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How Quantales Emerge by Introducing Induction within the Operational Approach

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Abstract. We formally introduce and study a notion of 'soft induction' on entities with an operationally motivated logico-algebraic description, and in particular the derived notions of 'induced state transition' and 'induced property transition'. We study the meaningful collections of these soft inductions which all have a quantale structure due to the introduction of temporal composition and arbitrary choice on the level of these state transitions and the corresponding property transitions.

1 Introduction

The essential physical concepts that lie at the base of this paper are the notion of a property according to the Geneva school operational approach [2, 10, 13] and the idea that measurements on an entity provoke a real change of the state of the system [3], i.e., a change of its 'actual' properties [2, 10, 13]. Within this conceptual context, we introduce the notion of an 'induction', and in particular of 'soft inductions'. For other aspects related to these inductions we refer to [5, 7]. The essential mathematical object that emerges when these soft inductions are introduced are quantales, originally introduced in the late thirties in order to translate ring-theoretical ideas to lattices. The name quantale itself has been introduced by Mulvey in [11] where he studied them in relation to C^*-algebras in order to build a constructive base for quantum mechanics. For an explicit definition we refer to the third appendix at the end of this paper. In a second appendix we study maps on a state space and their join preserving extensions. In a first appendix we give some basic categorical notions.

Let us consider an entity Ξ. If we consider a 'particular realization of this entity' we shall refer to it as 'a particular Ξ' [10]. It can be physically argued that Ξ is described by a complete meet semi-lattice $\mathcal{L}$, if we identify properties as equivalence classes of so called 'definite experimental projects' [10, 13]. Given a particular Ξ, we can determine all of its actual properties, i.e., those properties for which a corresponding definite experimental project, if performed, would give yes with certainty, and their meet will be called the strongest actual property of this particular Ξ. We denote by Σ the set of all the strongest actual properties of Ξ. The collection Σ generates the lattice $\mathcal{L}$, i.e., $\forall a \in \mathcal{L} : a = \vee\{p \in \Sigma \mid p \leq a\}$, and it is an axiom of the Geneva school operational approach that Σ are exactly the atoms of $\mathcal{L}$, denoted as $\mathcal{A}(\mathcal{L})$. Now following [8, 9]:

Definition 1 *A morphism from a complete atomistic lattice $\mathcal{L}_1$ into a complete atomistic lattice $\mathcal{L}_2$ is a map $f : \mathcal{L}_1 \to \mathcal{L}_2$ which satisfies (1) $\forall A \subseteq \mathcal{L}_1 : f(\vee A) = \vee f(A)$; (2) $f(\mathcal{A}(\mathcal{L}_1)) \subseteq \mathcal{A}(\mathcal{L}_2) \cup \{0\}$. A closure space is a set X together with a closure operator $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$: (1) $\forall A \subseteq X : A \subseteq \mathcal{C}(A)$; (2) $A \subseteq B \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B)$; (3) $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$. The closure is called T_1 (or 'simple') if (4) $\mathcal{C}(\emptyset) = \emptyset$ and $\forall x \in \Sigma : \mathcal{C}(x) = \{x\}$. A subset $F \subseteq X$ is called closed if $\mathcal{C}(F) = F$, and we denote the collection of all closed subsets by $\mathcal{F}(X)$. A morphism from a closure space $(X_1, \mathcal{C}_1)$ into a closure space $(X_2, \mathcal{C}_2)$ is a partially defined map $f : X_1 \setminus K \to X_2$, defined on the complement of some $K \subseteq X_1$ (called kernel of f), which satisfies for every $A \subseteq X_1$: $f(\mathcal{C}_1(A) \setminus K) \subseteq \mathcal{C}_2(f(A \setminus K))$.*

Denote **T$_1$SPACE** for the category of T_1-closure spaces and their morphisms and **CALAT** for the complete atomistic lattices.

Theorem 1 **T$_1$SPACE** *is categorically equivalent with* **CALAT**.

Proofs can be found in [8, 9]. When starting from a lattice of properties $\mathcal{L}$ that is axiomatized to be atomistic, we obtain, via the categorical equivalence, a T_1-closure space, the points of which are exactly the strongest actual properties of $\mathcal{L}$, called *states*, and the closure space is then called *state space*, denoted by $(\Sigma, \mathcal{C})$. The underlying set Σ will be called *state set*. The isomorphism of lattices $\mu : \mathcal{L} \to \mathcal{F}(\Sigma)$ is a Cartan map [14]: it sends every property $a \in \mathcal{L}$ to the $F \in \mathcal{F}(\Sigma)$ that contains exactly those states that make the property actual. Working 'dually', i.e., starting from a state set Σ, then imposing on Σ a T_1-closure $\mathcal{C}$ with class of closed subsets $\mathcal{F}(\Sigma)$, thus obtaining a lattice of properties through the map $F \mapsto \vee F$ for $F \in \mathcal{F}(\Sigma)$, can be interpreted as adding an 'operational resolution' to the set Σ by means of the closure $\mathcal{C}$: for any $T \subseteq \Sigma$ we have that $\mathcal{C}(T)$ is the smallest subset of Σ for which there is a property, namely $\vee\mathcal{C}(T)$, such that exactly those states in $\mathcal{C}(T)$ have this property.

2 Soft induction

Definition 2 *We talk about an induction in case of an externally imposed change of a particular entity. This change might modify the collection of actual properties of Ξ, i.e., its* *state, as well as the whole collection of properties, i.e., Ξ itself; in the case that Ξ is preserved with certainty we speak of a soft induction, otherwise of a hard induction.*

We denote the collection of soft inductions on Ξ by $\mathcal{E}(\Xi)$. Preservation of Ξ implies on the formal level that the state space $(\Sigma, \mathcal{C})$ is not altered. Thus, the setup induces a 'state transition' *in* Σ. We observe that there exist two operations on collections of elements of $\mathcal{E}(\Xi)$:

- Finite composition of inductions: for $e_1, e_2, ..., e_n \in \mathcal{E}(\Xi)$ we have that $e_n \circ ... \circ e_2 \circ e_1 \in \mathcal{E}(\Xi)$ is the induction consisting of first performing e_1, then e_2, then ... until e_n;
- Arbitrary choice of induction: for $\{e_i\}_i \subseteq \mathcal{E}(\Xi)$ we have that $\vee_i e_i \in \mathcal{E}(\Xi)$ is the induction consisting of performing one of the e_i, chosen in any possible way.

For an induction $e \in \mathcal{E}(\Xi)$, we denote Σ_e for the set of all states that may result when performing e, i.e., that are not excluded by the performed procedure. This set Σ_e will be called 'set of outcome states' for the induction e. Note that any $e \in \mathcal{E}(\Xi)$ can be performed on any $s \in \Sigma$.

Proposition 1 *We have that $\Sigma_{e_n \circ ... \circ e_1} \subseteq \Sigma_{e_n}$ and $\Sigma_{\vee_i e_i} = \cup_i \Sigma_{e_i}$.*

The inclusion is not necessarily an equality since the last performed induction e_n can only have states in $\Sigma_{e_{n-1} \circ ... \circ e_1}$ as possible initial states such that some states in Σ_{e_n} might become excluded as outcome state for $e_n \circ ... \circ e_1$ and thus, not contained in $\Sigma_{e_n \circ ... \circ e_1}$. Remark that defining the arbitrary choice of of induction by a join in stead of a meet is due to the fact that we consider *those outcome states that are not excluded*; a meet would only be relevant if we consider *certain outcome states*.

2.1 State transitions

State transition is the concept of 'going from one state to another'. We will now interpret and formalize what soft induction means in relation to this concept of state transition. To every induction $e \in \mathcal{E}(\Xi)$, we can associate an atomic map $\tilde{e}' : \Sigma \to \mathcal{P}(\Sigma) : s \mapsto \tilde{e}'(s)$, where for $s \in \Sigma$, $\tilde{e}'(s) \subseteq \Sigma_e \subseteq \Sigma$ is exactly the set of outcome states when performing the induction e on Ξ in state s. We write $\tilde{\mathcal{E}}'(\Xi)$ for the collection of these atomic maps. Noting that $(\mathcal{P}(\Sigma), \subseteq, \cap, \cup)$ is a complete atomistic lattice with atoms Σ, and using the material from the second appendix, we can develop a 1-1 correspondence between these atomic maps $\tilde{e}' : \Sigma \to \mathcal{P}(\Sigma)$ and the atomically generated maps $\tilde{e} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto \tilde{e}(T) = \cup\{\tilde{e}'(t) \mid t \in \Sigma \cap T\}$. We then have that $\tilde{e} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ preserves unions. This preservation of unions has the following significance: whenever we consider more than one possible initial state (for example due to a lack of knowledge on the initial state) the set of possible outcome states for an induction on this entity consist of the union of the outcome state sets for the possible initial states. Remark that since $ker(\tilde{e}') = \emptyset$ (every $e \in \mathcal{E}(\Xi)$ preserves Ξ and thus assures an outcome state in Σ), we also have that $\forall T \in \mathcal{P}(\Sigma) : \tilde{e}(T) = \emptyset \Leftrightarrow T = \emptyset$. We denote $\tilde{\mathcal{E}}(\Xi) = \{\tilde{e} \mid e \in \mathcal{E}(\Xi)\}$ and an element of $\tilde{\mathcal{E}}(\Xi)$ is called 'state transition'. Next, considering the map $\tilde{}: \mathcal{E}(\Xi) \to \tilde{\mathcal{E}}(\Xi) : e \mapsto \tilde{e}$, we define two operations on $\tilde{\mathcal{E}}(\Xi)$:

$$\tilde{e}_2 \circ \tilde{e}_1 = \tilde{}(e_2 \circ e_1) \tag{2.1}$$

$$\vee_i \tilde{e}_i = \tilde{}(\vee_i e_i) \tag{2.2}$$

Note the importance of the fact that $\forall e \in \mathcal{E}(\Xi) : dom(\tilde{e}) = cod(\tilde{e}) = \mathcal{P}(\Sigma)$, for eq.(2.1). We can interpret that: (i) Finite composition of inductions corresponds to composition of maps; (ii) Arbitrary choice of inductions corresponds to the join of maps relative to the pointwise ordering of these maps, i.e., $\forall T \in \mathcal{P}(\Sigma) : (\vee_i \tilde{e}_i)(T) = \cup_i(\tilde{e}_i(T))$. Adopting the notation $\& : \tilde{\mathcal{E}}(\Xi) \times \tilde{\mathcal{E}}(\Xi) \to \tilde{\mathcal{E}}(\Xi) : (\tilde{e}_1, \tilde{e}_2) \mapsto \tilde{e}_1 \& \tilde{e}_2 = \tilde{e}_2 \circ \tilde{e}_1$, we have that:

