j\phi$ $(j = 1, 2)$ are the gauge-covariant derivatives, $\mathrm{i} = \sqrt{-1}$, and $F_{12} = \partial_1 A_2 - \partial_2 A_1$ is the curvature tensor or magnetic field. We will look for an N-soliton solution of the above system so that ϕ vanishes exactly at the arbitrarily prescribed points $p_1, p_2, \cdots, p_N \in \mathbb{R}^2$ and

$$
|\phi| \to 1, \quad (1-|\phi|^2)(|D_1\phi| + |D_2\phi|) \to 0
$$

as $|x| \to \infty$ due to the standard finite energy requirement. The integer N in fact corresponds to the homotopy class of the solution in the framework of a well defined topological classification [11].

By (2.1), we may rewrite (2.2) in the form

$$
(1-|\phi|^2)F_{12} = -|D_1\phi|^2 - |D_2\phi|^2 + \frac{1}{2}\lambda(1-|\phi|^2)^2|\phi|^2, \tag{2.3}
$$

On the other hand, with the notation

$$
\partial = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad a = A_1 + \mathrm{i}A_2,
$$

we can put (2.1) into the form

$$2\mathrm{i}\partial\phi = \overline{a}\phi \quad \text{or} \quad \overline{a} = 2\mathrm{i}\partial \ln \phi. \tag{2.4}$$

Consequently, away from the zeros of ϕ, the curvature F_{12} becomes

$$F_{12} = -\mathrm{i}(\partial a - \overline{\partial}\overline{a}) = -2(\partial\overline{\partial}\ln\overline{\phi} + \overline{\partial}\partial\ln\phi) = -2\partial\overline{\partial}\ln|\phi|^2 = -\frac{1}{2}\Delta\ln|\phi|^2.$$

The equation (2.1) implies that, locally, up to a vanishing factor, ϕ is analytic in the variable $z = x^1 - \mathrm{i}x^2$ (see [11]). Hence there are finitely many zeros of ϕ, say $p_1, p_2, \cdots, p_N$ in $\mathbb{R}^2 = \mathbb{C}$.

Here is our main existence result for multi-soliton solutions of the Chern–Simons equations (2.1), (2.2).

Theorem 2.1. *For given the points $p_1, p_2, \cdots, p_N \in \mathbb{R}^2$, the system (2.1), (2.2) has a solution (ϕ, A) so that ϕ vanishes precisely at $p_1, p_2, \cdots, p_N$ and the solution is characterized by the topological asymptotic property*

$$|\phi|^2 = 1 + O(e^{-\sqrt{2\lambda}(1-\varepsilon)|x|}), \quad (1-|\phi|^2)(|D_1\phi| + |D_2\phi|) = O(e^{-\sqrt{2\lambda}(1-\varepsilon)|x|}),$$

as $|x| \to \infty$, where ε is any number lying in the interval $(0,1)$. Moreover, the energy of the solution is quantized and is proportional to the number of zeros of ϕ, N.

3 Reduction to PDE's

We note that, if ϕ is written locally as $\phi = e^{\sigma + \mathrm{i}\omega}$ where σ and ω are real-valued functions, then (2.4) implies the useful relations

$$D_1\phi = (\partial + \overline{\partial})\phi - \left(\frac{\partial\phi}{\phi} - \frac{\overline{\partial\phi}}{\overline{\phi}}\right)\phi = 2\phi(\overline{\partial}\sigma), \tag{3.1}$$

$$D_2\phi = \mathrm{i}(\partial - \overline{\partial})\phi - \mathrm{i}\left(\frac{\partial\phi}{\phi} - \frac{\overline{\partial\phi}}{\overline{\phi}}\right)\phi = -2\mathrm{i}\phi\overline{\partial}\sigma. \tag{3.2}$$

Introduce now the real variable $u = \ln|\phi|^2$. From (3.1) and (3.2), we have

$$|D_1\phi|^2 + |D_2\phi|^2 = 4|\phi|^2(|\partial\sigma|^2 + |\overline{\partial}\sigma|^2) = \frac{1}{2}e^u|\nabla u|^2. \tag{3.3}$$

Substituting (3.3) into (2.3), we arrive at the following quasilinear elliptic governing equation

$$(1-e^u)\Delta u - e^u|\nabla u|^2 = -\lambda(1-e^u)^2 e^u + 4\pi\sum_{j=1}^{N}\delta_{p_j} \quad \text{in } \mathbb{R}^2, \tag{3.4}$$

where δ_p is the Dirac distribution concentrated at the point $p \in \mathbb{R}^2$ and the unknown u is subject to the boundary condition $u(x) \to 0$ as $|x| \to \infty$.

