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Autor: Riss, Uwe Volker
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Extension of the Hilbert Space by J–Unitary Transformations

By Uwe Volker Riss

Theoretische Chemie, Physikalisch–Chemisches Institut, Universität Heidelberg,
Im Neuenheimer Feld 253, 69120 Heidelberg, Germany

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Abstract. A theory of non–unitary unbounded similarity transformation operators is developed. To this end the class of J–unitary operators U is introduced. These operators are similar to unitary operators in their algebraic aspects but differ in their topological properties. It is shown how J–unitary operators are related to so–called J–biorthonormal systems and J–selfadjoint projections. Families $\{U_\alpha\}$ of J–unitary operators define in a natural way a Fréchet subspace of the Hilbert space $\mathcal{H}$, the dual space of which constitutes an extension of $\mathcal{H}$. J–unitary transformations of a J–selfadjoint Hamilton operator H can be regarded as representations of H in different Hilbert spaces all including the same Fréchet subspace. The J–selfadjoint Hamilton operator H can also be regarded as a restriction of an operator H' defined on the extension of the Hilbert space. The advantages of a J–unitary transformation theory and the relation to other approaches in scattering theory are discussed.

1 Introduction

In the recent years methods applying complex symmetric Hamilton operators have become very popular in various branches of scattering theory. This concerns both conceptual pictures of quantum mechanics: the time–dependent as well as the time–independent one.

In time-independent quantum mechanics the relation between resonances and complex poles of the analytical continuation of the S matrix to the non–physical sheet has been a well known fact for many years [1]. These poles can also be related to generalized (non square–integrable) Gamow–Siegert eigenfunctions of the Hamilton operator which belong to complex eigenvalues [2, 3]. An efficient computation of these states was made feasible by the introduction of the complex scaling method [4, 5, 6]. Here a rotation of the spatial coordinate from the real axis into the complex plane causes the Gamow–Siegert eigenfunctions to become square–integrable. As a novel result of this rotation one obtains a non–Hermitian Hamilton operator. Consequently the usual spectral theory for selfadjoint operators [7] can no longer be applied. However, it is remarkable that here the development has lead in a direct way from the mathematical theory to practical applications [8, 9, 10, 11].

The complex scaling method is based essentially on analytical continuation techniques for the Hamilton operator. This is convenient for many theoretical considerations but constitutes an obstacle for practical applications. Therefore other related approaches remove the continuation of the operator to a continuation of the basis functions, as in the complex basis function method [12, 13], the method of unbounded similarity transformations (UST) as considered in ref. [14] or the transformative complex absorbing potential method [15]. All these approaches lead to complex symmetric representations of the Hamilton operator.

In time–dependent quantum mechanics complex perturbations, e. g. complex absorbing potentials [16, 17, 18, 19, 20], lead to complex symmetric Hamilton operators as well. These complex potentials have been introduced to avoid artificial boundary reflections in finite basis set or grid calculations but mathematical investigations of such perturbations are rare. In fact, it was found that the understanding of USTs is also important for the understanding of these complex absorbing potentials [21]. This development demonstrates the interchangeability of these different complex symmetric Hamiltonian methods and the decisive role of USTs in the theory of complex symmetric Hamiltonians. Therefore a theoretical frame combining the central ideas of USTs would be desirable.

Most attempts to study unbounded similarity transformations are rather technical or regard mainly algebraic aspects [22, 23]. Moreover, they are mainly concentrated on the complex scaling approach. Nevertheless, topological aspects are very important and the ansatz can be formulated in a much more general way. This becomes obvious if one compares USTs to usual unitary Hilbert space transformations. Unfortunately, rigorous investigations on unbounded similarity transformations are rather rare. Recently, Löwdin [24] has given an illuminating overview on this subject but many questions about the nature of USTs remain open. It seems to be promising to take a rather abstract point of view to this problem to recognize the essential features. By this abstract access rather technical considerations can be avoided and the main structure becomes apparent. It is the aim of the present paper to provide the mathematical means for a rigorous foundation of the theory of USTs. This is not only necessary for a well–founded transformation theory of Hamilton operators with respect to USTs, which is still missing, but it also reveals possible computational applications.

If we look for an abstract structure of Hamilton operators, which is preserved by usual USTs, we find a property which is known as J–symmetry. J–symmetric operators were introduced by Glazman [25] in the investigation of complex boundary–value problems. An investigation of the spectral properties of J–symmetric operators was performed by Race [26]. We will see that this structure is the key for the understanding of USTs. Here it is important to note that most physical Hamilton operators are not only symmetric but also J–

symmetric. This means that the theory can be applied to a large class of physically relevant Hamilton operators.

In section 2 we will give the basic notations of J–symmetric operators as described in refs. [25, 26]. The following sections comprise two different topics. In the first part the general properties of J–unitary operators are examined. Thus in section 3 the terms J–isometric and J–unitary will be introduced and characterized. In the sections 4 and 5 we will introduce J–biorthonormal systems and J–projections in analogy to orthonormal systems (ONS) and to orthogonal projections, respectively. In the second part it will be demonstrated how J–unitary operators generate extensions of the Hilbert space. Thus in section 6 it will be shown how families of J–unitary operators define a family of (distinct) Hilbert spaces containing a common locally convex vectorspace, and that this construction leads in a natural way to an extension of the original Hilbert space. Based on this construction a J–unitary transformation theory will be introduced in section 7. In section 8 a discussion of the results will be given.

In the following $\mathcal{H}$ denotes a separable complex Hilbert space of infinite dimension endowed with the scalar product $< \cdot \mid \cdot >$, linear in the second component; all operators on $\mathcal{H}$ are assumed to be linear and densely defined.

2 Preliminaries

Let H be an operator in $\mathcal{H}$. $\mathcal{D}(H)$ denotes the domain of H, $\mathcal{N}(H)$ denotes its nullspace, and $\mathcal{R}an(H)$ denotes its range. $H' \subset H$ means that the operator H is an extension of the operator H', i. e. $\mathcal{D}(H') \subset \mathcal{D}(H)$ and $H'x = Hx$ for all $x \in \mathcal{D}(H')$. Additionally, some terms are to be introduced which refer particularly to J–symmetric operators. The expression J–symmetric operator stems from a conjugation operator J in a complex Hilbert space $\mathcal{H}$. A *conjugation* J is characterized as an antilinear involution, i. e. $J^2 = I$ and $J(\lambda x) = \lambda^* Jx$, $x \in \mathcal{H}$, which obeys $< Jx|Jy > = < y|x >$ for all $x, y \in \mathcal{H}$. A vector $x \in \mathcal{H}$ is said to be *J–real* if $Jx = x$. A linear operator H in $\mathcal{H}$ is said to be *J–symmetric* if its domain $\mathcal{D}(H)$ is dense in $\mathcal{H}$ and H satisfies

$$H \subset JH^\dagger J \tag{2.1}$$

where $H^\dagger$ denotes the Hilbert space adjoint of H. If the equation

$$H = JH^\dagger J \tag{2.2}$$

is satisfied, i. e. $\mathcal{D}(H) = \mathcal{D}(JH^\dagger J)$, then the operator H is said to be *J–selfadjoint*. The best known example of a conjugation is the usual complex conjugation $\psi(\vec{r}) \mapsto \psi^*(\vec{r})$ in $L^2(\mathbb{R}^n)$, the Hilbert space of square–integrable functions on $\mathbb{R}^n$. But also $\psi(\vec{r}) \mapsto \psi^*(-\vec{r})$, $\vec{r} \in \mathbb{R}^n$, defines a conjugation that corresponds to the complex conjugation in momentum space. A J–symmetric operator H is said to be *essentially J–selfadjoint* if its closure $\overline{H}$ is J–selfadjoint. An operator H in $\mathcal{H}$ is said to be *J–real* if $J\mathcal{D}(H) \subset \mathcal{D}(H)$ and

$$JH = HJ. \tag{2.3}$$

In order to simplify the notation we introduce two further terms. First, we define a *J-product* by

$$(x|y) :=< Jx|y > \qquad \forall\, x, y \in \mathcal{H}\ . \tag{2.4}$$

This J–product coincides with the c–product introduced by Moiseyev [27, 28] if we take J as the complex conjugation. Second, instead of the adjoint operator $H^\dagger$ we regard the *transposed operator* H^T to H which is defined by

$$H^T := JH^\dagger J. \tag{2.5}$$

This definition fits better to the structure of J–products since it satisfies

$$(Hx|y) = (x|H^T y) \qquad \forall\, x \in \mathcal{D}(H), y \in \mathcal{D}(H^\dagger J) = \mathcal{D}(H^T)\ . \tag{2.6}$$

The transposed operator H^T has similar properties to the adjoint operator.