Proposition 2 *$(\tilde{\mathcal{E}}(\Xi), \vee, \&)$ is a unitary quantale.*

Proof: (o) for any $\tilde{e}_1, \tilde{e}_2, \tilde{e}_i \in \tilde{\mathcal{E}}(\Xi)$ we have that both $\tilde{e}_1 \& \tilde{e}_2$ and $\vee_i \tilde{e}_i$ are in $\tilde{\mathcal{E}}(\Xi)$: this can easily be seen when considering eq.(2.1), eq.(2.2) and taking into account that $\mathcal{E}(\Xi)$ is closed under the operations arbitrary choice and finite composition; (i) $(\tilde{\mathcal{E}}(\Xi), \vee)$ is a complete join semi-lattice, because in $\mathcal{E}(\Xi)$ all arbitrary choices exist and eq.(2.2); (ii) by definition we have that & is an associative product; (iii) the identity map $id : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto T$ is the unit element of $\tilde{\mathcal{E}}(\Xi)$, due to the trivial induction "doing nothing"; (iv) the first distribution law is proved as follows: for all $\tilde{e}_i, \tilde{e} \in \tilde{\mathcal{E}}(\Xi)$, we have that $(\vee_i \tilde{e}_i) \& \tilde{e} = \tilde{e} \circ (\vee_i \tilde{e}_i) = \vee_i(\tilde{e} \circ \tilde{e}_i) = \vee_i(\tilde{e}_i \& \tilde{e})$ and the second distribution law can be shown analogously•

We formulate two conditions that apply on maps $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$:

- $AS_\cup$ stands for $\forall \{T_i\}_i \subseteq \mathcal{P}(\Sigma) : f(\cup_i T_i) = \cup_i f(T_i)$
- $AS_\emptyset$ stands for $\forall T \in \mathcal{P}(\Sigma) : f(T) = \emptyset \Leftrightarrow T = \emptyset$

Now we use these two conditions to define the following set:

- $\mathcal{R}_{\emptyset,\cup}(\Sigma) = \{f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid f \textit{ meets } AS_\cup, f \textit{ meets } AS_\emptyset\}$

We call an $f \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$ a 'carrier of state transition'. From the above we know that there are two evident operations on $\mathcal{R}_{\emptyset,\cup}(\Sigma)$, namely composition of maps, denoted by $\circ$, where we will also adopt the notation $f \& g = g \circ f$, and the map $\vee_i f_i$ (for $\{f_i\}_i \subseteq \mathcal{R}_{\emptyset,\cup}(\Sigma)$) that consists of choosing one of the maps $\{f_i\}_i$ in any possible way and applying it, which means that $\vee_i f_i$ is the join of the maps $\{f_i\}_i$ relative to their pointwise ordering.

Proposition 3 *$(\mathcal{R}_{\emptyset,\cup}(\Sigma), \vee, \&)$ is a unitary quantale and $(\tilde{\mathcal{E}}(\Xi), \vee, \&)$ is a unitary subquantale of $(\mathcal{R}_{\emptyset,\cup}(\Sigma), \vee, \&)$.*

Proof: Let $i : \tilde{\mathcal{E}}(\Xi) \to \mathcal{R}_{\emptyset,\cup}(\Sigma) : \tilde{e} \mapsto \tilde{e}$ be the set-wise inclusion, then we have that (i) $i(\tilde{e}_2 \circ \tilde{e}_1) = i(\tilde{(}e_2 \circ e_1)) = \tilde{(}e_2 \circ e_1) = \tilde{e}_2 \circ \tilde{e}_1$, and (ii) $i(\vee_i \tilde{e}_i) = i(\tilde{(}\vee_i e_i)) = \vee_i \tilde{e}_i$. Also, quite evidently, $i : \tilde{\mathcal{E}}(\Xi) \to \mathcal{R}_{\emptyset,\cup}(\Sigma)$ identifies the identities of $\tilde{\mathcal{E}}(\Xi)$ and $\mathcal{R}_{\emptyset,\cup}(\Sigma)$. Hence, the set-wise inclusion is a unitary quantale morphism, which proves our claim•

We can summarize the results in this section as:

$$\mathcal{E}(\Xi) \to \tilde{\mathcal{E}}'(\Xi) \rightleftharpoons \tilde{\mathcal{E}}(\Xi) \hookrightarrow \mathcal{R}_{\emptyset,\cup}(\Sigma) \tag{2.3}$$

and we can read from left to right that: (i) the investigation of a soft induction $e \in \mathcal{E}(\Xi)$ gives us an atomic map $\tilde{e}' : \Sigma \to \mathcal{P}(\Sigma)$ with empty kernel; (ii) we have a 1-1 correspondence with atomically generated maps $\tilde{e} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that are $\cup$-preserving and $\tilde{e}(T) = \emptyset \Leftrightarrow T = \emptyset$; (iii) denoting these conditions by $AS_\cup$ and $AS_\emptyset$ and considering the set of all maps $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that meet $AS_\emptyset$ and $AS_\cup$, we obtain an inclusion of unitary quantales: the inclusion tells us which carriers of state transition are in fact state transitions of Ξ, i.e., for which carriers of state transition we have a soft induction setup available in $\mathcal{E}(\Xi)$.

2.2 Property transitions

In this section, we explore induction in relation to the properties of Ξ. To every induction $e \in \mathcal{E}(\Xi)$ we can associate an atomic map $\bar{e}' : \Sigma \to \mathcal{F}(\Sigma) : s \mapsto \bar{e}'(s)$, where for $s \in \Sigma$, $\bar{e}'(s) \subseteq \mathcal{C}(\Sigma_e) \subseteq \Sigma$ is the smallest $\mathcal{C}$-closed subset of Σ in which all of the outcome states lie. Remembering that $\mathcal{F}(\Sigma) \cong \mathcal{L}$ by means of the map $\vee : \mathcal{F}(\Sigma) \to \mathcal{L} : F \mapsto \vee F$, where $\mathcal{L}$ stands for the property lattice, we see that $\vee(\bar{e}'(s))$ is in fact the property that characterizes the smallest distinguishable set of states that contains all the outcome states of the induction $e \in \mathcal{E}(\Xi)$ performed on Ξ in state s. We write $\bar{\mathcal{E}}'(\Xi)$ for the collection of these atomic maps. Noting that $(\mathcal{F}(\Sigma), \subseteq, \cap, \vee)$ is a complete atomistic lattice, with atoms Σ, we can again develop a 1-1 correspondence between the atomic maps $\bar{e}' : \Sigma \to \mathcal{F}(\Sigma) : s \mapsto \bar{e}'(s)$ and the atomically generated maps $\bar{e} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \vee\{\bar{e}'(s) \mid s \in \Sigma \cap F\}$. We then have that these maps preserve the $\mathcal{F}(\Sigma)$-$\vee$. The significance of this is the following: Whenever we consider more than one property as a possible initial one, i.e., the strongest property about which we are certain before the induction is the join of a given collection, then we have to consider all possible property transitions for all properties in this collection, so the smallest property about which we are certain after the induction is the join of their images. We also have that $\forall F \in \mathcal{F}(\Sigma) : \bar{e}(F) = \emptyset \Leftrightarrow F = \emptyset$, since $ker(\bar{e}') = \emptyset$. We denote $\bar{\mathcal{E}}(\Xi) = \{\bar{e} \mid e \in \mathcal{E}(\Xi)\}$ and call an element of $\bar{\mathcal{E}}(\Xi)$ a 'property transition'. Next, considering the map $^- : \mathcal{E}(\Xi) \to \bar{\mathcal{E}}(\Xi) : e \mapsto \bar{e}$, again we have the evident definitions for operations on $\bar{\mathcal{E}}(\Xi)$ by:

$$\bar{e}_2 \circ \bar{e}_1 = \bar{(}e_2 \circ e_1) \tag{2.4}$$

$$\vee_i \bar{e}_i = \bar{(}\vee_i e_i) \tag{2.5}$$

Again, it can be interpreted that the finite composition of inductions corresponds to (finite) composition of maps and that the arbitrary choice of inductions corresponds to the join of maps relative to their pointwise ordering, i.e., $\forall F \in \mathcal{F}(\Sigma) : (\vee_i \bar{e}_i)(F) = \vee_i(\bar{e}_i(F))$, where this time in the right hand side of the equation the join refers to the $\mathcal{F}(\Sigma)$-join. Adopting the notation $\& : \bar{\mathcal{E}}(\Xi) \times \bar{\mathcal{E}}(\Xi) \to \bar{\mathcal{E}}(\Xi) : (\bar{e}_1, \bar{e}_2) \mapsto \bar{e}_1 \& \bar{e}_2 = \bar{e}_2 \circ \bar{e}_1$, we clearly have in analogy to Proposition 2:

Proposition 4 $(\bar{\mathcal{E}}(\Xi), \vee, \&)$ *is a unitary quantale.*

Analogously to the section state transitions, we now formulate two conditions that apply on maps $f : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)$:

- $AP_\vee$ stands for $\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : f(\vee_i F_i) = \vee_i f(F_i)$
- $AP_\emptyset$ stands for $\forall F \in \mathcal{F}(\Sigma) : f(F) = \emptyset \Leftrightarrow F = \emptyset$

Next, we use these conditions to define the following set:

- $\mathcal{S}_{\emptyset,\vee}(\Sigma) = \{f : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) \mid f \text{ meets } AP_\vee, f \text{ meets } AP_\emptyset\}$

We call an $f \in \mathcal{S}_{\emptyset,\vee}(\Sigma)$ a 'carrier of property transition'. $\mathcal{S}_{\emptyset,\vee}(\Sigma)$ is equipped with the operations finite composition of maps, with $f \& g = g \circ f$ as its notation, and the map $\vee_i f_i$ for $\{f_i\}_i \subseteq \mathcal{S}_{\emptyset,\vee}(\Sigma)$, i.e., the join of maps relative to their pointwise ordering. We we have in analogy to Proposition 3:

Proposition 5 $(\bar{\mathcal{E}}(\Xi), \vee, \&)$ *is a unitary subquantale of* $(\mathcal{S}_{\emptyset,\vee}(\Sigma), \vee, \&)$.

The results of this section can be summarized as:

$$\mathcal{E}(\Xi) \to \bar{\mathcal{E}}'(\Xi) \rightleftharpoons \bar{\mathcal{E}}(\Xi) \hookrightarrow \mathcal{S}_{\emptyset,\vee}(\Sigma) \tag{2.6}$$

and again, we can read from left to right: (i) a soft induction $e \in \mathcal{E}(\Xi)$ gives rise to an atomic map $\bar{e}' : \Sigma \to \mathcal{F}(\Sigma)$ with empty kernel; (ii) there is a 1-1 correspondence with atomically generated maps $\bar{e} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)$ that preserve $\vee$ and meet the condition $\bar{e}(F) = \emptyset \Leftrightarrow F = \emptyset$; (iii) the carriers of property transition are defined as maps that meet the two conditions $AP_\vee$ and $AP_\emptyset$, and the inclusion $\bar{\mathcal{E}}(\Xi) \hookrightarrow \mathcal{S}_{\emptyset,\vee}(\Sigma)$ then says which of the carriers of property transition are in fact property transitions of Ξ.

2.3 From state transitions to property transitions

From Theorem 1 it follows that a closure $\mathcal{C}$ imposed on Σ, and axiomatized to be T_1 yields a complete atomistic lattice $(\mathcal{F}(\Sigma), \subseteq, \cap, \vee)$ with atoms Σ, and $\mathcal{F}(\Sigma) \cong \mathcal{L}$ by means of the map $\mu : \mathcal{L} \to \mathcal{F}(\Sigma) : a \mapsto \{s \in \Sigma \mid s \leq a\}$. We can interpret the introduction of $\mathcal{C}$ as the presence of an operational resolution[1]: For any $S \subseteq \Sigma$, we have that $\mathcal{C}(S)$ is the smallest operationally distinguishable set of states that contains S, i.e., there is a property that is actual for exactly those states that are in $\mathcal{C}(S)$. We will now investigate how such an operational resolution links the ideas of state transitions and property transitions. Introducing such an operational resolution can be done on the level of the atomic maps (i.e., $\tilde{\mathcal{E}}'(\Xi)$ and $\bar{\mathcal{E}}'(\Xi)$), on the level of atomically generated maps (i.e., $\tilde{\mathcal{E}}(\Xi)$ and $\bar{\mathcal{E}}(\Xi)$) and on the level of the carriers. We will first investigate how $\mathcal{S}_{\emptyset,\vee}(\Sigma)$ can be obtained from $\mathcal{R}_{\emptyset,\cup}(\Sigma)$. $\forall f \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$, define $f_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \mathcal{C}(f(F))$. Referring to the notation just above, we have the following:

Lemma 1 *$dom(f_{pr}) = cod(f_{pr}) = \mathcal{F}(\Sigma)$ and $f_{pr}(F) = \emptyset \Leftrightarrow F = \emptyset$.*

Proof: The first equation is true by definition. $f_{pr}(F) = \emptyset \Leftrightarrow \mathcal{C}(f(F)) = \emptyset \Leftrightarrow f(F) = \emptyset \Leftrightarrow F = \emptyset$•

One could think that $\forall f \in \mathcal{R}_{\emptyset,\cup}(\Sigma) : f_{pr} \in \mathcal{S}_{\emptyset,\vee}(\Sigma)$ but this is manifestly not the case since in general an f_{pr} does not preserve the $\mathcal{F}(\Sigma)$-join:

Lemma 2 *We have:*

$$\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : f_{pr}(\vee_i F_i) = \vee_i f_{pr}(F_i)$$
$$\Updownarrow$$
$$\forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T)) \subseteq \mathcal{C}(f(T))$$

Proof: (i) $\vee_i f(F_i) = \vee_i f_{pr}(F_i)$ is always true: the inclusion $\subseteq$ is easy to see: $\cup_i f(F_i) \subseteq \cup_i \mathcal{C}(f(F_i)) \Rightarrow \mathcal{C}(\cup_i f(F_i)) \subseteq \mathcal{C}(\cup_i \mathcal{C}(f(F_i))) \Rightarrow \vee_i f(F_i) \subseteq \vee_i f_{pr}(F_i)$. The converse inclusion $\supseteq$ requires not much more: $\forall i : f(F_i) \subseteq \cup_i f(F_i) \Rightarrow \forall i : \mathcal{C}(f(F_i)) \subseteq \mathcal{C}(\cup_i f(F_i)) \Rightarrow \cup_i \mathcal{C}(f(F_i)) \subseteq \mathcal{C}(\cup_i f(F_i)) \Rightarrow \mathcal{C}(\cup_i \mathcal{C}(f(F_i)) \subseteq \mathcal{C}(\cup_i f(F_i)) \Rightarrow \vee_i f_{pr}(F_i) \subseteq \vee_i f(F_i)$. (ii) $\vee_i f(F_i) \subseteq f_{pr}(\vee_i F_i)$ is always true: $\cup_i F_i \subseteq \mathcal{C}(\cup_i F_i) \Rightarrow f(\cup_i F_i) \subseteq f(\mathcal{C}(\cup_i F_i))$ (we use that any $f \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$ is isotone on $\mathcal{P}(\Sigma)$, since it preserves $\cup$) $\Rightarrow \mathcal{C}(f(\cup_i F_i)) \subseteq \mathcal{C}(f(\mathcal{C}(\cup_i F_i))) \Rightarrow \mathcal{C}(\cup_i f(F_i)) \subseteq \mathcal{C}(f(\mathcal{C}(\cup_i F_i))) \Rightarrow \vee_i f(F_i) \subseteq f_{pr}(\vee_i F_i)$. (iii) Now we have: $[\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : f_{pr}(\vee_i F_i) \subseteq \vee_i f(F_i)] \Leftrightarrow [\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : \mathcal{C}(f(\mathcal{C}(\cup_i F_i))) \subseteq \mathcal{C}(\cup_i f(F_i))] \Leftrightarrow [\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : \mathcal{C}(f(\mathcal{C}(\cup_i F_i)) \subseteq \mathcal{C}(f(\cup_i F_i))] \Leftrightarrow [\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : f(\mathcal{C}(\cup_i F_i)) \subseteq \mathcal{C}(f(\cup_i F_i))] \Leftrightarrow [\forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T)) \subseteq \mathcal{C}(f(T))]$. In the last step we use on the one hand ($\Leftarrow$) that $\forall \{F_i\}_i \subseteq \mathcal{F}(\Sigma) : \cup_i F_i \in \mathcal{P}(\Sigma)$, and on the other hand ($\Rightarrow$) that $\forall T \in \mathcal{P}(\Sigma) : T = \cup\{t \mid t \in T\} = \cup\{\mathcal{C}(t) \mid t \in T\}$ (we use that $\mathcal{C}$ is T_1, which implies that $\mathcal{C}(t) = \{t\}$)•

It is easy to give a counterexample for $\forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T) \subseteq \mathcal{C}(f(T))$. Consider $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that maps (a) $\emptyset \mapsto \emptyset$; (b) $T \mapsto \{s\}$ for all subsets of $T \in \mathcal{P}(\Sigma) \setminus \mathcal{F}(\Sigma)$ and a certain $s \in \Sigma$; and (c) $S \mapsto \Sigma$ for all $S \in \mathcal{P}(\Sigma) \setminus \{\mathcal{P}(T), \emptyset\}$. Then we have that $f \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$ since it preserves $\cup$ and meets $f(X) = \emptyset \Leftrightarrow X = \emptyset$, but $f(\mathcal{C}(T)) = \Sigma$ and $\mathcal{C}(f(T)) = \{s\}$.