The second-order scalar equation (3.4) and the first-order system (2.1), (2.2) are equivalent. In fact, if u solves (3.4), then define the complex scalar function ϕ by

$$\phi(z) = \exp\left(\frac{1}{2}u(z) + \mathrm{i}\sum_{j=1}^{N}\arg(z-p_j)\right), \quad z = x^1 - \mathrm{i}x^2 \tag{3.5}$$

and the vector field A by (2.4). It can be examined [11] that (ϕ, A) is a smooth solution of the system (2.1), (2.2). Hence we may focus on (3.4).

Theorem 3.1. *For any $\lambda > 0$ and the prescribed points $p_1, p_2, \cdots, p_N \in \mathbb{R}^2$, the equation (3.4) has a negative solution that vanishes at infinity according to the rate*

$$|u| + |1-e^u||\nabla u| = O(e^{-\sqrt{2\lambda}(1-\varepsilon)|x|}),$$

where ε is an arbitrary constant lying in the interval $(0,1)$. Moreover, there holds the quantized integral

$$\lambda\int_{\mathbb{R}^2}(1-e^u)^2e^u = 4\pi N.$$

In order to solve (3.4), we consider a new dependent variable w defined by

$$w = F(u) = 1 + u - e^u. \tag{3.6}$$

Then, formally, the equation (3.4) is transformed into the following semilinear equation

$$\Delta w = -\lambda(1-e^{G(w)})^2e^{G(w)} + 4\pi\sum_{j=1}^{N}\delta_{p_j} \quad x \in \mathbb{R}^2 \tag{3.7}$$

subject to the same boundary condition, $w(x) \to 0$ as $|x| \to \infty$, where G is the inverse function of F: $G(w) = F^{-1}(w)$. Clearly, both F and G are 1-1 from the interval $(-\infty, 0]$ to itself. Consequently, in the category of negative solutions, (3.4) and (3.7) are equivalent. We are to find a negative solution of the boundary value problem: w solves (3.7) and fulfills the condition $w(x) \to 0$ as $|x| \to \infty$.

Theorem 3.2. *For any $\lambda > 0$ and prescribed points $p_1, p_2, \cdots, p_N \in \mathbb{R}^2$, the equation (3.7) has a negative solution that vanishes at infinity according to the rate*

$$|w(x)| + |\nabla w(x)| = O(e^{-\sqrt{2\lambda}(1-\varepsilon)|x|}),$$

where ε is an arbitrary constant lying in the interval $(0,1)$. Moreover, there holds the quantized integral

$$\lambda\int_{\mathbb{R}^2}(1-e^{G(w)})^2e^{G(w)} = 4\pi N.$$

We shall first prove Theorem 3.2 in detail and then derive the corresponding implications stated in Theorem 3.1 and Theorem 2.1.

4 Existence proofs

There are two difficulties with (3.7). The first one is that, although (3.7) has a variational principle (see the functional I defined by (4.8)), the function G is only meaningful on the half line $(-\infty, 0]$ and any minimization sequence has to stay within such a constraint. The second one is that, although (3.7) has a convenient supersolution, it has been elusive to obtain a comparable subsolution, which obstructs the method of monotone iterations. In our proof, we combine the favorable aspects of these two features to obtain a proof which may be described as follows. We begin by using the supersolution to start an iterative sequence $\{v_n\}$ so that each v_n stays within our required constraint range. We then use the variational structure to control the sequence $\{v_n\}$ from below. Finally, we take limit to arrive at a solution.