3 J–isometric and J–unitary operators

Let us commence with the consideration of the complex scaling transformation which can be regarded as the prototype of the class of transformations that we are going to examine. In the complex scaling theory [29] one considers a Hamilton operator H defined in $L^2(\mathbb{R})$ (for sake of simplicity we only regard the one–dimensional case) and a unitary transformation U_θ. It is defined by the dilation group in $L^2(\mathbb{R})$:

$$U_\theta \psi(s) := e^{\theta/2}\psi(e^\theta s) \tag{3.1}$$

for $\theta \in \mathbb{R}_+ := [0, \infty)$ and $\psi \in L^2(\mathbb{R})$, $s \in \mathbb{R}$.

If we look for a structure which is preserved under the complex scaling transformation we find the equality [24]

$$\begin{aligned} \int_R U_\theta\psi(x)U_\theta\varphi(x)\ dx \ &= \ \int_R \psi(e^\theta x)\ \varphi(e^\theta x)\ e^\theta dx \ = \\ \int_{C=e^\theta R} \psi(z)\ \varphi(z)\ dz \ &= \ \int_R \psi(x)\ \varphi(x)\ dx \end{aligned} \tag{3.2}$$

for $\psi, \varphi \in L^2(\mathbb{R})$, $\theta \in \mathbb{R}_+$. By application of Cauchy's theorem this relation is valid also for complex θ as long as ϕ and φ are analytic functions in some open set containing the sector between $\mathbb{R}$ and $e^\theta\mathbb{R}$.

If we take a more abstract point of view this means that the transformation U_θ leaves the J–product invariant (in the example above J is just the complex conjugation). This property fits to operators which are symmetric with respect to the J–product. In fact, the Hamilton operators transformed by complex scaling transformations are J–symmetric, as are most selfadjoint Hamilton operators in quantum mechanics (they are J–real <u>and</u> symmetric). Therefore it seems promising to take this structure as the starting point of the investigation.

It is to be remarked that the consideration of a general conjugation is quite relevant. If we regard the momentum operator $-i\ d/dx$ we find that this operator is essentially selfadjoint if a suitable domain is chosen. But it is not J-real with respect to the usual complex conjugation. On the other hand, if we consider the Fourier transform of the momentum operator, which is just a multiplication operator, we find that this operator is J–real with respect to the complex conjugation in the momentum space. We see that there are also relevant conjugation operators beside the complex conjugation.

The simplest class of operators that provide the conservation of the J–product we will call J–isometric operators since they play that role for the J–product that isometric operators play for the usual inner product of the Hilbert space. These J–isometric operators mainly represent the algebraic aspect of USTs.

Definition 1 *An operator V in the Hilbert space $\mathcal{H}$ is called J–isometric if V leaves the J–product invariant:*

$$(Vx|Vy) = (x|y) \qquad \forall\ x, y \in \mathcal{D}(V)\ . \tag{3.3}$$

It is to be remarked that in order to show that V is J–isometric it is sufficient to prove that $(Vx|Vx) = (x|x)$ for all $x \in \mathcal{D}(V)$.

Although J–isometric operators seem to be very similar to isometric operators there is a decisive difference. Isometric operators are automatically bounded while J–isometric operators are in general unbounded. Therefore topological considerations are here essential. It is to be noted that every J–isometric operator V is injective.

J–isometric operators V leave the J–product invariant but for a transformation theory we also need the existence of a densely defined inverse V^{-1}. To secure this we need stronger conditions on the operator leading to the following definition:

Definition 2 *A J–isometric operator U in $\mathcal{H}$ the range $\mathcal{R}an(U)$ of which is dense in $\mathcal{H}$ is called J–unitary.*

In analogy to unitary operators [30] we can characterize J–unitary operators in the following way. Let U be an operator in $\mathcal{H}$. Then the following propositions are equivalent:

1. U is J–unitary;
2. U^{-1} is J–unitary;
3. U and U^{-1} are J–isometric;
4. U is J–isometric, U^T is injective;
5. U^{-1} is a densely defined operator and it holds that $U^{-1} \subset U^T$.

It is to be noticed that although J–unitary operators are not continuous they are regular in the sense that they are closeable (cf. theorem VIII.1 in ref. [31]). In the following sections we will examine the similarities and differences of J–unitary operators compared to unitary operators.

4 J–biorthonormal systems

We want to show how J–unitary operators can be constructed and represented. This construction is in some aspects comparable to that of unitary operators. It is well known that unitary operators transform one complete orthonormal system (ONB) to another (theorem 4.4 in ref. [30]). Similar assertions hold for J–unitary operators with the exception that the orthonormality must be substituted by a property fitting to J–products.

Definition 3 *A system $\{u_n\}_{n\in\mathbb{N}}$ of vectors in $\mathcal{H}$ is called a J-biorthonormal system (JBS) in $\mathcal{H}$ if it satisfies*

$$(u_n|u_m) = \delta_{n,m} \qquad \forall\ m,n \in \mathbb{N}\ . \tag{4.1}$$

A JBS is said to be complete or a J-biorthonormal basis (JBB) if span($\{u_n\}_{n\in\mathbb{N}}$), the linear span of the vectors u_n, $n \in \mathbb{N}$, is a dense subspace of $\mathcal{H}$.

In the case of the complex conjugation such systems are known as complex conjugate biorthonormal sets [24] . The expression biorthonormal is here employed in a particular sense. In general it is used to characterize two systems $\{v_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ which satisfy $< v_n|w_m >= \delta_{n,m}$. In our case the two systems $\{u_n\}_{n\in\mathbb{N}}$ and $\{Ju_n\}_{n\in\mathbb{N}}$ are biorthonormal in the usual sense. This motivates the term J-biorthonormal for the system $\{u_n\}_{n\in\mathbb{N}}$. In particular one finds that the vectors u_n of a JBS are always linear independent.

One can interpret the previous definition in such a way that J–biorthonormal systems define a certain geometrization of the Hilbert space. But the induced geometry is different from the geometry induced by the Hermitian inner product of the Hilbert space.

In an analogous way to how isometric operators perform the transformation of one orthonormal system to another, J–isometric operators transform one JBS to another. Let U be a J–isometric operator and $\{u_n\}_{n\in\mathbb{N}}$ a JBS in $\mathcal{D}(U)$. Then the system $\{Uu_n\}_{n\in\mathbb{N}}$ is also a JBS. The question of completeness is more difficult to answer. However, it is always possible to construct a JBB from any dense system of linear independent vectors.

Lemma 1 *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be systems of linear independent vectors with $(x_m|y_n) = 0$ for all $m,n \in \mathbb{N}$ and*

$$\overline{span(\{x_n\}_{n\in\mathbb{N}} \cup \{y_n\}_{n\in\mathbb{N}})} = \mathcal{H}. \tag{4.2}$$

Then there exist bijections $\mu : \mathbb{N} \to \mathbb{N}$, $\nu : \mathbb{N} \to \mathbb{N}$, and J-biorthnormal systems $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ such that $\{u_n\}_{n\in\mathbb{N}} \cup \{v_n\}_{n\in\mathbb{N}}$ is a JBB and

$$\begin{aligned} span(\{x_{\nu(n)}\}_{n=0,\cdots,N}) &= span(\{u_n\}_{n=0,\cdots,N})\ , \\ span(\{y_{\nu(n)}\}_{n=0,\cdots,N'}) &= span(\{v_n\}_{n=0,\cdots,N'}) \end{aligned} \tag{4.3}$$

for an infinity of $N, N' \in I\!N$.

PROOF. We start with the consideration of the vector x_0. Let us further regard the two cases (1.) $(x_0|x_0) \neq 0$ and (2.) $(x_0|x_0) = 0$ separately.

In case (1.) we define $u_0 := (x_0|x_0)^{-1/2}x_0$ and $x'_{j-1} := x_j - (u_0|x_j)u_0$ for all $j \geq 1$. The system $\{x'_n\}_{n\in I\!N}$ is still linear independent and it holds that $(x'_n|u_0) = 0$ for all $n \in I\!N$. Every x'_n is a linear combination of x_{n+1} and x_0. Thus eq. (4.3) is valid for $N = 0$. Moreover, we can decompose $\mathcal{H}$ into

$$\mathcal{H} = \mathrm{span}(\{u_0\}) \oplus \mathcal{H}^0 \tag{4.4}$$

where

$$\mathcal{H}^0 := \{x \in \mathcal{H} \ ; \ (u_0|x) = 0\} \tag{4.5}$$

is a closed subspace of $\mathcal{H}$ which can therefore be regarded as a Hilbert space whith

$$\overline{\mathrm{span}(\{x'_n\}_{n\in I\!N})} = \mathcal{H}^0. \tag{4.6}$$

We can proceed by applying the same procedure to $\{x'_n\}_{n\in I\!N}$.