[1]On this idea of an operational resolution is elaborated in [7].

Corollary 1 *For $f \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$, we have that $f_{pr} \in \mathcal{S}_{\emptyset,\vee}(\Sigma)$ if and only if f meets the condition $\forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T) \subseteq \mathcal{C}(f(T))$. For such an f we then have that $f_{pr}(\vee_i F_i) = \vee_i f_{pr}(F_i) = \vee_i f(F_i)$.*

For $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$:

- AS_* stands for $\forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T) \subseteq \mathcal{C}(f(T))$

How should this condition AS_* be interpreted in the presence of $AS_\cup$ and $AS_\emptyset$? We have the following equivalences: $\neg AS_* \Leftrightarrow [\exists\, T \in \mathcal{P}(\Sigma), \exists\, x \in \Sigma : (a)\ x \notin \mathcal{C}(f(T)); (b)\ x \in f(\mathcal{C}(T))] \Leftrightarrow [\exists\, T \in \mathcal{P}(\Sigma), \exists\, x \in \Sigma : (a)\ \exists\, F_0 \in \mathcal{F}(\Sigma)$ such that $f(T) \subseteq F_0, x \notin F_0; (b)\ \exists\, y \in \Sigma$ such that $\forall F \in \mathcal{F}(\Sigma)$ with $T \subseteq F$, $y \in F$ and $x \in f(y)] \Leftrightarrow$[there exists a state y, not in a certain $T \in \mathcal{P}(\Sigma)$, but nevertheless indistinguishable from the states in T, the image of which through f, i.e., $f(y)$, contains a state x that is distinguishable from $f(T)$]. Hence we have:

Proposition 6 $\forall e \in \mathcal{E}(\Xi) : \tilde{e}$ *meets the condition AS_*.*

We define the following subset of $\mathcal{R}_{\emptyset,\cup}(\Sigma)$:

- $\mathcal{R}^*_{\emptyset,\cup}(\Sigma) = \{f \in \mathcal{R}_{\emptyset,\cup}(\Sigma) \mid f \textit{ meets } AS_*\}$

Of course, $\mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ inherits the operations $\vee$ and $\&$ from $\mathcal{R}_{\emptyset,\cup}(\Sigma)$.

Proposition 7 *$(\mathcal{R}^*_{\emptyset,\cup}(\Sigma), \vee, \&)$ is a unitary subquantale of $(\mathcal{R}_{\emptyset,\cup}(\Sigma), \vee, \&)$ and $(\tilde{\mathcal{E}}(\Xi), \vee, \&)$ is a unitary subquantale of $(\mathcal{R}^*_{\emptyset,\cup}(\Sigma), \vee, \&)$.*

Proof: An obvious consequence of Proposition 7•

Proposition 8 *The map $F_{pr} : \mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma) : f \mapsto f_{pr}$ is a unitary quantale morphism.*

Proof: (o) By Lemma 1 and Lemma 2, we have that $\forall f \in \mathcal{R}^*_{\emptyset,\cup}(\Sigma) : f_{pr} \in \mathcal{S}_{\emptyset,\vee}(\Sigma)$; (i) Since we have that $F_{pr} : [id : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto T] \mapsto [id_{|\mathcal{F}(\Sigma)} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto F]$, we see that F_{pr} maps the unit of $\mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ to the unit of $\mathcal{S}_{\emptyset,\vee}(\Sigma)$; (ii) for any $f, g \in \mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ and any $F \in \mathcal{F}(\Sigma)$, we have that $(f \circ g)_{pr}(F) = \mathcal{C}(f(g(F)))$ and that $(f_{pr} \circ g_{pr})(F) = \mathcal{C}(f(\mathcal{C}(g(F)))$, hence we need to show that $\forall F \in \mathcal{F}(\Sigma) : \mathcal{C}(f(g(F))) = \mathcal{C}(f(\mathcal{C}(g(F)))$. The proof of $\subseteq$ is evident, using the fact that, since any $f \in \mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ preserves unions of elements of $\mathcal{P}(\Sigma)$, such an f is isotone. To prove $\supseteq$, it is sufficient that $\forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T)) \subseteq \mathcal{C}(f(T))$, which is true $\forall f \in \mathcal{R}^*_{\emptyset,\cup}(\Sigma)$; (iii) $\forall f \in \mathcal{R}^*_{\emptyset,\cup}(\Sigma), \forall F \in \mathcal{F}(\Sigma)$ we have that $(\vee_i f_i)_{pr}(F) = \vee_i(f_{i,pr}(F))$: on the one hand we have that $(\vee_i f_i)_{pr}(F) = \mathcal{C}((\vee_i f_i)(F)) = \mathcal{C}(\vee_i(f_i(F))) = \vee_i(f_i(F)) = \mathcal{C}(\cup_i f_i(F))$

and on the other that $(\vee_i f_{i,pr})(F) = \vee_i(f_{i,pr}(F)) = \vee_i(\mathcal{C}(f_i(F))) = \mathcal{C}(\cup_i(\mathcal{C}(f_i(F))))$, the equality of which can be verified analogously to the proof of Lemma 2•

AS_* is the condition by which we select elements of $\mathcal{R}_{\emptyset,\cup}(\Sigma)$ to form the unitary subquantale $\mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ (cfr. Proposition 7). According to Corollary 1 we have that AS_* is exactly the condition that ensures that $\mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ is the domain for the (set-wise) map $F_{pr} : \cdot \to \mathcal{S}_{\emptyset,\vee}(\Sigma)$. It is clear from the proofs that the core of this matter is in fact that, if you have a set of maps $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that preserve $\cup$, and by applying $F_{pr} : f \mapsto f_{pr}$ you want to obtain a set of maps $f_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)$ that preserve $\vee$, then the largest possible domain of F_{pr} is the set of maps $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that preserve $\cup$ and meet the condition AS_*. In Proposition 8 it is then shown that the set-wise map $F_{pr} : \mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma)$ is in fact a unitary quantale homomorphism. In part (ii) of the proof of Proposition 8, it is apparent that it is again AS_* that plays a crucial role: it is a sufficient condition for $F_{pr} : \mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma)$ to preserve finite composition. But here, we can also show necessity:

Proposition 9 *AS_* is a necessary condition for $F_{pr} : \mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma)$ to preserve finite composition.*

Proof: $[\forall\, f, g \in \mathcal{R}_{\emptyset,\cup}(\Sigma), \forall\, F \in \mathcal{F}(\Sigma) : (f_{pr} \circ g_{pr})(F) = (f \circ g)_{pr}(F)] \Rightarrow [\forall\, f \in \mathcal{R}_{\emptyset,\cup}(\Sigma) : \forall\, g \in \mathcal{R}_{\emptyset,\cup}(\Sigma), \forall\, F \in \mathcal{F}(\Sigma),\ \mathcal{C}(f(\mathcal{C}(g(F)))) = \mathcal{C}(f(g(F)))] \Rightarrow [\forall\, f \in \mathcal{R}_{\emptyset,\cup}(\Sigma) : \forall\, g \in \mathcal{R}_{\emptyset,\cup}(\Sigma), \forall\, F \in \mathcal{F}(\Sigma),\ f(\mathcal{C}(g(F))) \subseteq \mathcal{C}(f(g(F)))]$. Now, because of the arbitrariness of $F \in \mathcal{F}(\Sigma)$ and $g \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$, we have that $\{T \mid T \in \mathcal{P}(\Sigma)\} \subseteq \{g(F) \mid F \in \mathcal{F}(\Sigma), g \in \mathcal{R}_{\emptyset,\cup}(\Sigma)\}$:for any $T \subseteq \Sigma$, consider for all $t \in T$ the maps $g_t : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that map (a) $\emptyset \mapsto \emptyset$ and (b) $S \mapsto t, \forall\, S \in \mathcal{P}(\Sigma) \setminus \{\emptyset\}$; hence it can be verified that $\vee_t\, g_t \in \mathcal{R}_{\emptyset,\cup}(\Sigma)$ and that for any $F \in \mathcal{F}(\Sigma) \setminus \{\emptyset\}$, we have that $(\vee_t\, g_t)(F) = T$. Hence we conclude that $[\forall\, f \in \mathcal{R}_{\emptyset,\cup}(\Sigma) : \forall\, T \in \mathcal{P}(\Sigma), f(\mathcal{C}(T)) \subseteq \mathcal{C}(f(T))]$•

Again, the core of Proposition 9 can be seen as: If you have a set of maps $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ equipped with the operations arbitrary choice and finite composition, and by application of $F_{pr} : f \mapsto f_{pr}$ you want to obtain a set of maps $f_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)$, equipped with the operation finite composition, in such a way that finite composition is preserved by F_{pr}, then the largest possible domain of F_{pr} is the set of maps $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that meet AS_*, equipped with the arbitrary choice and the finite composition.