To proceed, we use the background functions

$$w_0 = -\sum_{j=1}^{N} \ln(1+|x-p_j|^{-2}), \quad g_0 = 4\sum_{j=1}^{N}(1+|x-p_j|^2)^{-2} \tag{4.1}$$

and the substitution $w = w_0 + v$ to recast (3.7) into the form

$$\Delta v = -\lambda(1-e^{G(w_0+v)})^2 e^{G(w_0+v)} + g_0. \tag{4.2}$$

By the definition of w_0, we need to find a solution of (4.2) that vanishes at infinity.

To solve (4.2), we consider the iterative scheme over $\mathbb{R}^2$ defined by

$$(\Delta - K)v_{n+1} = -\lambda(1-e^{G(w_0+v_n)})^2 e^{G(w_0+v_n)} - Kv_n + g_0, \tag{4.3}$$

$$v_{n+1} \to 0 \quad \text{as } |x|\to\infty, \quad n = 0,1,2,\cdots, \tag{4.4}$$

$$v_0 = -w_0, \tag{4.5}$$

where the constant K is so large that $K \geq 2\lambda$. We will show that (4.3)–(4.5) give us an approximation sequence that goes to an exact solution of (4.2) as $n \to \infty$.

Step 1. The scheme described in (4.3)–(4.5) defines a monotone sequence $\{v_n\}$ in the space $W^{2,2}(\mathbb{R}^2)$ which satisfies the property

$$-w_0 = v_0 > v_1 > v_2 > \cdots > v_n > \cdots. \tag{4.6}$$

We use the notation $r = |x|$. First note that $v_0 = -w_0 = \mathrm{O}(r^{-2})$ near infinity. So $v_0 \in L^2(\mathbb{R}^2)$. On the other hand, since $G(0) = 0$, v_1 satisfies

$$(\Delta - K)v_1 = -Kv_0 + g_0.$$

By L^2-theory for elliptic equations, we see that $v_1 \in W^{2,2}(\mathbb{R}^2)$. In particular, $v_1 = 0$ at infinity. Besides, since v_0 satisfies

$$(\Delta - K)v_0 = -Kv_0 + g_0 - 4\pi\sum_{j=1}^{N}\delta_{p_j},$$

we have $(\Delta - K)(v_1 - v_0) \geq 0$. By the maximum principle and $v_1 - v_0 \to 0$ as $|x| \to \infty$, we have $v_1 - v_0 < 0$.

In general, we assume that $v_0 > v_1 > \cdots > v_n$, $v_1, \cdots, v_n \in W^{2,2}(\mathbb{R}^2)$, and we consider v_{n+1}. To show that $v_{n+1} \in W^{2,2}(\mathbb{R}^2)$, it suffices to see that the right-hand side of (4.3) belongs to $L^2(\mathbb{R}^2)$. Since $G(w_0 + v_n) < 0$, we need only to show that $(1 - e^{G(w_0+v_n)})^2 \in L^2(\mathbb{R}^2)$. Of course, we have only to check what happens at infinity. To this end, we observe the finite limit

$$\lim_{s\to 0^-} \frac{(1-e^{G(s)})^2}{s} = -\lim_{s\to 0^-} 2(1-e^{G(s)})e^{G(s)}G'(s) = -2 \tag{4.7}$$

which suggests that, away from a local region, $(1 - e^{G(w_0+v_n)})^2 \sim w_0 + v_n \in L^2(\mathbb{R}^2)$ as expected.

Define the function $P(s) = -\lambda(1 - e^{G(s)})^2 e^{G(s)}$, $s \leq 0$. Then (4.7) says that $P'(0^-) = 2\lambda$. Besides, for $s < 0$, we have

$$P'(s) = 2\lambda e^{2G(s)} - \lambda(1 - e^{G(s)})e^{G(s)} < 2\lambda$$

because $G(s) < 0$. Hence we have in general $P'(s) \leq 2\lambda$, $s \in (-\infty, 0]$. Therefore we obtain $(\Delta - K)(v_{n+1} - v_n) = P(w_0 + v_n) - P(w_0 + v_{n-1}) - K(v_n - v_{n-1}) = (P'(\xi) - K)(v_n - v_{n-1}) \geq 0$ by $K \geq 2\lambda$ and $v_{n-1} > v_n$, where ξ lies between $w_0 + v_n$ and $w_0 + v_{n-1}$. Thus the maximum principle gives us $v_n > v_{n+1}$.