In case (2.) we have to prove that there exists a vector x_n for some $n \in I\!N$ such that $(x_0|x_n) \neq 0$. If $(x_0|x_n) = 0$ is assumed to be valid for all $n \in I\!N$ then it follows that $x_0 = 0$ according to the fact that $(x_0|y_n) = 0$ for all $n \in I\!N$ and (4.2). This would be a contradiction to the linear independence of the vectors x_n, $n \in I\!N$. Therefore we can assume without loss of generality that $(x_0|x_1) \neq 0$. Let us now substitute $x_1 - 1/2\,(x_1|x_1)\cdot(x_0|x_1)^{-1}x_0$ for x_1 such that we can also assume $(x_1|x_1) = 0$ without loss of generality. Let us further consider the vectors

$$u_0 := \frac{i}{2\,(x_0|x_1)}\,x_0 \ - \ ix_1 \tag{4.7}$$

and

$$u_1 := \frac{1}{2\,(x_0|x_1)}\,x_0 \ + \ x_1. \tag{4.8}$$

We calculate $(u_0|u_0) = (u_1|u_1) = 1$ and $(u_0|u_1) = 0$. Defining the vectors $x'_{j-2} := x_j - (u_0|x_j)u_0 - (u_1|x_j)u_1$ for all $j \geq 2$ we obtain a result analogous to (1.).

The mapping $\mu : I\!N \to I\!N$ is defined by the exchange of the vectors x_n due to the construction above. The same procedure can be applied to the system $\{y_n\}_{n\in I\!N}$ obtaining $\{v_{\nu(n)}\}_{n\in I\!N}$ and $\nu : I\!N \to I\!N$. Since $\mathrm{span}(\{x_n\}_{n\in I\!N} \cup \{y_n\}_{n\in I\!N})$ is dense in $\mathcal{H}$ the same is true for $\mathrm{span}(\{u_{\mu(n)}\}_{n\in I\!N} \cup \{v_{\nu(n)}\}_{n\in I\!N})$. Thus $\{u_{\nu(n)}\}_{n\in I\!N} \cup \{v_{\nu(n)}\}_{n\in I\!N}$ is a JBB. QED

It is to be remarked that naturally the lemma holds analogously if one of the systems $\{x_n\}_{n\in I\!N}$ or $\{y_n\}_{n\in I\!N}$ is substituted by a finite system.

In the case of orthonormal systems one has the property that every maximal orthonormal system is complete. This assertion is no longer true for J–biorthonormal systems. Let α and β be complex numbers with $\alpha^2 + \beta^2 = 1$ and $|\beta|, |\alpha| \geq 1$. We regard the Hilbert space $\ell^2(I\!N)$

of all square–summable sequences. Then we can define a JBS in $\ell^2(\mathbb{N})$ by

$$\begin{aligned} u_0 &= (-\beta, \alpha, 0, \cdots) \\ u_1 &= (-\beta\alpha, -\beta^2, \alpha, 0, \cdots) \\ u_2 &= (-\beta\alpha^2, -\beta^2\alpha, -\beta^2, \alpha, 0, \cdots) \\ u_3 &= (-\beta\alpha^3, -\beta^2\alpha^2, -\beta^2\alpha, -\beta^2, \alpha, 0, \cdots) \\ &\vdots \end{aligned} \tag{4.9}$$

We will show that the system $\{u_n\}_{n\in\mathbb{N}}$ is maximal but not complete.

Let us consider the vector $e_1 = (1, 0, 0, \cdots)$. One finds that $e_1 \notin \overline{\text{span}(\{u_n\}_{n\in\mathbb{N}})}$. To prove this we construct the system

$$\begin{aligned} v_0 &= (1, -\alpha/\beta, 0, \cdots) \\ v_1 &= (1, 0, -\alpha^2/\beta, 0, \cdots) \\ v_2 &= (1, 0, 0, -\alpha^3/\beta, 0, \cdots) \\ &\vdots \end{aligned} \tag{4.10}$$

where the vectors v_n are linear combinations of $u_0, \cdots, u_n$, respectively. Let us assume that $e_1 \in \overline{\text{span}(\{u_n\}_{n\in\mathbb{N}})} = \overline{\text{span}(\{v_n\}_{n\in\mathbb{N}})}$. Then we find a linear combination of vectors $v_0, \cdots, v_N$ and complex numbers $\alpha_0, \cdots, \alpha_N$ for every $1 > \epsilon > 0$ and some suitable $N \in \mathbb{N}$ such that

$$\|e_1 - \sum_{k=0}^{N} \alpha_n v_n\|^2 < \epsilon\ . \tag{4.11}$$

From eq. (4.11) we obtain that

$$|1 - \sum_{k=0}^{N} \alpha_n|^2 < \epsilon \tag{4.12}$$

and

$$\sum_{k=1}^{N} \left|\alpha_n^2 \frac{\alpha^{2n}}{\beta^2}\right| < \epsilon\ . \tag{4.13}$$

¿From eq. (4.12) we conclude that

$$\sum_{k=0}^{N} |\alpha_n|^2 \geq \left|\sum_{k=0}^{N} \alpha_n^2\right| > 1 - 3\epsilon\ . \tag{4.14}$$

With eq. (4.13) this leads to the contradiction

$$\epsilon > \sum_{k=1}^{N} \left|\alpha_n^2 \frac{\alpha^{2n}}{\beta^2}\right| > \left|\frac{\alpha}{\beta}\right|^2 (1 - 3\epsilon)\ . \tag{4.15}$$

Therefore e_1 is not an element of $\overline{\text{span}(\{u_n\}_{n\in\mathbb{N}})}$ and the system $\{u_n\}_{n\in\mathbb{N}}$ is not complete. Obviously the vector e_1 is sufficient to complete the linear span, i.e.

$$\overline{\text{span}(\{u_n\}_{n\in\mathbb{N}} \cup \{e_1\})} = \mathcal{H}\ . \tag{4.16}$$

We now want to show that the system $\{u_n\}_{n\in I\!N}$ is maximal. Let us assume that it is not maximal. Then there is a vector $v \in \mathcal{H}$ with $(v|v) = 1$ and $(v|u_n) = 0$ for all $n \in I\!N$. By the latter condition it is now possible to construct this vector $v = (\gamma_0, \gamma_1, \cdots)$, $\gamma_n \in \mathbb{C}$. Starting with the condition $(v|u_1) = 0$ one calculates that

$$v = (\gamma_0, \gamma_0\beta/\alpha, \gamma_2, \gamma_3, \cdots) . \tag{4.17}$$

By componentwise proceeding and choosing $\gamma_0 = 1$ one finally obtains

$$v = (1, \beta/\alpha, \beta/\alpha^2, \cdots) . \tag{4.18}$$

Obviously this is an element of $\ell^2(I\!N)$. Here one calculates that $(v|v) = 0$ in contradiction to the assumption that the system is maximal. In particular

$$\overline{\mathrm{span}(\{u_n\}_{n\in I\!N} \cup \{Jv\})} = \mathcal{H} \tag{4.19}$$

since

$$\frac{1}{2}Jv - \frac{\beta^2}{2\alpha^2}\sum_{k=0}^{n}\frac{1}{\alpha^{2k}}v_k \to e_1 \qquad \text{for} \qquad n \to \infty . \tag{4.20}$$

As a consequence the system $\{u_n\}_{n\in I\!N}$ cannot be extended, i.e. it is maximal but not complete. The following theorem says that this type of incomplete maximal JBS is the only one that can be found.

Theorem 1 *For every incomplete JBS $\{u_n\}_{n\in I\!N}$ there is a vector $v \in \mathcal{H}-\{0\}$ with $(v|u_n) = 0$ for all $n \in I\!N$. Then either it holds that $(v|v) \neq 0$ and $v \in \mathcal{H} - \overline{span(\{u_n\}_{n\in I\!N})}$ such that $\{u_n\}_{n\in I\!N}$ can be complemented and is not maximal or for all $w \in \mathcal{H}$ with $(w|u_n) = 0$, $n \in I\!N$, it holds that $(w|w) = 0$ such that $\{u_n\}_{n\in I\!N}$ is maximal.*

PROOF. From the system $\{u_n\}_{n\in I\!N}$ one can construct an orthonormal system $\{w_n\}_{n\in I\!N}$ by Gram–Schmidt orthonormalization. Since $\{u_n\}_{n\in I\!N}$ is not complete $\{w_n\}_{n\in I\!N}$ is not complete either. Therefore there is a vector $v \in \mathcal{H}$ with $< v\, |w_n >= 0$ for all $n \in I\!N$ for the conjugate Jv of which one calculates

$$(u_n|Jv) = \sum_{k=0}^{n}\alpha_k^n(w_k|Jv) = \sum_{k=0}^{n}\alpha_k^n < Jw_k|Jv > = \sum_{k=0}^{n}\alpha_k^n < v|w_k > = 0 \tag{4.21}$$

for suitable complex number $\alpha_0^n, \cdots, \alpha_n^n$.

Let us assume that $(Jv|Jv) \neq 0$ and $Jv \in \overline{\mathrm{span}(\{u_n\}_{n\in I\!N})}$. In this case we find vectors $x_n \in \mathrm{span}(\{u_n\}_{n\in I\!N})$ with $\|x_n - Jv\| \to 0$ for $n \to \infty$. But this leads to the contradiction $0 = (x_n|Jv) \to (Jv|Jv) \neq 0$. It follows that $Jv \in \mathcal{H} - \overline{\mathrm{span}(\{u_n\}_{n\in I\!N})}$. QED

There is a well-known relation between unitary operators and complete orthonormal systems: A unitary operator maps every complete orthonormal systems to another complete orthonormal system and every two orthonormal systems $\{u_n\}_{n\in I\!N}$ and $\{v_n\}_{n\in I\!N}$ define a unique unitary operator with $Uu_n = v_n$, $n \in I\!N$, (cf. Theorem 4.4 [30]). Here the question arises if there are corresponding assertions for J-unitary operators. Let us start with the first point.