Now we will investigate how this map F_{pr} acts on the atomic maps and the atomically generated maps related to the inductions on Ξ. We have the obvious definitions:

$$
\begin{aligned}
F_{pr} : \tilde{\mathcal{E}}'(\Xi) \to \tilde{\mathcal{E}}'(\Xi) :\ & [\tilde{e}' : \Sigma \to \mathcal{P}(\Sigma) : s \mapsto \tilde{e}'(s)] \mapsto \\
& \quad [\tilde{e}'_{pr} : \Sigma \to \mathcal{F}(\Sigma) : s \mapsto \tilde{e}'_{pr}(s) = \mathcal{C}(\tilde{e}'(s))] \\
F_{pr} : \tilde{\mathcal{E}}(\Xi) \to \tilde{\mathcal{E}}(\Xi) :\ & [\tilde{e} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto \tilde{e}(T)] \mapsto \\
& \quad [\tilde{e}_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \tilde{e}_{pr}(F) = \mathcal{C}(\tilde{e}(F))]
\end{aligned}
$$

Starting from an $\tilde{e}' \in \tilde{\mathcal{E}}'(\Xi)$, one can construct an element of $\tilde{\mathcal{E}}(\Xi)$ in two ways: (a) first one applies $F_{pr} : \tilde{\mathcal{E}}'(\Xi) \to \tilde{\mathcal{E}}'(\Xi)$ and then one constructs the atomically generated map according to the obtained $\bar{e}'_{pr} \in \bar{\mathcal{E}}'(\Xi)$; or (b) first one makes the atomically generated map according to $\tilde{e}' \in \tilde{\mathcal{E}}'(\Xi)$, and then one applies $F_{pr} : \tilde{\mathcal{E}}(\Xi) \to \bar{\mathcal{E}}(\Xi)$ on the obtained $\tilde{e} \in \tilde{\mathcal{E}}(\Xi)$. One easily verifies that (a) = (b), i.e., when considering $\bar{e}^a_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \vee\{\bar{e}'_{pr}(s) \mid s \in F \cap \Sigma\} = \mathcal{C}(\cup\{\mathcal{C}(\tilde{e}'(s)) \mid s \in F \cap \Sigma\})$ and $\bar{e}^b_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \mathcal{C}(\tilde{e}(F)) = \mathcal{C}(\vee\{\tilde{e}'(s) \mid s \in F \cap \Sigma\}) = \mathcal{C}(\cup\{\tilde{e}'(s) \mid s \in F \cap \Sigma\})$ we have $\bar{e}^a_{pr} = \bar{e}^b_{pr}$.

Theorem 2 $F_{pr} : \tilde{\mathcal{E}}(\Xi) \to \bar{\mathcal{E}}(\Xi)$ *is a unitary quantale morphism.*

Proof: This is shown from the fact that, due to both unitary quantale inclusions $i : \tilde{\mathcal{E}}(\Xi) \hookrightarrow \mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ and $i : \bar{\mathcal{E}}(\Xi) \hookrightarrow \mathcal{S}_{\emptyset,\vee}(\Sigma)$, $F_{pr} : \tilde{\mathcal{E}}(\Xi) \to \bar{\mathcal{E}}(\Xi)$ is the restriction of the unitary quantale morphism $F_{pr} : \mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma)$ to the unitary quantale $\tilde{\mathcal{E}}(\Xi)$•

The results of this section can be summarized in a schematic overview, in which all squares commute:

$$\begin{array}{ccccccc} & & \bar{\mathcal{E}}'(\Xi) & \rightleftharpoons & \bar{\mathcal{E}}(\Xi) & \hookrightarrow & \mathcal{S}_{\emptyset,\vee}(\Sigma) & & \\ & \nearrow & & & & & & & \\ \mathcal{E}(\Xi) & & \uparrow F_{pr} & & \uparrow F_{pr} & & \uparrow F_{pr} & & \\ & \searrow & & & & & & & \\ & & \tilde{\mathcal{E}}'(\Xi) & \rightleftharpoons & \tilde{\mathcal{E}}(\Xi) & \hookrightarrow & \mathcal{R}^*_{\emptyset,\cup}(\Sigma) & \hookrightarrow & \mathcal{R}_{\emptyset,\cup}(\Sigma) \end{array} \tag{2.7}$$

From this scheme we can read that the map F_{pr} provides a duality between the sets $\{f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid f \textit{ meets } AS_\cup, AS_\emptyset, AS_*\}$ and $\{f : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) \mid f \textit{ meets } AP_\vee, AP_\emptyset\}$. Further we see that these sets include respectively state transitions and property transitions, sets that are F_{pr}-dual too.

2.4 A remark on the carriers

We will extend the above scheme by adding structures that emerge as purely mathematical generalizations of the entity-related quantales above. Then, a "backwards reading" of the resulting scheme provides an analysis of the conditions $AS_\cup, AS_\emptyset, AS_*$ and $AP_\vee, AP_\emptyset$. We will use the following notations that combine the notations of the previous sections where $\vee$ will now refer to the join of the considered collection:

- $A_\emptyset$ stands for $\forall T \in dom(f) : f(T) = \emptyset \Leftrightarrow T = \emptyset$;
- $A_\vee$ stands for $\forall\{T_i\}_i \subseteq dom(f) : f(\vee_i T_i) = \vee_i f(T_i)$;
- A_* stands for $\forall T \in dom(f) : f(\mathcal{C}(T) \subseteq \mathcal{C}(f(T))$;

and consider the following collections of maps:

- $\mathcal{R}^*_\cup(\Sigma) = \{f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid f \textit{ meets } A_\vee, f \textit{ meets } A_*\}$;

- $\mathcal{R}^*_{\emptyset,\cup}(\Sigma) = \{f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid f \textit{ meets } A_\emptyset, f \textit{ meets } A_\vee, f \textit{ meets } A_*\}$.
- $\mathcal{S}_\vee(\Sigma) = \{f : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) \mid f \textit{ meets } A_\vee\}$;
- $\mathcal{S}_{\emptyset,\vee}(\Sigma) = \{f : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) \mid f \textit{ meets } A_\emptyset, f \textit{ meets } A_\vee\}$.

All these sets can be equipped with $\vee_i f_i$ and $f\&g$. We have:

Proposition 10 *The above sets, equipped with $\vee$ and $\&$ are unitary quantales and all set-wise inclusions are unitary quantale inclusions.*

Much as how we linked $\mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ and $\mathcal{S}_{\emptyset,\vee}(\Sigma)$ in the section 2.3 by means of the map F_{pr} : $f \mapsto f_{pr}$ (cfr. Proposition 8), we can link the larger quantales that we have just introduced. We will do this by means of the same map $F_{pr} : f \mapsto f_{pr}$. We can immediately remark that from Proposition 9 it is clear that A_* is a necessary condition on the domain of the map F_{pr} for it to preserve $\&$. From Proposition 8 it is then clear that A_* is the exact condition on an f that is $\vee$-preserving to yield an f_{pr} that is $\vee$-preserving.

Proposition 11 *The map $F_{pr} : [f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)] \mapsto [f_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)]$, with $f_{pr}(F) = C(f(F))$, yields the following commuting square of unitary quantale homomorphisms:*

$$\begin{array}{ccc} \mathcal{R}^*_{\emptyset,\cup}(\Sigma) & \hookrightarrow & \mathcal{R}^*_{\cup}(\Sigma) \\ \downarrow F_{pr} & & \downarrow F_{pr} \\ \mathcal{S}_{\emptyset,\vee}(\Sigma) & \hookrightarrow & \mathcal{S}_{\vee}(\Sigma) \end{array}$$

3 Hard induction

The dual conditions $AS_\emptyset$ and $AP_\emptyset$ on atomically generated maps are equivalent with saying that the respective atomic maps have an empty kernel (cfr. Proposition 19). In this section we will investigate how $AS_\emptyset$ and $AP_\emptyset$ can be dropped, enabling us to say something more about state transitions and property transitions when we also consider hard inductions (see Definition 2). Essentially, one can consider two kinds of hard inductions that still seem relevant within the development of this paper: (i) those that essentially behave as soft inductions for every possible initial state, but are such that an occasional change of state space (i.e., vanishing of the entity) is not excluded (for example, an induction device with a non-zero probability for absorption of the entity); (ii) those that preserve the state space for some initial states, but not for others (an example of such an induction is a preparation procedure through 'filtering'). We denote the collection of hard inductions on Ξ by $\Phi(\Xi)$. Since a hard induction is performed *on* Ξ, the description of it can still be done via atomic maps. However, for a hard induction $\varphi \in \Phi(\Xi)$ we are no longer certain that Ξ is preserved: the outcome state set Σ_φ can still be understood as containing those states that may result when performing φ on Ξ, but we can no longer say that one of the states in Σ_φ will result. In particular, if $\Sigma_\varphi = \{s\}$ for some $\varphi \in \Phi(\Xi)$ and some $s \in \Sigma$, then we can only say that: If Ξ is preserved by φ then after the performance of φ we will find Ξ in state s. In fact, the only hard induction of which we can say with certainty what will result after the performance, is the $\varphi \in \Phi(\Xi)$ with $\Sigma_\varphi = \emptyset$: the performance of φ on Ξ implies non-preservation of Ξ. We denote it throughout this paper by φ_D. As on $\mathcal{E}(\Xi)$, there are the operations finite composition of inductions and arbitrary choice of induction on $\Phi(\Xi)$ for which we will use the same notations.

Proposition 12 *We have that* $\Sigma_{\varphi_n \circ \ldots \circ \varphi_1} \subseteq \Sigma_{\varphi_n}$ *and* $\Sigma_{\vee_i \varphi_i} = \cup_i \Sigma_{\varphi_i}$.