Since it is hard to obtain a suitable subsolution to bound $\{v_n\}$ from below, we turn to a method using energy estimates.

Step 2. We formulate the energy functional

$$I(v) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla v|^2 + \frac{1}{4}\lambda(1 - e^{G(w_0+v)})^4 + g_0 v \right\}, \quad v \in W^{1,2}(\mathbb{R}^2). \tag{4.8}$$

It may be shown formally that (4.2) is the variational equation of (4.8). However, due to the fact that G is only defined for $w_0 + v \leq 0$ and the inconvenient nonlinearity present, it is difficult to minimize (4.8) directly. Here we consider the values of I over the sequence $\{v_n\}$ instead.

We can establish the following monotonicity property

$$\cdots < I(v_n) < \cdots < I(v_2) < I(v_1) < \infty. \tag{4.9}$$

We have already shown that $(1 - e^{G(w_0+v_n)})^2 \in L^2(\mathbb{R}^2)$. Hence the finiteness of $I(v_n)$ follows. Next, multiplying (4.3) by $v_{n+1} - v_n$ and integrating by parts, we have

$$\int_{\mathbb{R}^2} \{|\nabla v_{n+1}|^2 - \nabla v_{n+1} \cdot \nabla v_n + K(v_{n+1} - v_n)^2\}$$

$$= \lambda \int_{\mathbb{R}^2} (v_{n+1} - v_n)(1 - e^{G(w_0+v_n)})^2 e^{G(w_0+v_n)} - \int_{\mathbb{R}^2} g_0(v_{n+1} - v_n). \tag{4.10}$$

We consider the function

$$\Psi(s) = \frac{1}{4}\lambda(1 - e^{G(w_0+s)})^4 - \frac{1}{2}Ks^2, \quad w_0 + s \leq 0.$$

It can be examined that $\Psi(s)$ is concave down: $\Psi''(s) < 0$. Hence

$$\Psi(s_2) < \Psi(s_1) + \Psi'(s_1)(s_2 - s_1), \quad s_1 \neq s_2. \tag{4.11}$$

Inserting $s_2 = v_{n+1}$, $s_1 = v_n$ into (4.11), we get

$$\begin{aligned}\frac{\lambda}{4}(1 - e^{G(w_0+v_{n+1})})^4 \; &< \; \frac{\lambda}{4}(1 - e^{G(w_0+v_n)})^4 + \frac{K}{2}(v_{n+1} - v_n)^2 \\ &\quad -\lambda(1 - e^{G(w_0+v_n)})^2 e^{G(w_0+v_n)}(v_{n+1} - v_n). \end{aligned}\tag{4.12}$$

Using (4.10), (4.12), and $|\nabla v_{n+1} \cdot \nabla v_n| \leq \frac{1}{2}|\nabla v_{n+1}|^2 + \frac{1}{2}|\nabla v_n|^2$, we get

$$\begin{aligned}\int_{\mathbb{R}^2} \left\{\frac{1}{2}|\nabla v_{n+1}|^2 \; - \; \frac{1}{2}|\nabla v_n|^2 + K(v_{n+1} - v_n)^2\right\} \\ < -\frac{\lambda}{4}\int_{\mathbb{R}^2}([1 - e^{G(w_0+v_{n+1})}]^4 - [1 - e^{G(w_0+v_n)}]^4) \\ + \int_{\mathbb{R}^2} \frac{K}{2}(v_{n+1} - v_n)^2 - \int_{\mathbb{R}^2} g_0(v_{n+1} - v_n),\end{aligned}$$

or, in other words,

$$I(v_{n+1}) + \frac{K}{2}\int_{\mathbb{R}^2}(v_{n+1} - v_n)^2 < I(v_n), \tag{4.13}$$

which proves the inequality (4.9).