Lemma 2 *Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be two JBBs. Then the operator U defined by*

$$Ux := \sum_n (v_n|x)\, u_n \qquad \forall\, x \in \mathcal{D}(U) := span(\{v_n\}_{n\in\mathbb{N}}) \tag{4.22}$$

is J–unitary.

The proof of this lemma is obvious.

Let us now concentrate on the question if a J–unitary operator U always transforms a JBB $\{u_n\}_{n\in\mathbb{N}}$ to another JBB $\{Uu_n\}_{n\in\mathbb{N}}$. Also here we obtain a property which corresponds to the unitary case but the proof is more complicated.

Theorem 2 *Let U be a J–unitary operator in $\mathcal{H}$. Then there exists a JBB in $\mathcal{D}(U)$ and U transforms every JBB in $\mathcal{D}(U)$ to a JBB in $\mathcal{R}an(U)$.*

PROOF. Since U is closeable the graph of U

$$\Gamma(U) := \{(x, Ux) \in \mathcal{H}\times\mathcal{H} \;:\; x \in \mathcal{D}(U)\} \tag{4.23}$$

can be closed as a subspace of the Hilbert space $\mathcal{H}\times\mathcal{H}$. One endows $\overline{\Gamma(U)}$ with the inner product

$$< (x, Ux) \mid (y, Uy) >_{\Gamma(U)} \;:=\; < x|y > + < Ux|Uy > \tag{4.24}$$

for all $(x, Ux), (y, Uy) \in \overline{\Gamma(U)}$. This inner product makes $\overline{\Gamma(U)}$ a Hilbert space. A bilinear form in $\overline{\Gamma(U)}$ is given by

$$(\,(x, Ux) \mid (y, Uy)\,)_{\Gamma(U)} := (x|y) = (Ux|Uy)\;. \tag{4.25}$$

Since this bilinear form defines a continuous linear functional $(\,(x, Ux) \mid \cdot\,)_{\Gamma(U)}$ for every $(x, Ux) \in \overline{\Gamma(U)}$ the theorem of Riesz states that there is an element $(y, Uy) \in \overline{\Gamma(U)}$ with $(\,(x, Ux) \mid \cdot\,)_{\Gamma(U)} = \;< (y, Uy) \mid \cdot\; >_{\Gamma(U)}$. This way we can define the conjugation $\mathcal{J}$ in $\overline{\Gamma(U)}$ by $\mathcal{J}(y, Uy) := (x, Ux)$ such that (4.25) becomes the corresponding $\mathcal{J}$–product.

The existence of a JBB in $\mathcal{D}(U)$ follows directly from lemma 1 since $\mathcal{H}$ and therefore $\Gamma(U)$ are separable. Consequently there is a complete system $\{\,(u_n, Uu_n)\,\}_{n\in\mathbb{N}}$ in $\Gamma(U)$ that can be transformed to a JBB.

Let us now assume that $\{u_n\}_{n\in\mathbb{N}}$ is a JBB in $\mathcal{D}(U)$ but $\underline{\{Uu_n\}_{n\in\mathbb{N}}}$ does not form a dense subspace of $\mathcal{H}$ and neither does $\underline{\{\,(u_n, Uu_n)\,\}_{n\in\mathbb{N}}}$ with respect to $\overline{\Gamma(U)}$. Then it follows from theorem 1 that there is an element $(v, Uv) \in \overline{\Gamma(U)}$ with $(v, Uv) \neq (0,0)$ and

$$(\,(u_n, Uu_n) \mid (v, Uv)\,)_{\Gamma(U)} = 0 \tag{4.26}$$

for all $n \in \mathbb{N}$. Since $(u_n|v) = 0$ for all $n \in \mathbb{N}$ and $\{u_n\}_{n\in\mathbb{N}}$ was assumed to be complete it follows that $v = 0$ in contradiction to the condition $(v, Uv)) \neq (0,0)$. QED

The last proposition shows that although J–unitary operators are unbounded and thus rather different from unitary operators the central properties are similar. In particular they conserve a certain kind of coordinate systems that are represented by JBBs. The following section will emphasize this similarity further.

5 J–projections

The simplest class of J–selfadjoint operators consists in appropriate projections. We will study this class as a prototype of the class of J–selfadjoint operators since here the connection between JBBs and representations of the operators is rather obvious. A central term in the theory of orthogonal projections is that of the orthogonal complement. We will therefore begin with an equivalent definition for the J–product.

Definition 4 *For a subset $A \subset \mathcal{H}$ we denote by $A^{\perp(J)}$ the set*

$$A^{\perp(J)} := \{x \in \mathcal{H} \ : \ (x|y) = 0 \ \ \forall y \in A\} \ . \tag{5.1}$$

$A^{\perp(J)}$ is called the J-biorthogonal complement of A.

The J–biorthogonal complement of A is related to the orthognal complement by

$$A^{\perp(J)} \ = \ (JA)^{\perp} \ = \ J\,A^{\perp} \ . \tag{5.2}$$

This means that many properties of the orthogonal complement can directly be transfered to the J–biorthogonal complement:

1. For every subset $A \in \mathcal{H}$ the J–biorthogonal complement is a closed subspace of $\mathcal{H}$ and it holds that
$$A^{\perp(J)} \ = \ \mathrm{span}(A)^{\perp(J)} \ = \ \overline{\mathrm{span}(A)}^{\perp(J)} \tag{5.3}$$
and
$$(A^{\perp(J)})^{\perp(J)} \ = \ \overline{\mathrm{span}(A)} \ . \tag{5.4}$$
2. From $A \subset B$ it follows that $B^{\perp(J)} \subset A^{\perp(J)}$.
3. The set $A \subset \mathcal{H}$ is dense in $\mathcal{H}$ if and only if $A^{\perp(J)} = \{0\}$.

Nevertheless there is a decisive difference to the orthogonal complement. In the latter case every closed subspace M of $\mathcal{H}$ allows a unique decompositon of the Hilbert space $\mathcal{H} = M \oplus M^{\perp}$ into M and its orthogonal complement $M^{\perp}$. In the case of the J–biorthogonal complement this is only possible for a special class of closed subspaces of $\mathcal{H}$.

Definition 5 *A decomposition of the Hilbert space $\mathcal{H}$ into two subspaces M and N is called J-decomposition if $\mathcal{H} = \overline{M + N}$ and $(x|y) = 0$ is valid for all $x \in M$, $y \in N$. In this case we write*

$$\mathcal{H} \ = \ M \overset{J}{\oplus} N \ . \tag{5.5}$$

A subset $A \subset \mathcal{H}$ is said to be J-projective if it obeys $A \cap A^{\perp(J)} \subset \{0\}$.

There are simple examples of sets which are not J–projective. Let us consider a vector $u \in \mathcal{H} - \{0\}$ with $(u|u) = 0$. The set $\{u\}$ is not J–projective since $u \in \{u\}^{\perp(J)}$. We will see that only J–projective subspaces allow a decomposition (5.5).

Lemma 3 *For every J–projective subspace M of $\mathcal{H}$ there is the J–decompostion*

$$\mathcal{H} \;=\; M \overset{J}{\oplus} M^{\perp(J)} \; . \tag{5.6}$$

For two closed subspaces M and N of $\mathcal{H}$ for which eq. (5.5) is valid it holds that $N = M^{\perp(J)}$ and both subspaces are J–projectivs.

PROOF. In order to prove eq. (5.6) we only have to show that $M + M^{\perp(J)}$ is dense in $\mathcal{H}$. This is equivalent to the assertion that $(M + M^{\perp(J)})^{\perp(J)} = \{0\}$. But this is a consequence of the fact that M is J–projective since

$$(M + M^{\perp(J)})^{\perp(J)} \;=\; M^{\perp(J)} \cap (M^{\perp(J)})^{\perp(J)} \;=\; M^{\perp(J)} \cap M \;=\; \{0\} \; . \tag{5.7}$$

For the second part it holds that $N \subset M^{\perp(J)}$. Since $M + N$ is dense in $\mathcal{H}$ and N is closed we obtain $N = M^{\perp(J)}$. Since (5.7) is valid M is J–projective. The same follows for N. QED

We now turn to the relation between J–decompositions and J–unitary operators. From theorem 2 and lemma 1 we obtain the following characterization of J–unitary operators:

Lemma 4 *A (densely defined) operator U is J–unitary if and only if U transfers every J–decomposition*

$$\mathcal{H} \;=\; M \overset{J}{\oplus} N \tag{5.8}$$

with $M + N \subset \mathcal{D}(U)$ to a J–decomposition

$$\mathcal{H} \;=\; U(M) \overset{J}{\oplus} U(N) \; . \tag{5.9}$$

Since $\mathcal{D}(U)$ is dense in $\mathcal{H}$ lemma 1 secures that there is a sufficient number of J–decompostions of $\mathcal{D}(U)$ to make the assertion meaningful. The previous lemma roughly says that J–unitary operators are typified by the conservation of J–decompositions.