The second equation of this proposition reveals a subtlety of hard induction as we formalize it here, i.e., fitting it in the scheme/perspective of soft induction: for any $\varphi \in \Phi(\Xi)$ we cannot distinguish between φ and $\varphi \vee \varphi_D$, since $\Sigma_{\varphi \vee \varphi_D} = \Sigma_\varphi \cup \Sigma_{\varphi_D} = \Sigma_\varphi \cup \emptyset = \Sigma_\varphi$. At first sight it may seem that we can consider any soft induction $e \in \mathcal{E}(\Xi)$ as 'a hard induction for which non-preservation of Ξ is excluded'. However, if this were true, then considered as a hard induction, e would be indistinguishable from $e \vee \varphi_D$, which is a 'true' hard induction in the sense that we cannot exclude non-preservation of Ξ. All this goes to show that soft induction and hard induction should be understood as 'complementary' notions, which makes it in some sense remarkable that it seems to be possible to fit both of them within the same formal scheme.

3.1 State and property transitions, the F_{pr}-duality

To every hard induction $\varphi \in \Phi(\Xi)$ we associate two atomic maps $\tilde{\varphi}' : \Sigma \to \mathcal{P}(\Sigma) : s \mapsto \tilde{\varphi}'(s)$ where for $s \in \Sigma$, $\tilde{\varphi}'(s)$ is exactly the set of outcome states, i.e., if Ξ is preserved then the outcome state is an element of $\tilde{\varphi}'(s)$, and $\bar{\varphi}' : \Sigma \to \mathcal{P}(\Sigma) : s \mapsto \bar{\varphi}'(s)$ where for $s \in \Sigma$, $\bar{\varphi}'(s)$ is the smallest $\mathcal{C}$-closed subset that contains $\tilde{\varphi}'(s)$. The kernels of these atomic maps are equal and contain exactly those $s \in \Sigma$ for which it is certain that if φ is performed on Ξ in state s then Ξ is not preserved. Remark that it is emphatically not true that if $ker(\tilde{\varphi}') = ker(\bar{\varphi}') = \emptyset$ then φ preserves Ξ, it only implies that for no $s \in \Sigma$ it is certain that Ξ is not preserved. We will denote $\tilde{\Phi}'(\Xi) = \{\tilde{\varphi}' \mid \varphi \in \Phi(\Xi)\}$ and $\bar{\Phi}'(\Xi) = \{\bar{\varphi}' \mid \varphi \in \Phi(\Xi)\}$. It is then clear that we have, like before, the map:

$$F_{pr} : \tilde{\Phi}'(\Xi) \to \bar{\Phi}'(\Xi) : \tilde{\varphi}' \mapsto [\tilde{\varphi}'_{pr} : \Sigma \to \mathcal{F}(\Sigma) : s \mapsto \mathcal{C}(\tilde{\varphi}'(s))]$$

Using the material of the second appendix, we consider the atomically generated maps $\tilde{\varphi} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto \cup\{\tilde{\varphi}'(t) \mid t \in T\}$ and $\bar{\varphi} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \vee\{\bar{\varphi}'(z) \mid z \in F\}$. Then we denote $\tilde{\Phi}(\Xi) = \{\tilde{\varphi} \mid \tilde{\varphi}' \in \tilde{\Phi}'(\Xi)\}$ and $\bar{\Phi}(\Xi) = \{\bar{\varphi} \mid \bar{\varphi}' \in \bar{\Phi}'(\Xi)\}$, and again we have the (set-wise) map:

$$F_{pr} : \tilde{\Phi}(\Xi) \to \bar{\Phi}(\Xi) : \tilde{\varphi} \mapsto [\tilde{\varphi}_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \mathcal{C}(\tilde{\varphi}(F))]$$

for which it can easily be verified that we obtain the following commuting square:

$$\begin{array}{ccc} \tilde{\Phi}'(\Xi) & \rightleftharpoons & \tilde{\Phi}(\Xi) \\ \downarrow F_{pr} & & \downarrow F_{pr} \\ \bar{\Phi}'(\Xi) & \rightleftharpoons & \bar{\Phi}(\Xi) \end{array}$$

If we define $\tilde{\varphi_1}\&\tilde{\varphi_2} = \urcorner(\varphi_1\&\varphi_2)$, $\vee_i\tilde{\varphi_i} = \urcorner(\vee_i\varphi_i)$, $\bar{\varphi_1}\&\bar{\varphi_2} = \urcorner(\varphi_1\&\varphi_2)$ and $\vee_i\bar{\varphi_i} = \urcorner(\vee_i\varphi_i)$ we have the analogues of Propositions 2 and 4:

Proposition 13 *$(\tilde{\Phi}(\Xi), \vee, \&)$ and $(\bar{\Phi}(\Xi), \vee, \&)$ are unitary quantales and $F_{pr} : \tilde{\Phi}(\Xi) \to \bar{\Phi}(\Xi) : \tilde{\varphi} \mapsto \tilde{\varphi}_{pr}$ is a unitary quantale morphism.*

Remark that although in both cases, the unit is due to the trivial induction "doing nothing", it is still possible that Ξ is 'destroyed' when 'performing' the unitary hard induction. Along the lines of the previous section it can be verified that we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & \bar{\Phi}'(\Xi) & \rightleftharpoons & \bar{\Phi}(\Xi) & \hookrightarrow & \mathcal{S}_{\vee}(\Sigma) & & \\ & \nearrow & & & & & & & \\ \Phi(\Xi) & & \uparrow F_{pr} & & \uparrow F_{pr} & & \uparrow F_{pr} & & \\ & \searrow & & & & & & & \\ & & \tilde{\Phi}'(\Xi) & \rightleftharpoons & \tilde{\Phi}(\Xi) & \hookrightarrow & \mathcal{R}^*_{\cup}(\Sigma) & \hookrightarrow & \mathcal{R}_{\cup}(\Sigma) \end{array} \tag{3.1}$$

3.2 Soft induction versus hard induction

It was already pointed out that a soft induction $e \in \mathcal{E}(\Xi)$ cannot simply be considered as 'a hard induction that certainly preserves Ξ'. It was indicated that the only way to interpret a soft induction e within the framework of hard inductions, is by considering $e \vee \varphi_D$. This comes down to 'giving up the certainty' that e preserves Ξ. Here we will briefly formalize this idea. For reasons of formal simplicity we will work on the level of the carriers of transition. We introduce the following notation:

- $A_{\mathcal{K}}$ stands for $\{T \in dom(f) \mid f(T) = \emptyset\} = \mathcal{K}$

Proposition 14 *If $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ meets $A_{\vee}, A_{\mathcal{K}}$ then $\exists! K \in \mathcal{P}(\Sigma) : \mathcal{K} = \{T \in \mathcal{P}(\Sigma) \mid T \subseteq K\}$ and if it meets $A_{\vee}, A_{\mathcal{K}}, A_*$ then $\exists! K \in \mathcal{F}(\Sigma) : \mathcal{K} = \{T \in \mathcal{P}(\Sigma) \mid T \subseteq K\}$. If $f : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)$ meets $A_{\vee}, A_{\mathcal{K}}$ then $\exists! K \in \mathcal{F}(\Sigma) : \mathcal{K} = \{F \in \mathcal{F}(\Sigma) \mid F \subseteq K\}$.*

Proof: Resp. (i) Set $K = \cup\mathcal{K}$; (ii) Considering $K = \cup\mathcal{K}$, we have $f(\mathcal{C}(K)) \subseteq \mathcal{C}(f(K)) = \emptyset$, hence $\mathcal{C}(K) = K$ by uniqueness of K; (iii) Set $K = \vee\mathcal{K}\bullet$

The above propositions show that it is permitted to write A_K instead of $A_{\mathcal{K}}$. Remark that A_K is a generalization of the condition $A_\emptyset$ (concerning maps that meet $A_\vee$ and A_*). Evidently, $\mathcal{R}^*_{K,\cup}(\Sigma) \subseteq \mathcal{R}^*_\cup(\Sigma)$ and $\mathcal{S}_{K,\vee}(\Sigma) \subseteq \mathcal{S}_\vee(\Sigma)$, and of course the operations $\vee$ and & are inherited.

Proposition 15 *$\forall K \in \mathcal{F}(\Sigma)$ we have $(\mathcal{R}^*_{K,\cup}(\Sigma), \vee)$, resp. $(\mathcal{S}_{K,\vee}(\Sigma), \vee)$, is a complete join subsemilattice of $(\mathcal{R}^*_\cup(\Sigma), \vee)$, resp. $(\mathcal{S}_\vee(\Sigma), \vee)$.*

Proof: We need to show that $\vee$ respects the condition A_K: For $\{f_i\}_i \subseteq \mathcal{R}^*_{K,\cup}(\Sigma) : \{T \in \mathcal{P}(\Sigma) \mid (\vee_i f_i)(T) = \emptyset\} = \{T \in \mathcal{P}(\Sigma) \mid \forall i : f_i(T) = \emptyset\} = \cap_i \{T \in \mathcal{P}(\Sigma) \mid f_i(T) = \emptyset\} = \mathcal{K}$•

In general, $\mathcal{R}^*_{K,\cup}(\Sigma)$ is not a subquantale of $\mathcal{R}^*_\cup(\Sigma)$, let alone a unitary subquantale of $\mathcal{R}^*_\cup(\Sigma)$. This is due to the fact that & does not preserve the condition A_K unless $K = \emptyset$. Also note that $\mathcal{R}^*_{K,\cup}(\Sigma)$ does not contain the unit element unless $K = \emptyset$, for the unit element has $K = \emptyset$. However, we have that $\forall K \in \mathcal{F}(\Sigma)$ we have $F_{pr} : \mathcal{R}^*_{K,\cup}(\Sigma) \to \mathcal{S}_{K,\vee}(\Sigma)$ is a morphism of complete join semi-lattices with as a limiting case that $F_{pr} : \mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma)$ is a unitary quantale morphism. Essentially, we conclude this section by stating that the formal correspondence expressed in the fact that $\mathcal{R}^*_{\emptyset,\cup}(\Sigma) \to \mathcal{R}^*_{\emptyset,\cup}(\Sigma) : f \mapsto f \vee \tilde{\varphi}_D$, resp., $\mathcal{S}_{\emptyset,\vee}(\Sigma) \to \mathcal{S}_{\emptyset,\vee}(\Sigma) : f \mapsto f \vee \bar{\varphi}_D$, are unitary quantale isomorphisms has no physical counterpart on the level of $\tilde{\mathcal{E}}$ and $\tilde{\Phi}$, resp., $\bar{\mathcal{E}}$ and $\bar{\Phi}$, that yields a commuting diagram when the above stated isomorphisms are combined with eq.(2.3) & eq.(2.6) (or eq.(2.7)) and eq.(3.1).