Step 3. We now estimate the sequence $\{I(v_n)\}$ from below. In particular, we obtain $W^{1,2}$-boundedness of the sequence $\{v_n\}$.

Using (3.6), we have $w = 1 + G(w) - e^{G(w)}$. Therefore

$$1 - e^{G(w)} - (1 - e^w) = e^w - e^{G(w)} = e^\xi(w - G(w)) = e^\xi(1 - e^{G(w)}) > 0, \quad w < 0,$$

where ξ lies between w and $G(w)$. Hence, the above leads us to the following useful comparison,

$$0 \leq 1 - e^w \leq 1 - e^{G(w)}, \quad w \leq 0. \tag{4.14}$$

On the other hand, for the function

$$\eta(w) = (1 - e^{G(w)})^2 + w, \quad w \leq 0,$$

it can be checked that $\eta'(w) < 0$ when $w \in (G^{-1}(-\ln 2), 0]$ and $\eta(0) = 0$. Hence

$$\eta(w) \geq 0 \quad \text{or} \quad (1 - e^{G(w)})^2 \geq |w|, \quad w \in [G^{-1}(-\ln 2), 0]. \tag{4.15}$$

Moreover, in view of (4.14), we have

$$(1 - e^{G(w)})^4 \geq (1 - e^{G(w)})^2(1 - e^w)^2 \geq \frac{1}{4}(1 - e^w)^2, \quad w < G^{-1}(-\ln 2). \tag{4.16}$$

Besides, it is straightforward to show that

$$|1 - e^w| \geq \frac{|w|}{1 + |w|}, \quad \forall w. \tag{4.17}$$

As a consequence of (4.14)–(4.17), we obtain for the function $w_0 + v \leq 0$,

$$\begin{aligned}
\int_{\mathbb{R}^2} (1 - e^{G(w_0+v)})^4 &= \left(\int_{w_0+v<G^{-1}(-\ln 2)} + \int_{G^{-1}(-\ln 2)\leq w_0+v\leq 0} \right) (1 - e^{G(w_0+v)})^4 \\
&\geq \frac{1}{4} \int_{w_0+v<G^{-1}(-\ln 2)} (1 - e^{w_0+v})^2 + \int_{G^{-1}(-\ln 2)\leq w_0+v\leq 0} (w_0 + v)^2 \\
&\geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|w_0 + v|^2}{(1 + |w_0 + v|)^2} \geq \frac{1}{8} \int_{\mathbb{R}^2} \frac{v^2}{(1 + |w_0| + |v|)^2} - C_1, \quad (4.18)
\end{aligned}$$

by noting that w_0 decay like r^{-2} at infinity and a simple interpolation technique.

Let $\|\cdot\|_p$ denote the standard L^p norm over $\mathbb{R}^n$. We recall the following well known Nirenberg–Gagliardo interpolation inequality in $\mathbb{R}^n$:

$$\|D^j u\|_p \leq C\|D^m u\|_r^t \|u\|_q^{1-t},$$

where j, m are integers so that $0 \leq j < m$ and t satisfies

$$\frac{1}{p} = \frac{j}{m} + t\left(\frac{1}{r} - \frac{m}{n}\right) + (1-t)\frac{1}{q}, \quad \frac{j}{m} \leq a < 1,$$

and $C > 0$ is a constant depending only on j, m, n, p, q, t. The useful special case for us is when $n = 2, j = 0, m = 1, p = 4, r = 2, q = 2$. Hence $t = 1/2$ and we have

$$\|u\|_4^4 \leq C\|\nabla u\|_2^2 \|u\|_2^2. \tag{4.19}$$

We will use (4.18) and (4.19) to obtain a desired lower estimate for the sequence $\{I(v_n)\}$. In this following, we denote by C any positive constant which may assume different values at different places.