Another term which is closely related to orthonormal systems is that of the orthogonal projection. A similar term can be introduced here. Let us consider the following example. For some fixed $u \in \mathcal{H}$ with $(u|u) = 1$ we define the operator

$$P_u x \;:=\; (u|x)\, u \qquad \forall\, x \in \mathcal{H} \; . \tag{5.10}$$

The operator P_u is a projection, i.e. $P_u^2 = P_u$ and it is J–selfadjoint, i.e. $P_u^T = P_u$. But we also find differences to orthogonal projections. For example the norm of such a J–projection can be arbitrarily large: $\|P_u\| = \|u\|^2 \geq 1$.

If we consider the case of a vector $u \in \mathcal{H}$ with $u \neq 0$ and $(u|u) \;=\; 0$ as discussed by Moiseyev [27] we find that such vectors do not define J–projections since $P_u(Ju) = \|u\|\, u \neq 0$ but $P_u^2(Ju) = 0$ in contradiction to the projection property $P_u^2 = P_u$.

Definition 6 *A densely defined operator P in $\mathcal{H}$ is called a J–projection if it satisfies*

$$P^2 = P \quad \text{and} \quad (x|y) = 0 \qquad \forall\, x \in \mathcal{R}an(P), y \in \mathcal{N}(P). \tag{5.11}$$

If P is a J–projection, then we obtain the J–decomposition

$$\mathcal{H} = \mathcal{R}an(P) \overset{J}{\oplus} \mathcal{R}an(I-P) \;=\; \mathcal{R}an(P) \overset{J}{\oplus} \mathcal{N}(P)\ . \tag{5.12}$$

By similar arguments to those in the orthogonal case one finds:

Lemma 5 *A projection P in $\mathcal{H}$ is a J–projection if and only if P is essentially J–selfadjoint. In particular every J–projection is closeable and every closed J–projection is J–selfadjoint.*

Actually every J–projection P corresponds to a J–projective subspace of $\mathcal{H}$ as the following proposition shows:

Lemma 6 *A subspace M of $\mathcal{H}$ is J–projective if and only if there is a J–projection P_M with $\mathcal{R}an(P_M) \;=\; M$.*

Proof. Let M be a J–projective subspace of $\mathcal{H}$. Then we can choose the dense subspace $M + M^{\perp(J)}$ of $\mathcal{H}$ as the domain of P_M. Since $M \cap M^{\perp(J)} = \{0\}$ every decomposition of $x \in M + M^{\perp(J)}$ into $x = x_1 + x_2$, $x_1 \in M$, $x_2 \in M^{\perp(J)}$ is unique and $P_M(x) := x_1$ is a well defined J–projection. By construction it is $\mathcal{R}an(P_M) = M$ and $\mathcal{N}(P_M) = M^{\perp(J)}$. The other implication follows from (5.12) and lemma 3. QED

Finally we derive a representation theorem for J–projections:

Corollary 1 *A closed operator P in $\mathcal{H}$ is a J–projection if and only if there exist a JBB $\{u_n\}_{n\in\mathbb{N}}$ and a set $A \subset \mathbb{N}$ such that*

$$Px \;=\; \sum_{n\in A} P_{u_n} \qquad \forall\, x \in span(\{u_n\}_{n\in\mathbb{N}})\ . \tag{5.13}$$

In particular it holds that $P \;=\; \overline{\sum_{n\in A} P_{u_n}}$.

Proof. Since eq. (5.12) is valid lemma 1 allows the construction of a suitable JBB $\{u_n\}_{n\in\mathbb{N}}$ with $\{u_n\}_{n\in A} \subset \mathcal{R}an(P)$ and $\{u_n\}_{n\in\mathbb{N}-A} \subset \mathcal{N}(P)$. Conversely, it is clear that eq. (5.13) defines a J–projection. QED

If we turn to J–selfadjoint operators H the previous corollary tells us that a spectral resolution of H can at best be expected on certain dense subspaces but not on the entire Hilbert space. Beside such small differences like this, the last two sections have mainly presented similarities between the unitary and the J–unitary case. To show that there are also decisive differences is the central point of the following sections.

6 Extension of the Hilbert space by means of J–unitary operators

In the following it is to be shown that J–unitary operators play a role that essentially exceeds that of unitary operators. Thus a unitary transformation of the Hilbert space always leads to the "same" Hilbert space. This is different if one applies J–unitary transformations. Here the unboundedness of the transformation is crucial.

It is to be demonstrated how J–unitary operators define in a natural way a certain locally convex subspace of the Hilbert space. To this end we use J–unitary operators to construct continuous linear functionals on a certain subspace of $\mathcal{H}$. This vectorspace can be extended to a Hilbert space endowed with a structure that differs from the original one. This leads to a new Hilbert space isomorphic to $\mathcal{H}$ but not identical to it.

In the following we consider a family of J–unitary operators. For example, in the complex scaling theory one can regard all transformations belonging to different coordinate rotation angles. Let us start with the definition of a reasonable operator family.

Definition 7 *Let $\{U_\alpha\}_{\alpha\in A}$ be a countable family of operators on $\mathcal{H}$. This family is said to be J–unitary if it satisfies*

1. *$0 \in A$ and $U_0 = I$;*
2. *For $\mathcal{D}_A := \bigcap_{\alpha\in A} \mathcal{D}(U_\alpha)$ all spaces $U_\alpha(\mathcal{D}_A)$ are dense in $\mathcal{H}$ for $\alpha \in A$;*
3. *All operator U_α are J–unitary.*

For example every J–unitary operator U defines the J–unitary family $\{I, U\}$.

For the following investigation let us consider the (also J–unitary) operators U_α^{-1}. Let us start from the relation $(U_\alpha^{-1}x|y) = (x|U_\alpha y)$ for all $x \in \mathcal{R}an(U_\alpha)$ and $y \in \mathcal{D}(U_\alpha)$. For every fixed $\alpha \in A$ we define

$$[U_\alpha^{-1}x](y) := (x|U_\alpha y) \qquad \forall\, x \in \mathcal{H}, y \in \mathcal{D}_A\,. \tag{6.1}$$

$[U_\alpha^{-1}x]$ describes a linear functional on $\mathcal{D}_A$. Although U_α^{-1} is only defined on the subspace $\mathcal{R}an(U_\alpha)$ as an operator in $\mathcal{H}$ the functionals $[U_\alpha^{-1}x]$ are defined for <u>every</u> $x \in \mathcal{H}$. The linear space $\mathcal{H}_\alpha$ shall inherit the vectorspace structure of $\mathcal{H}$. Thus $\mathcal{H}_\alpha$ is in a natural way isomorphic to $\mathcal{H}$:

$$I_{\mathcal{H}_\alpha,\mathcal{H}} \;:\; H \ni x \mapsto [U_\alpha^{-1}x] \in \mathcal{H}_\alpha\,. \tag{6.2}$$

By construction $I_{\mathcal{H}_\alpha,\mathcal{H}}$ is surjective. Since $U_\alpha(\mathcal{D}_A)$ is dense in $\mathcal{H}$ it is even bijective. We use this bijectivity to transfer the Hilbert space structure of $\mathcal{H}$ to $\mathcal{H}_\alpha$. To this end we define a bilinear form on $\mathcal{H}_\alpha \times \mathcal{H}_\alpha$:

$$(\,[U_\alpha^{-1}x] \mid [U_\alpha^{-1}y]\,)_{\mathcal{H}_\alpha} := (x|y)_{\mathcal{H}} \qquad \forall\, x, y \in \mathcal{H}\,. \tag{6.3}$$

The linear space $\mathcal{H}_\alpha$ can finally be endowed with a Hilbert space structure explaining a conjugation operator J_α in $\mathcal{H}_\alpha$ by

$$J_\alpha[U_\alpha^{-1}x] := [U_\alpha^{-1}(Jx)] \qquad \forall\, x \in \mathcal{H}\,. \tag{6.4}$$

J is the conjugation operator in $\mathcal{H}$ which belongs to $(\cdot \mid \cdot)_\mathcal{H}$. The topology of the Hilbert space $\mathcal{H}_\alpha$ then is to be given by the norm

$$\|\, [U_\alpha^{-1}x] \,\|^2_{\mathcal{H}_\alpha} \;:=\; (\; J_\alpha[U_\alpha^{-1}x] \mid [U_\alpha^{-1}x]\;)_{\mathcal{H}_\alpha} \;=\; (Jx|x)_\mathcal{H} \;=\; \|x\|^2_\mathcal{H} \tag{6.5}$$

for all $x \in \mathcal{H}$. Thus $I_{\mathcal{H}_\alpha,\mathcal{H}}$ becomes an isometry. In particular also every element $x \in \mathcal{H}$ can be identified with a linear functional on $\mathcal{D}_A$ by

$$[U_0^{-1}x](y) \;:=\; (x|U_0y) \;=\; (x|y) \qquad \forall\, x \in \mathcal{H}, y \in \mathcal{D}_A\,. \tag{6.6}$$

It is important to note that the two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_\alpha$ are <u>different</u> as spaces of linear functionals $[U_0^{-1}x]$ and $[U_\alpha^{-1}x]$ on $\mathcal{D}_A$ (for the same vector $x \in \mathcal{H}$), respectively. But they are <u>isomorphic</u> as Hilbert spaces due to the isometry $I_{\mathcal{H}_\alpha,\mathcal{H}}$. We will use the difference as spaces of functionals to induce a new topology on $\mathcal{D}_A$.