4 Further aims

In this paper we intensively studied the collection of soft inductions $\mathcal{E}(\Xi)$ and we showed that quantale structures emerge in a natural way due to temporal composition and arbitrary choice of non-deterministic state transitions and the associated notion of a property transition. We also considered the case of hard inductions, but only relative to the scheme for soft inductions developed in this paper. A more evolved scheme could be developed by taking into account how the entity changes due to a hard induction. Attempts in this direction can be found in [5, 7], and in particular we showed there how the concept of induction can be seen as a starting point for the description of compound entities. Further papers on this topic are in preparation.

Appendix 1: Categories

Here we provide only those definitions of category theory that we use in this paper. For more details, see for example [1].

Definition 3 *A category is a quadruple $(Ob, Hom, id, \circ)$ consisting of:*

(1) A class Ob of objects;

(2) For each ordered pair (A, B) of objects a set $Hom(A, B)$ of morphisms;

(3) For each object A a morphism $id_A \in Hom(A, A)$;

(4) A composition law associating to each pair of morphisms $f \in Hom(A, B)$ and $g \in Hom(B, C)$ a morphism $g \circ f \in Hom(A, C)$ which is such that:

(4.1) Composition is associative;

(4.2) $id_B \circ f = f = f \circ id_A$ for all $f \in Hom(A, B)$;

(4.3) the sets $Hom(A, B)$ are pairwise disjoint.

In the first section of this paper we discuss not only the two categories **CALAT** and **T$_1$SPACE**, but also, and foremost, how they are related.

Definition 4 *A functor from the category* $\mathbf{X}$ *to the category* $\mathbf{Y}$ *is a family of maps F which associates to each object A in* $\mathbf{X}$ *an object FA in* $\mathbf{Y}$*, and to each morphism $f \in Hom(A, B)$ a morphism $Ff \in Hom(FA, FB)$, fulfilling:*

(1) $Fid_A = id_{FA}$ for all $A \in Ob$;

(2) $F(g \circ f) = Fg \circ Ff$ for all $f \in Hom(A, B), g \in Hom(B, C)$.

A trivial example of a functor is the identity functor on a category $\mathbf{X}$, denoted by $id_{\mathbf{X}}$. The next step is to consider all functors between given categories $\mathbf{X}$ and $\mathbf{Y}$ as "objects", and then define a "morphism" between functors as:

Definition 5 *A natural transformation from the functor $F : \mathbf{X} \to \mathbf{Y}$ to another functor $G : \mathbf{X} \to \mathbf{Y}$ is a map θ which assigns to each object A of* $\mathbf{X}$ *a morphism $\theta_A \in Hom(FA, GA)$ in* $\mathbf{Y}$*, such that for each $f \in Hom(A, B)$ in* $\mathbf{X}$ *we have that $\theta_B \circ Ff = Gf \circ \theta_A$.*

In this paper, we use two functors: (1) $\mathcal{F}$: **T$_1$SPACE** $\to$ **CALAT** that associates to every T_1-closure space $(X, \mathcal{C})$ a complete atomistic lattice of closed subsets $\mathcal{F}(X)$ and works 'functorial' on morphisms; (2) $\mathcal{A}$: **CALAT** $\to$ **T$_1$SPACE** that associates to every complete atomistic lattice the set of its atoms and an appropriate T_1-closure, with the obvious extension to the morphisms.

Definition 6 *We define $F : \mathbf{X} \to \mathbf{Y}$ to be the left adjoint of $G : \mathbf{Y} \to \mathbf{X}$ (and G is then right adjoint to F), written $F \dashv G$, if there exist natural transformations $\eta : id_{\mathbf{X}} \to G \circ F$ and $\varepsilon : F \circ G \to id_{\mathbf{Y}}$ such that $\varepsilon F \circ F\eta = idF$ and $G\varepsilon \circ \eta G = idG$.*

The above definition applies in the case that we consider $\mathcal{F}$: **T$_1$SPACE** $\to$ **CALAT** and $\mathcal{A}$: **CALAT** $\to$ **T$_1$SPACE**. But also have that for each complete atomistic lattice $\mathcal{L}$, that

$\mathcal{L} \cong \mathcal{F}(\mathcal{A}(\mathcal{L}))$ in a natural way, and also, for each T_1-closure space $(X, \mathcal{C})$, that $X \cong \mathcal{A}(\mathcal{F}(X))$ in a natural way.

Definition 7 *If, referring to Definition 6, η and ε are natural isomorphisms, then F and G are said to define a categorical equivalence.*

Appendix 2: Atomically generated maps

In this section, we elaborate on atomically generated maps, as introduced in [6]. However, we will work in a more general fashion.

Proposition 16 *Let $\mathcal{L}$ be a poset with minimal element 0 and atoms Σ, let $\mathcal{M}$ be a complete join semi-lattice. Given a map $f' : \Sigma \to \mathcal{M}$, the map defined by $f : \mathcal{L} \to \mathcal{M} : a \mapsto \vee\{f'(s) \mid s \in \Sigma, s \leq a\}$ is: (i) isotone, (ii) maps 0 on 0, and (iii) is an extension of f' (i.e., $f_{|\Sigma} = f'$).*

Proof: (i) $a \leq b \Rightarrow \{s \in \Sigma \mid s \leq a\} \subseteq \{s \in \Sigma \mid s \leq b\} \Rightarrow \{f'(s) \mid s \in \Sigma, s \leq a\} \subseteq \{f'(s) \mid s \in \Sigma, s \leq b\} \Rightarrow f(a) \leq f(b)$; (ii) $f(0) = \vee\{f'(s) \mid s \in \Sigma, s \leq 0\} = \vee\emptyset = 0$; (iii) $\forall t \in \Sigma : f(t) = \vee\{f'(s) \mid s \in \Sigma, s \leq t\} = \vee\{f'(t)\} = f'(t)$•

Definition 8 *Referring to Proposition 16, we say that $f' : \Sigma \to \mathcal{M}$ is an atomic map, and that $f : \mathcal{L} \to \mathcal{M}$ is atomically generated by f'.*

Note that in general an isotone extension of an atomic map need not be unique. However, by uniqueness of joins, we have that there is a unique atomically generated map for each given atomic map. Conversely, in the light of Proposition 16, part (iii), we have that each atomically generated map is the extension of exactly one atomic map.

Proposition 17 *Let $\mathcal{L}$ be a poset with minimal element 0 and atoms Σ, let $\mathcal{M}$ be a complete join semi lattice. Then we have: $f : \mathcal{L} \to \mathcal{M}$ is an atomically generated map $\Leftrightarrow \forall a \in \mathcal{L} : \vee\{f(s) \mid s \in \Sigma, s \leq a\} = f(a)$.*

Proof: f is the atomically generated map with respect to $f' \equiv f_{|\Sigma}$•

In fact, for any isotone map $f : \mathcal{L} \to \mathcal{M}$, we have that $\forall a \in \mathcal{L} : \vee\{f(s) \mid s \in \Sigma, s \leq a\} \leq f(a)$ (since by isotonicity we have that $s \leq a \Rightarrow f(s) \leq f(a)$). The atomically generated maps are then exactly those maps $f : \mathcal{L} \to \mathcal{M}$ that saturate this inequality.