By (4.19) and the Schwarz inequality, we see that for any $\varepsilon > 0$ there holds

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} g_0 v \right| &\leq \|g_0\|_{4/3} \|v\|_4 \leq C\|v\|_4 \\
&\leq \varepsilon \|v\|_2 + \frac{C}{\varepsilon} \|\nabla v\|_2 + C \\
&\leq \varepsilon \|v\|_2 + \frac{1}{4} \|\nabla v\|_2^2 + \frac{C}{\varepsilon^2}. \quad (4.20)
\end{aligned}$$

Inserting (4.18) and (4.20) into (4.8), we have the lower bound

$$I(v) \geq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\lambda}{32} \int_{\mathbb{R}^2} \frac{v^2}{(1 + |w_0| + |v|)^2} - \varepsilon \|v\|_2 - \frac{C}{\varepsilon^2} - C. \tag{4.21}$$

To proceed, we further apply (4.19) to write down the estimate

$$\begin{aligned}
\|v\|_2^4 &= \left(\int_{\mathbb{R}^2}\frac{|v|}{1+|w_0|+|v|}(1+|w_0|+|v|)|v|\right)^2\\
&\leq C\int_{\mathbb{R}^2}\frac{v^2}{(1+|w_0|+|v|)^2}\int_{\mathbb{R}^2}(v^2+v^4+w_0^4)\\
&\leq C(\|v\|_2^2+\|v\|_2^2\|\nabla v\|_2^2+1)\int_{\mathbb{R}^2}\frac{v^2}{(1+|w_0|+|v|)^2}\\
&\leq \frac{1}{2}\|v\|_2^4+C\Big\{\Big(\int_{\mathbb{R}^2}\frac{v^2}{(1+|w_0|+|v|)^2}\Big)^4+\|\nabla v\|_2^8+1\Big\}. \qquad (4.22)
\end{aligned}$$

Combining (4.21) and (4.22), we arrive at

$$\|v\|_2\leq C\Big\{1+\int_{\mathbb{R}^2}\Big(|\nabla v|^2+\frac{v^2}{(1+|w_0|+|v|)^2}\Big)\Big\}. \qquad (4.23)$$

Substituting (4.23) into (4.21), we obtain

$$\|v\|_2\leq C\Big\{I(v)+\varepsilon\|v\|_2+\frac{1}{\varepsilon^2}+1\Big\}. \qquad (4.24)$$

From (4.24), we immediately deduce the lower bound

$$I(v)\geq C_1\|v\|_{W^{1,2}(\mathbb{R}^2)}-C_2, \qquad (4.25)$$

where $C_1, C_2>0$ are uniform constants.

Step 4. We can now achieve convergence of the sequence $\{v_n\}$ to a solution of the governing equation (4.2) that vanishes at infinity.

In fact, applying the inequalities (4.9) and (4.25) with $v=v_n$ $(n=1,2,\cdots)$, we see that $\{v_n\}$ is a bounded sequence in $W^{1,2}(\mathbb{R}^2)$, which must be weakly convergent to some $w\in W^{1,2}(\mathbb{R}^2)$ because of the monotonicity property (4.6). Moreover, when we recall that the sequence $\{v_n\}$ comes from (4.3), namely

$$(\Delta-K)v_n=P(w_0+v_{n-1})-Kv_{n-1}+g_0,\quad n=1,2,\cdots,$$

and $|P'(s)|\leq 3\lambda$, the L^2-theory for elliptic equations gives us $W^{2,2}$-weak convergence of the sequence $\{v_n\}$ whose limit w of course is a weak solution of the original equation (4.2). Since we are in two dimensions, any $W^{2,2}(\mathbb{R}^2)$ function vanishes at infinity.

Step 5. Put $w=w_0+v$. Then w solves (3.7) and $w=0$ at $r=\infty$.

Step 6. We now obtain the expected decay estimates near infinity.

Linearizing (3.7) around $w=0$ near infinity, we have $\Delta w=2\lambda w$. Hence, we may use a suitable comparison function to show that $w(x)=O(e^{-\sqrt{2\lambda}(1-\varepsilon)|x|})$ as $|x|\to\infty$. Furthermore, differentiating (4.2), we see that $V=\partial_j v$ satisfies the equation

$$\Delta V=P'(w_0+v)V+P'(w_0+v)(\partial_j w_0)+\partial_j g_0.$$

Hence a similar argument as before gives us $V \in W^{2,2}(\mathbb{R}^2)$. In particular, $V = 0$ at $r = 0$. Set $W = \partial_j w$ with $w = w_0 + v$. We see that $W = 0$ at $r = \infty$ as well.