One essential item will be that the set $\mathcal{D}_A$ can be regarded as a common subspace of the two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_\alpha$. While the embedding of $\mathcal{D}_A \subset \mathcal{H}$ is given, the embedding of $\mathcal{D}_A \subset \mathcal{H}_\alpha$ must be constructed explicitly. To this end we have to determine a subspace of $\mathcal{H}_\alpha$ which coincides with $\mathcal{D}_A$ in the functional sense. Let us regard some vector $x_0 \in \mathcal{D}_A$. We have to look for an element $[U_\alpha^{-1}x_\alpha] \in \mathcal{H}_\alpha$ that is equal to $x_0 \in \mathcal{H}$ in the sense of

$$[U_\alpha^{-1}x_\alpha](y) \;=\; [U_0^{-1}x_0](y) \qquad \forall\, y \in \mathcal{D}_A\,. \tag{6.7}$$

The relation (6.7) is satisfied if we choose $U_\alpha x_0$ for x_α. Since U_α is injective we can identify

$$\mathcal{H} \supset \mathcal{D}_A \equiv [U_\alpha^{-1}(U_\alpha\mathcal{D}_A)] \subset \mathcal{H}_\alpha \tag{6.8}$$

setting

$$x \equiv [U_\alpha^{-1}(U_\alpha x)] \tag{6.9}$$

for all $x \in \mathcal{D}_A$. We can now read $\mathcal{D}_A$ as a subspace of $\mathcal{H}_\alpha$ embedded via an injection j_α in such a manner that the elements of $\mathcal{D}_A$ (read as functionals on $\mathcal{D}_A$) coincide in both Hilbert spaces. Such a construction is possible for every $\alpha \in A$.

After this construction of a common subspace for all Hilbert spaces $\mathcal{H}_\alpha$ one can endow $\mathcal{D}_A$ with a new topology $\mathcal{T}_A$ that makes all injections j_α continuous. The coarsest topology that accomplishes this task is the projective topology of $\mathcal{D}_A$ with respect to the family $(\mathcal{H}_\alpha, \mathcal{T}_A, j_\alpha)_{\alpha\in A}$ where $\mathcal{T}_A$ is the usual Hilbert space topology of $\mathcal{H}_\alpha$ generated by the norm $\|\cdot\|_{\mathcal{H}_\alpha}$ [32].

For convenience let Φ_A denote the topological vectorspace $(\mathcal{D}_A, \mathcal{T}_A)$ where $\mathcal{D}_A$ denotes the subset of $\mathcal{H}$. The topology of Φ_A is characterized by the convergence relations

$$\begin{aligned} y_n \xrightarrow{\Phi_A} y \;&\Leftrightarrow j_\alpha y_n \xrightarrow{\mathcal{H}_\alpha} j_\alpha y \quad \text{for all} \quad \alpha \in A \\ &\Leftrightarrow U_\alpha y_n \xrightarrow{\mathcal{H}} U_\alpha y \quad \text{for all} \quad \alpha \in A \end{aligned} \tag{6.10}$$

for $n \to \infty$, $n \in I\!N$. The elements of Φ_A can be characterized by

$$\begin{aligned} y \in \Phi_A \;&\Leftrightarrow \| \, j_\alpha y \, \|_{\mathcal{H}_\alpha} < \infty \quad \text{for all} \quad \alpha \in A \\ &\Leftrightarrow \| \, U_\alpha y \, \|_{\mathcal{H}} < \infty \quad \text{for all} \quad \alpha \in A \, . \end{aligned} \tag{6.11}$$

This results in a relationship between the J–unitary operator U_α and the injection j_α which can be described by the following commutative diagram:

$$\begin{array}{ccc} \Phi_A \ni x & \xrightarrow{j_\alpha} & j_\alpha x = [U_\alpha^{-1}(U_\alpha j_0 x)] \in \mathcal{H}_\alpha \\ j_0 \downarrow & & \uparrow I_{\mathcal{H}_\alpha,\mathcal{H}} \\ \mathcal{H} \ni j_0 x & \xrightarrow{U_\alpha} & U_\alpha j_0 x \in \mathcal{H} \end{array} \tag{6.12}$$

The operators $I_{\mathcal{H},\mathcal{H}_\alpha}$ and j_α, as well as their compositions are continuous. This means that the operator $U_\alpha \circ j_0$ is continuous, too. Since j_0 is injective we obtain the representation

$$U_\alpha \;=\; I_{\mathcal{H},\mathcal{H}_\alpha} \circ j_\alpha \circ j_0^{-1} \, . \tag{6.13}$$

The discontinuity of U_α therefore results from the discontinuity of j_0^{-1}. In a similar way we obtain J–unitary operators for every Hilbert space extension of Φ_A which is isomorphic to $\mathcal{H}$ (as a Hilbert space). The class of these Hilbert space extensions is in general larger then $\{\mathcal{H}_\alpha\}_\alpha \in A$. But this subclass determines the topology of Φ_A. In particular $\mathcal{D}(U_\alpha)$ is independent of the isometry $I_{\mathcal{H},\mathcal{H}_\alpha}$ such that the topology of Φ_A is independent of the special realization of the isomorphy between $\mathcal{H}$ and $\mathcal{H}_\alpha$. Moreover, the consideration above exhibits that only unbounded J–unitary operators contribute to the topology of Φ_A.

According to its construction the space Φ_A is complete. Since A is countable the space Φ_A is metrizable. Summerizing the properties of Φ_A we find that it is a complete metrizable locally convex vectorspace, i. e. a Fréchet space [32]. Moreover, the construction is unique in the following sense:

Theorem 3 *Let $\{U_\alpha\}_\alpha \in A$ be a J–unitary family in $\mathcal{H}$. Then there exists a unique Fréchet space Φ_A, densely embedded in $\mathcal{H}$ via a continuous injection j, satisfying:*

1. *$\mathcal{D}_A \;=\; j(\Phi_A)$;*
2. *$U_\alpha \circ j$ is continuous for all $\alpha \in A$.*

PROOF. The main part of the proof follows from the construction above. In addition it must only be demonstrated that the construction is unique in the stated sense. Let us consider a second Fréchet space Φ satisfying the condition (1.) and (2.). As a set Φ can be identified with Φ_A according to condition (1.). The topology of Φ_A is the coarsest satisfying (2.). Therefore it follows $\Phi \subset \Phi_A$ as Fréchet spaces. Due to III.2.1. corollary 2 [32] the spaces Φ and Φ_A are identical. QED

Since the vectorspace Φ_A is not a Hilbert space it is not isomorphic to its (topological) dual space Φ'_A, consisting of all continuous linear functionals on Φ_A, where the space Φ'_A is endowed with the weak topology. We will make some remarks concerning this dual space in order to explain its relation to the Hilbert spaces $\mathcal{H}_\alpha$. Due to the theory of locally convex vectorspaces [33] the elements of Φ'_A are characterized by

$$\begin{aligned} F \in \Phi'_A \Leftrightarrow\ & \exists\, I \subset A \text{ finite and } C > 0 \ : \\ & |F(x)| \leq C \cdot \sum_{\alpha \in I} \|j_\alpha x\|^2_{\mathcal{H}_\alpha} \\ & \text{for all } x \in \Phi_A \ . \end{aligned} \tag{6.14}$$

Let us now consider the vectorspace $\mathcal{H}_I$ of all families $\{x_\alpha\}_{\alpha\in I}$, $x_\alpha \in \mathcal{H}_\alpha$, for some fixed finite index set $I \subset A$. The addition and scalar multiplication shall be componentwise. We can introduce an inner product for $\mathcal{H}_I$ by

$$(\{x_\alpha\}_{\alpha\in I} \mid \{y_\alpha\}_{\alpha\in I})_{\mathcal{H}_I} \ := \ \sum_{\alpha\in I} (j_\alpha x_\alpha \mid j_\alpha y_\alpha)_{\mathcal{H}_\alpha} \tag{6.15}$$

for all $\{x_\alpha\}_{\alpha\in I}, \{y_\alpha\}_{\alpha\in I} \in \mathcal{H}_I$. Thus $\mathcal{H}_I$ becomes a pre–Hilbert space. Since the index set I is finite $\mathcal{H}_I$ inherits the completeness of the Hilbert spaces $\mathcal{H}_\alpha$. The space Φ_A is densely embedded in $\mathcal{H}_I$ via the injection

$$j_I \ : \ \Phi_A \ni x \mapsto \{j_\alpha x\}_{\alpha\in I} \in \mathcal{H}_I \ . \tag{6.16}$$