Proposition 18 *Let $\mathcal{L}$ be a complete atomistic lattice, with atoms Σ, let $\mathcal{M}$ be a complete join semi-lattice. Then we have: $f : \mathcal{L} \to \mathcal{M}$ is an atomically generated map $\Leftrightarrow f : \mathcal{L} \to \mathcal{M}$ preserves joins of sets of the form $\{s \in \Sigma \mid s \leq a\}, \forall a \in \mathcal{L}$.*

Proof: $\vee\{f(s) \mid s \in \Sigma, s \leq a\} = f(\vee\{s \in \Sigma \mid s \leq a\}) = f(a)$, making use of isotonicity of f in the second equality, and atomicity of $\mathcal{L}$ in the third. In the light of Proposition 17, this proves our claim•

Definition 9 *For an atomic map $f' : \Sigma \to \mathcal{M}$, we define $ker(f') = \{s \in \Sigma \mid f'(s) = 0\}$ and call this the kernel of f'.*

Proposition 19 *Let $\mathcal{L}$ be a poset with minimal element 0 and atoms Σ, let $\mathcal{M}$ be a complete join semi-lattice. Then we have: there is a one-one correspondence between the atomic maps $f' : \Sigma \to \mathcal{M}$ with $ker(f') = \emptyset$ on the one hand, and atomically generated maps $f : \mathcal{L} \to \mathcal{M}$ with $\forall t \in \mathcal{L} : f(t) = 0 \Leftrightarrow t = 0$ on the other.*

Proof: $(\to)$ $\forall t \in \mathcal{L}$ we define $f(t) = \vee\{f'(s) \mid s \in \Sigma, s \leq t\}$, then $f(t) = 0 \Leftrightarrow \{f'(s) \mid s \in \Sigma, s \leq t\} = \{0\}$ or $\{f'(s) \mid s \in \Sigma, s \leq t\} = \emptyset$. We have: $\{f'(s) \mid s \in \Sigma, s \leq t\} = \{0\} \Leftrightarrow \forall s \in \Sigma, s \leq t : f'(s) = 0 \Leftrightarrow \forall s \in \Sigma, s \leq t : s \in ker(f')$, but $ker(f') = \emptyset$ by hypothesis, hence this is impossible. On the other hand we have: $\{f'(s) \mid s \in \Sigma, s \leq t\} = \emptyset \Leftrightarrow \not\exists s \in \Sigma : s \leq t \Leftrightarrow t = 0$. $(\leftarrow)$ Suppose that $ker(f') \neq \emptyset$, then $\exists s \in \Sigma : s \in ker(f') \Rightarrow f'(s) = 0 = f(s)$ (using that $f' \equiv f_{|\Sigma}$) $\Rightarrow s = 0$ (using the hypothesis that $\forall t \in \mathcal{L} : f(t) = 0 \Leftrightarrow t = 0$), which is impossible since $0 \notin \Sigma$•

Appendix 3: Quantales

We only give basic definitions. Detailed discussions can be found in [11, 15].

Definition 10 *A quantale is a complete join semi-lattice $(Q, \vee)$ equipped with an associative product, $\& : Q \times Q \to Q$, which satisfies $\forall a, b_i \in Q$: $a\&(\vee_i b_i) = \vee_i(a\&b_i)$, $(\vee_i b_i)\&a = \vee_i(b_i\&a)$. This quantale is called unitary if there exists a so-called unit element $e \in Q$ which satisfies $\forall a \in Q : e\&a = a = a\&e$. It is called involutive if it is equipped with an involution with an involution, i.e., with $* : Q \to Q : a \mapsto a^*$ satisfying for all a, b, a_i in Q: $a^{**} = a$, $(a\&b)^* = b^*\&a^*$, $(\vee_i a_i)^* = \vee_i a_i^*$. Given two quantales Q and Q', we call $\phi : Q \to Q'$ a quantale morphism if it preserves $\&$ and $\vee$; in the case of unitary quantales we also require $\phi(e) = e'$; in the case of involutive quantales we require preservation of the involution.*

Note that a morphism of quantales is a morphism of the underlying complete join semi-lattices, however this does not imply that it is a morphism of the complete lattice: it can still happen that $\phi(\wedge_i a_i) \neq \wedge_i(\phi(a_i))$.

Definition 11 *An element $a \in Q$ of a quantale is said to be right-sided if $a\&b \leq a$ for all $b \in Q$. Similarly, $a \in Q$ is said to be left-sided if $b\&a \leq a$ for all $b \in Q$.*

We denote $R(Q)$ for the right-sided elements of a quantale Q, and $L(Q)$ for its left-sided elements. It can easily be verified that $R(Q)$ and $L(Q)$ are quantales with respect to the operations of the quantale Q.

Definition 12 *A quantale Q will be said to be a Gelfand quantale if Q is unitary, involutive, and satisfies $\forall a \in R(Q) : a\&a^*\&a = a$.*

The above condition is equivalent to $\forall a \in L(Q) : a\&a^*\&a = a$ since $a \in R(Q) \Leftrightarrow a^* \in L(Q)$. The following property can also be found in [12].

Proposition 20 *Let $(\mathcal{L}, \vee, \perp)$ be a complete orthocomplemented join semi-lattice; let $Q(\mathcal{L})$ be the set of all $\vee$-preserving maps $f : \mathcal{L} \to \mathcal{L}$. Then define a join $\vee$ on $Q(\mathcal{L})$ relative to the pointwise order of the maps, and define a binary operation $\&$ as $f\&g = g \circ f$. Then we have that $(Q(\mathcal{L}), \vee, \&)$ is a Gelfand quantale.*

The proof is based on the following definition for an involution $* : Q(\mathcal{L}) \to Q(\mathcal{L})$: for $\varphi \in Q(\mathcal{L})$, for $m \in \mathcal{L}$, set $\varphi^*(m) = [\vee\{n \in \mathcal{L} \mid \varphi(n) \leq m^\perp\}]^\perp$, where on the right hand side of the equation the $\vee$ and the $\perp$ refer to the join and the orthocomplement of $\mathcal{L}$. Any Gelfand quantale Q that is isomorphic to a quantale $Q(\mathcal{L})$ as in Proposition 20 is called Hilbert quantale. Then, in [12], it is shown that these can be characterized:

Proposition 21 *A Gelfand quantale Q, of which the subquantale of right-sided elements is denoted by $R(Q)$, is a Hilbert quantale if and only if the map $\mu : Q \to Q(R(Q)) : a \mapsto \pi_a$, with $\pi_a : R(Q) \to R(Q) : b \mapsto a^*\&b$, is an isomorphism of quantales.*

In the proof (see [12]) it is stipulated that, for any complete orthocomplemented join semi-lattice $\mathcal{L}$, there is an isomorphism of complete orthocomplemented lattices: $\mathcal{L} \cong R(Q(\mathcal{L}))$, which yields an orthocomplement on $R(Q(\mathcal{L}))$. Thus, for any Hilbert quantale Q, an orthocomplement can be defined on $R(Q)$. Similar reasonings yield an orthocomplement on $L(Q)$. In connection to the quantales considered in this paper, it is clear that all this material applies on $\mathcal{R}_\cup(\Sigma)$, since $\mathcal{P}(\Sigma)$ is a complete orthocomplemented lattice with $\forall T \in \mathcal{P}(\Sigma) : T^\perp = \Sigma \backslash T$. Thus we have that $\mathcal{R}_\cup(\Sigma)$ is a Hilbert quantale, which means that we can define an orthocomplement on its elements. One then asks whether $\mathcal{R}^*_{\emptyset,\cup}(\Sigma)$ also is a Hilbert quantale. If yes, this would mean that an orthocomplement can be stated explicitly for each of its elements. Unfortunately, imposing condition $A_\emptyset$ puts a spoke in the wheel: the quantale $\mathcal{R}_{\emptyset,\cup}(\Sigma)$ is not even involutive! A counterexample would be the map $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that maps (i) $\emptyset \mapsto \emptyset$ and (ii) $\forall T \in \mathcal{P}(\Sigma) \setminus \{\emptyset\} : T \mapsto \{s\}$ for some $s \in \Sigma$. Then we have: $f^*(\{s\}^\perp) = [\cup\{T \mid f(T) \leq \{s\}^{\perp\perp}\}]^\perp = \Sigma^\perp = \emptyset$. Since $\{s\}^\perp \neq \emptyset$ we have that f^* does not meet $A_\emptyset$, hence $f^* \notin \mathcal{R}_{\emptyset,\cup}(\Sigma)$. An analogous reasoning holds for $\mathcal{S}_\vee$, in the case that the property lattice $\mathcal{L}$ is axiomatized to be orthocomplemented: then Proposition 20 says that $\mathcal{S}_\vee$ is a Hilbert quantale, but again imposing condition $A_\emptyset$ prevents $\mathcal{S}_{\emptyset,\vee}$ from being a Hilbert quantale.

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References

[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and Concrete Categories*, J. Wiley & Sons (1990).

[2] D. Aerts, *The One and the Many*, PhD-Thesis, Vrije Universiteit Brussel, TENA (1981); Found. Phys. **12**, 1131 (1982).

[3] D. Aerts, *Found. Phys.* **24**, 1227 (1994).

[4] G. Birkhoff, *Lattice Theory*, Am. Math. Soc., Providence (1967).

[5] B. Coecke, *Found. Phys.* **28**, 1109 (1998).

[6] B. Coecke and D.J. Moore, 'Decompositions of Probability Measures on Complete Lattices in Join Preserving Maps', in preparation (n.d.).

[7] B. Coecke and I. Stubbe, 'On a Duality of Quantales emerging from an Operational Resolution' and 'Operational Resolutions and State Transitions in a Categorical Setting', submitted for publication (n.d.).

[8] Cl.-A. Faure, *Cah. Topol. Geom. Diff. Cat.* **35**, 309 (1994).

[9] D.J. Moore, *Helv. Phys. Acta* **68**, 658 (1995).

[10] D.J. Moore, 'On State Spaces and Property Lattices', *Stud. Hist. Phil. Mod. Phys.*, to appear (1998).

[11] C.J. Mulvey, *Rend. Circ. Math. Palermo* **12**, 99 (1986).

[12] C.J. Mulvey and J.W. Pelletier, *Can. Math. Soc. Conference Proceeding* **13**, 345 (1992).

[13] C. Piron, *Foundations of Quantum Physics*, W. A. Benjamin, Inc. (1976).

[14] C. Piron, *Mécanique Quantique*, Presses Polytechniques et Universitaires Romandes (1990).

[15] K.I. Rosenthal, *Quantales and Their Applications*, Pitmann Research Notes in Mathematical Series **234**, Longmann Scientific & Technical, Essex (1990).