Differentiating (3.7), we obtain, away from a local region, the equation $\Delta W = P'(w)W$. Since $P(w) \to 2\lambda$ as $w \to 0$, we see that W satisfies the same exponential decay estimate as that for w.

Step 7. We finally derive the quantized integral.

To avoid confusion with singularities, we work on the regular version of the equation, (4.2). By the definition of g_0 given in (4.1), we easily see that

$$\int_{\mathbb{R}^2} g_0 = 4\pi N. \tag{4.26}$$

On the other hand, from $\partial_j w_0 = \mathrm{O}(r^{-3})$ at infinity and $\partial_j v = \partial_j w - \partial_j w_0$, we have $\partial_j v = \mathrm{O}(r^{-3})$ at infinity ($j = 1, 2$). Hence

$$\int_{\mathbb{R}^2} \Delta v = \lim_{\rho\to\infty} \oint_{|x|=\rho} \frac{\partial v}{\partial n}\, \mathrm{d}s = 0.$$

Integrating (4.2) and appling the above result, we obtained from (4.26) the quantized integral

$$\lambda \int_{\mathbb{R}^2} (1 - e^{G(w)})^2 e^{G(w)} = \lambda \int_{\mathbb{R}^2} (1 - e^{G(w_0+v)})^2 e^{G(w_0+v)} = 4\pi N.$$

The proof of Theorem 3.2 is complete.

Since the obtained solution w is negative, we can use $u = G(w)$ to get a solution of (3.4) which enjoys the decay estimate stated in Theorem 3.1. In fact, by (3.6), we have

$$u(1 + e^{\xi(u)}) = w; \quad (1 - e^u)(\partial_j u) = \partial_j w, \quad j = 1, 2,$$

where $\xi(u) \to 0$ as $u \to 0$, which yield the expected result. The quantized integral is a direct consequence of the result for (3.7). Hence Theorem 3.1 follows.

5 Quantized energy

Let u be the solution obtained in Theorem 3.1. Define the field configuration pair (ϕ, A) by (3.5) and (2.4). Then (ϕ, A) is an N-vortex solution of (2.1), (2.2). It can be checked that the asymptotic estimates obtained in Theorem 3.1 are enough for calculation of various physical and topological quantities in the model. As an illustration, we now calculate the energy of an N-vortex solution.

Following [3], we write down the energy associated with (ϕ, A) as follows,

$$\begin{aligned} E &= \int_{\mathbb{R}^2} \Big\{ \frac{1}{|\phi|^2}([1 - |\phi|^2]F_{12} - \mathrm{i}[D_1\phi\overline{D_2\phi} - D_2\phi\overline{D_1\phi}])^2 \\ &\quad + 2\lambda(1 - |\phi|^2)^2(|D_1\phi|^2 + |D_2\phi|^2) + \frac{1}{4}\lambda^2(1 - |\phi|^2)^4|\phi|^2 \Big\}. \end{aligned} \tag{5.1}$$

Inserting (2.1), (2.2) into (5.1), we have

$$\begin{aligned} E &= \lambda \int_{\mathbb{R}^2} \{(1-|\phi|^2)^3 F_{12} + 3(1-|\phi|^2)^2(|D_1\phi|^2 + |D_2\phi|^2)\} \\ &= -\frac{\lambda}{2} \lim_{\rho\to 0} \int_{\mathbb{R}^2 - \cup_{j=1}^N B_\rho(p_j)} \{(1-e^u)^3 \Delta u - 3(1-e^u)^2 e^u |\nabla u|^2\} \\ &= -\frac{\lambda}{2} \lim_{\rho\to 0} \int_{\mathbb{R}^2 - \cup_{j=1}^N B_\rho(p_j)} \nabla\cdot([1-e^u]^3 \nabla u) \\ &= \frac{\lambda}{2} \sum_{j=1}^N \lim_{\rho\to 0} \oint_{\partial B_\rho(p_j)} (1-e^u)^3\,(-\partial_2 u\,\mathrm{d}x^1 + \partial_1 u\,\mathrm{d}x^2), \end{aligned} \tag{5.2}$$