We can use this for another characterization of the dual space Φ'_A:

$$\begin{aligned} F \in \Phi'_A \Leftrightarrow\ & \exists\, I \subset A \text{ finite and } C > 0 \ : \\ & |F(x)| \leq C \cdot \|j_I x\|^2_{\mathcal{H}_I} \\ & \text{for all } x \in \Phi_A \ . \end{aligned} \tag{6.17}$$

According to the theorem of Hahn–Banach [32] every continuous linear functional F on Φ_A can be extended to a continuous linear functional $F_I \in \mathcal{H}'_I$. Due to the theorem of Riesz F_I possesses the representation

$$F_I(\ \{x_\alpha\}_{\alpha\in I}\ ) \ = \ \sum_{\alpha\in I} (x_\alpha^{(F)} \mid x_\alpha)_{\mathcal{H}_\alpha} \tag{6.18}$$

for all $\{x_\alpha\}_{\alpha\in I} \in \mathcal{H}_I$ and some suitable $x_\alpha^{(F)} \in \mathcal{H}_\alpha$ depending on F. This means that the dual space Φ'_A is just the locally convex direct sum $\oplus_{\alpha\in A}\mathcal{H}'_\alpha$ of the dual Hilbert spaces $\mathcal{H}'_\alpha$ [32] and every element $F \in \Phi'_A$ can be represented by

$$F \ = \ \sum_{\alpha\in I} (\ x_\alpha^{(F)} \mid j_\alpha(\cdot)\ )_{\mathcal{H}_\alpha} \ . \tag{6.19}$$

As a consequence of this representation the space Φ'_A is separable and complete (II.6.2. [32]).

The convergence in Φ'_A can be reduced to the weak convergence in the Hilbert spaces $\mathcal{H}_\alpha$. It is characterized in the following way:

$$\begin{aligned} F_n \xrightarrow{\Phi'_A} F \Leftrightarrow\ & \exists\, n_0 \in I\!N \text{ and } I \subset A \text{ finite} \\ & \forall\, n \geq n_0 \text{ and } \alpha \in I \quad \exists\, {y_\alpha^{(F)}}_n\ ,\ y_\alpha^{(F)} \in \mathcal{H}_\alpha \ : \\ & F_n(x) \ = \ \textstyle\sum_{\alpha\in I}\ ({y_\alpha^{(F)}}_n \mid j_\alpha x)_{\mathcal{H}_\alpha}\ , \\ & F(x) \ = \ \textstyle\sum_{\alpha\in I}\ (y_\alpha^{(F)} \mid j_\alpha x)_{\mathcal{H}_\alpha}\ , \\ & ({y_\alpha^{(F)}}_n \mid j_\alpha x)_{\mathcal{H}_\alpha} \ \longrightarrow \ (y_\alpha^{(F)} \mid j_\alpha x)_{\mathcal{H}_\alpha}\ ,\ \text{for all } x \in \Phi_A \ . \end{aligned} \tag{6.20}$$

Some final remarks concerning the dual system (Φ'_A, Φ_A) shall be added. For this system we have a canonical bilinear form defined by

$$(F \mid y)_{\Phi'_A,\Phi_A} := F(y) \qquad \forall\, F \in \Phi'_A, y \in \Phi_A\ . \tag{6.21}$$

According to the structure of Φ_A this bilinear form actually originates from some bilinear form on $\Phi_A \times \Phi_A$. This bilinear form on $\Phi_A \times \Phi_A$ is induced by the J–product in $\mathcal{H}$

$$\begin{aligned}(x \mid y)_{\Phi_A} &:= (U_\alpha x \mid U_\alpha y)_{\mathcal{H}} \\ &= (j_\alpha x \mid j_\alpha y)_{\mathcal{H}_\alpha}\end{aligned} \tag{6.22}$$

for all $x, y \in \Phi_A$. It is well defined due to the fact that it does not depend on $\alpha \in A$ since U_α is J–unitary. Hence a continuous injection $h\ :\ \Phi_A \hookrightarrow \Phi'_A$ is defined:

$$h\ :\ \Phi_A \in x \mapsto (x \mid \ \cdot\ )_{\Phi_A} \in \Phi'_A\ . \tag{6.23}$$

The range $h(\Phi_A)$ is dense in Φ'_A. The injection can be equivalently expressed by the adjoint injection $j_\alpha^\dagger\ :\ \mathcal{H}'_\alpha \hookrightarrow \Phi'_A$ given by

$$[j_\alpha^\dagger F](y) := F(j_\alpha y) \qquad \forall\, F \in \mathcal{H}'_\alpha, y \in \Phi_A\ . \tag{6.24}$$

This yields $h = j_\alpha^\dagger j_\alpha$ (independent of $\alpha \in A$).

It is to be remarked that the bilinear form $(\cdot \mid \cdot)_{\Phi_A}$ does not originate from the topological structure of the Fréchet space. It actually defines an additional geometrical structure on Φ_A. This geometrical structure is the same in all Hilbert spaces $\mathcal{H}_\alpha$ and hence a common link.

The dual space Φ'_A is actually a countable Hilbert space [34] which can be regarded as the common hull of all Hilbert spaces $\mathcal{H}_\alpha$. It is interesting to note that the theory presented here leads to similar structures as the rigged Hilbert space theory [35, 36, 37] and the Hilbert subspace theory [38, 39, 40, 41] where also an encapsulation of the Hilbert space between a locally convex vectorspace and its dual space is considered. We will return to this point later.

7 J–unitary transformations of J–selfadjoint operators

We want to investigate how J–selfadjoint operators can be transformed with respect to J–unitary transformations. The fact that a J–selfadjoint operator is J–unitarily transformable can open opportunity to consider this operator on the Fréchet space Φ_A and its dual space Φ'_A. To make sure that the J–unitary transformation of the J–selfadjoint operator is reasonable the operator has to satisfy some general conditions. The transformation must fit to the operator in such a way that the denseness of the domains is conserved. In contrast to unitary transformations this is not generally fulfilled.

Let us first introduce a term which is once more borrowed from the theory of selfadjoint operators. In general J–selfadjoint operators cannot directly be transformed to J–selfadjoint operators and the transformation can only be performed on a certain subspace of $\mathcal{D}_A$ explicitly. Only a core of the operator can be transformed directly. The J–selfadjoint operator must then be reconstructed from this core by closure. This is supplied by essentially J–selfadjoint operators. In order to check if a J–symmetric operator is essentially J–selfadjoint one can here apply similar criteria as for selfadjoint operators [42]. The following definition states the conditions necessary for the transformation:

Definition 8 *Let H be a J-selfadjoint operator in $\mathcal{H}$. A J-unitary family $\{U_\alpha\}_{\alpha\in A}$ is said to be admissible to H if the following conditions are satisfied:*

1. *There is a subspace $\mathcal{D}_H \subset \mathcal{D}(H) \cap \mathcal{D}_A$ dense in $\mathcal{H}$ wrt $\mathcal{T}_A$ such that $H(\mathcal{D}_H) \subset \mathcal{D}_A$;*
2. *$U_\alpha H U_\alpha^{-1}$ defined on $U_\alpha(\mathcal{D}_H)$ is essentially J-selfadjoint;*

In this case we call $H_{U_\alpha} := \overline{U_\alpha H U_\alpha^{-1}}$ the (J-unitary) transform of H wrt U_α.
We say that U is admissible to H if $\{I, U\}$ is admissible to H.

As the simplest example we consider the transform of a J–projection:

Lemma 7 *Let U be a J-unitary operator in $\mathcal{H}$ and $M \subset \mathcal{D}(U)$ a closed J-projective subspace. Then U is admissible to P_M and*

$$(P_M)_U \;=\; P_{\overline{U(M)}} \,. \tag{7.1}$$

PROOF. The fact that U is admissible to P_M is a direct consequence of lemma 4. Eq. (7.1) follows directly from $UP_MU^{-1} \;=\; P_{U(M)}$ and $\overline{P_{U(M)}} \;=\; P_{\overline{U(M)}}$. QED

Another simple case is given by J–selfadjoint operators that possess a representation with respect to J–biorthonormal systems.

Lemma 8 *Let H be a J-selfadjoint operator that can be represented as*

$$H \;=\; \overline{\sum_n \alpha_n \, P_{u_n}} \tag{7.2}$$

where $\{u_n\}_{n\in I\!N}$ is a JBB. Let further $\{U_\alpha\}_{\alpha\in A}$ be a J-unitary family. Then the system $\{U_\alpha\}_{\alpha\in A}$ is admissible to H if and only if $u_n \in \mathcal{D}_A$ for all $n \in I\!N$. In this case it is

$$H_{U_\alpha} \;=\; \overline{\sum_n \alpha_n \, P_{U_\alpha u_n}} \,. \tag{7.3}$$

A consequence of the two previous lemmas concerns the change of the discrete spectrum by J–unitary transformations.