where $B_\rho(p_j)$ is the disk in $\mathbb{R}^2$ centered at p_j with radius $\rho > 0$ ($j = 1, 2, \cdots, N$) and the path integrals are all taken counterclockwise. Note that the above is valid because the path integral along a circle around infinity vanishes due to the exponential decay estimate obtained in Theorem 3.1.

Since we can write u near $x = p_j$ in the form

$$u(x) = \ln|x-p_j|^2 + f_j(x), \quad f_j \in C^\infty(B_\rho(p_j)), \quad j = 1, 2, \cdots, N, \tag{5.3}$$

where $\rho > 0$ is small, we get from inserting (5.3) into (5.2) that $E = 2\lambda\pi N$ as stated in Theorem 2.1.

6 On a general Abelian Higgs model

The multi-soliton solution obtained earlier for the Chern–Simons model can conveniently be used to construct a solution for the general self-dual Abelian Higgs equations discovered in [3]. To see this, we rewrite these equations as follows,

$$D_1\phi = \mathrm{i}D_2\phi, \tag{6.1}$$

$$F_{12} = \mathrm{i}(D_1\phi\overline{D_2\phi} - D_2\phi\overline{D_1\phi}) + \frac{1}{2}\lambda(1-|\phi|^2)^2. \tag{6.2}$$

Although this system is similar to (2.1), (2.2), it is hard to approach it directly because there is lack of a variational structure. Indeed, we may follow the same procedure as that for the Chern–Simons system to transform (6.1), (6.2) into the elliptic equation

$$(1-e^u)\Delta u - e^u|\nabla u|^2 = -\lambda(1-e^u)^2 + 4\pi\sum_{j=1}^N \delta_{p_j} \tag{6.3}$$

over $\mathbb{R}^2$. Using the substitution (3.6) again, (6.3) becomes its equivalent form,

$$\Delta w = -\lambda(1-e^{G(w)})^2 + 4\pi\sum_{j=1}^N \delta_{p_j} \tag{6.4}$$

in the category of negative solutions. Thus, with (4.1) and $w = w_0 + v$, (6.4) becomes

$$\Delta v = -\lambda(1 - e^{G(w_0+v)})^2 + g_0. \tag{6.5}$$

Unlike (4.2), which is the variational equation of the energy functional (4.8), (6.5) does not enjoy such a structure. Thus the energy method used earlier to control the iterative sequence defined by (4.3)–(4.5) fails here. Nevertheless, we now show that we can avoid such an approach by using a convenient subsolution of (6.5) obtained for the Chern–Simons equation.

In fact, let $v_{\rm CS}$ be the solution of (4.2) satisfying $w_0 + v_{\rm CS} < 0$. Since $e^{G(w_0+v_{\rm CS})} < 1$, we have

$$\Delta v_{\rm CS} > -\lambda(1 - e^{G(w_0+v_{\rm CS})})^2 + g_0,$$

which implies that $v_{\rm CS}$ is a subsolution of (6.5). As before, we already know that $-w_0$ is a (distributional) supersolution of (6.5). Therefore we can slightly modify the scheme (4.3)–(4.5) to get a solution, v, for (6.5), satisfying $v_{\rm CS} < v < -w_0$. In particular, $v = 0$ at infinity. In this way the governing equation (6.4) or (6.3) is again solved. Other details are omitted.

We remark that our procedure here works also for the classical Abelian Higgs model solved previously by a direct variational method in [11]. Indeed, we may use a topological solution [18] of the self-dual Chern–Simons equations [9, 10] as a subsolution and a solution to the Abelian Higgs model is thus reproduced.

The discussion of this section suggests that the self-dual Abelian Higgs equations may be regarded as been covered by the Chern–Simons equations, classical or general.

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