Corollary 2 *Let H be a J-selfadjoint operator as in (7.2) with discrete spectrum $\sigma(U)$ and $\{U_\alpha\}_{\alpha\in A}$ a J-unitary family that is admissible to H. Then it is $\sigma(H) = \sigma(H_{U_\alpha})$ for all $\alpha \in A$.*

It is well known that the situation is more complicated if the operator H possesses a continuous spectrum [29]. Then the spectrum is in general not conserved under J-unitary transformation.

It follows directly from the symmetry in definition 8 that the different transformations of H with respect to U_α are interchangeable.

Lemma 9 *Let $\{U_\alpha\}_{\alpha\in A}$ a J-unitary family admissible to the J-selfadjoint operator H in $\mathcal{H}$. Then for every $\beta \in A$ it holds that:*

1. *$\{U_\alpha U_\beta^{-1}\}_{\alpha\in A}$ is admissible to H_{U_β};*
2. *$(H_{U_\beta})_{U_\alpha U_\beta^{-1}} = H_{U_\alpha}$.*

In particular this is true for $U_\alpha = I$.

This lemma clearly shows the equivalence of the operators H_{U_α}. The decisive point is that the spectral properties of these equivalent representations can be completely different. This make them a useful tool in the study of generalized spectral properties of H.

In a next step we want to show how the J-unitarily transformed operators H_{U_α} are related to the Hilbert spaces $\mathcal{H}_\alpha$ constructed in the previous section. Here we have to pay attention to the fact that the operator H is only defined on the dense (wrt $\mathcal{T}_A$) subspace $\mathcal{D}_H$ of $\mathcal{D}_A$. Since $\mathcal{D}_A$ is densely embedded in every Hilbert space $\mathcal{H}_\alpha$ also the space $\mathcal{D}_H$ is densely embedded in $\mathcal{H}_\alpha$. Therefore the operator $H|_{\mathcal{D}_H}$ can be transfered to every Hilbert space $\mathcal{H}_\alpha$ by definition of an operator

$$H_\alpha \ : \ [U_\alpha^{-1}(U_\alpha x)] \mapsto [U_\alpha^{-1}(U_\alpha(Hx))] \tag{7.4}$$

for all $x \in \mathcal{D}_H$. It is easy to see that the operator H_α is J-symmetric. This means that the operators H_{U_α} are actually related to representations of H in the Hilbert spaces $\mathcal{H}_\alpha$:

Lemma 10 *Let $\{U_\alpha\}_{\alpha\in A}$ a J-unitary family admissible to the J-selfadjoint operator H in $\mathcal{H}$. Then the operator H_α is essentially J_α-selfadjoint with*

$$[U_\alpha^{-1}y] \in \mathcal{D}(\overline{H_\alpha}) \ \Leftrightarrow \ y \in \mathcal{D}(H_{U_\alpha}) \ . \tag{7.5}$$

It holds that

$$\overline{H_\alpha}[U_\alpha^{-1}y] \ = \ [U_\alpha^{-1}(H_{U_\alpha}y)] \ . \tag{7.6}$$

H_α is called the representation of H in $\mathcal{H}_\alpha$.

PROOF. If $x \in \mathcal{D}_H$ then the operator H_α is defined according to (7.4). We now have to prove that H_α can be extended to $[U_\alpha^{-1}\mathcal{D}(H_{U_\alpha})]$. If $y \in \mathcal{D}(H_{U_\alpha})$ then there is a family $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}_H$ such that $U_\alpha x_n \to y$ and $U_\alpha H x_n \to H_{U_\alpha} y$ for $n \to \infty$. Therefore we have $[U_\alpha^{-1}(U_\alpha x_n)] \to [U_\alpha^{-1} y]$ for $n \to \infty$ due to the definition of $\mathcal{H}_\alpha$ and thus

$$H_\alpha[U_\alpha^{-1}(U_\alpha x_n)] \;=\; [U_\alpha^{-1}(U_\alpha H x_n)] \;\to\; [U_\alpha^{-1}(H_{U_\alpha} y)] \;=\; \overline{H_\alpha}[U_\alpha^{-1} y]\,. \tag{7.7}$$

In a similar manner one can prove that the operator $\overline{H_\alpha}$ is J$_\alpha$–selfadjoint in $\mathcal{H}_\alpha$ by transfering this property from H_{U_α} to $\overline{H_\alpha}$. QED

According to (7.4) the operator H does not only induce an operator H_α in every Hilbert space $\mathcal{H}_\alpha$ but also an operator $\hat{H} : \Phi_H \to \Phi_A$ where $\Phi_H := j_0^{-1}(\mathcal{D}_H)$ is a dense subspace of Φ_A (wrt $\mathcal{T}_A$). In order to clarify how the operator $\hat{H}$ acts on Φ_H let us represent every element $x \in \Phi_H$ by a family $\{U_\alpha x\}_{\alpha\in A}$. This way the operator $\hat{H}$ can be defined by

$$\hat{H} \;:\; x \equiv \{U_\alpha x\}_{\alpha\in A} \mapsto \hat{H}\, x \equiv \{U_\alpha H x\}_{\alpha\in A} \tag{7.8}$$

for every $x \in \Phi_H$. According to the previous lemma and the definition of $\mathcal{D}_H$ every operator H_α is a J$_\alpha$–selfadjoint extension of $\hat{H}$ in the Hilbert space $\mathcal{H}_\alpha$ with

$$H_\alpha\, j_\alpha \;=\; j_\alpha\, \hat{H} \tag{7.9}$$

on Φ_H. Since Φ'_A is isomorhpic to $\oplus_{\alpha\in A}\mathcal{H}'_\alpha$ we can also define an extension of $\hat{H}$ on a dense subspace Φ'_H of Φ'_A since every $F \in \Phi'_A$ allows the representation (6.19). We define the operator H' by

$$[H'F](x) \;:=\; \sum_{\alpha\in I}(H_\alpha x_\alpha^{(F)} \mid j_\alpha x)_{\mathcal{H}_\alpha} \qquad \forall\, x \in \mathcal{D}_A \tag{7.10}$$

if the condition $x_\alpha^{(F)} \in \mathcal{D}(H_\alpha)$ is fulfilled. All functionals (6.19) for which the relation (7.10) holds form the vectorspace Φ'_H. A simple calculation shows that H' is well defined. One proves that for $z \in \Phi_H$ the generalized symmetry condition

$$[H'F](z) \;=\; F(\hat{H}z) \tag{7.11}$$

is valid such that the definition of H' by eq. (7.10) is independent of the particular representation of F, at least on Φ_H. But since Φ_H is dense in Φ_A (wrt $\mathcal{T}_A$) and F is a continuous linear functional on Φ_A this independence of the particular representation holds for all $z \in \Phi_A$.

If we restrict the general representation (7.10) of H' to the case that $F = j_\alpha^\dagger y$, $y \in \mathcal{D}(H_\alpha)$ we can use the relations (6.24), the J$_\alpha$–symmetry of H_α, and the generalized symmetry condition (7.11) to obtain

$$\begin{aligned}[][j_\alpha^\dagger(H_\alpha y)](z) &= (H_\alpha y|j_\alpha z)_{\mathcal{H}_\alpha} \\ &= (y|j_\alpha \hat{H} z)_{\mathcal{H}_\alpha} \\ &= [j_\alpha^\dagger y](\hat{H}z) \\ &= [H'(j_\alpha^\dagger y)](z)\end{aligned} \tag{7.12}$$

for all $z \in \Phi_H$. This relation remains valid for all $z \in \Phi_A$ and H' can be considered as the extension of H_α to the space Φ'_H. Thus eqs. (7.12) result in

$$H'\, j_\alpha^\dagger \;=\; j_\alpha^\dagger\, H_\alpha\,. \tag{7.13}$$

According to lemma 9 and 10 the operator $\hat{H}$ contains the entire structure of H such that the extension H' can be constructed from $\hat{H}$ unambiguously. Let us summerize these results:

Theorem 4 *Let $\{U_\alpha\}_{aA}$ be a J-unitary family admissible to the J-selfadjoint operator H in $\mathcal{H}$. Then there exists a dense subspace Φ_H of Φ_A wrt $\mathcal{T}_A$ and an unique operator $\hat{H}$: $\Phi_H \to \Phi_A$ which is symmetric with respect to the bilinear form $< \cdot | \cdot >_{\Phi_A}$ allowing unique J_α-selfadjoint extensions H_α in $\mathcal{H}_\alpha$. Moreover, there exists a common extension H' of the operators H_α in Φ'_A that fulfills the generalized symmetry condition (7.11).*

Theorem 4 does not only express the equivalence of the different Hilbert space extensions H_α in $\mathcal{H}_\alpha$ but it states that their equivalence stems from a common extension H' on Φ'_A. That means that there is actually only one operator which is restricted to different Hilbert spaces.

Finally let us regard an example where an abstractly defined J-unitary operator is used to transform a generalized eigenvector, i.e. a formal eigensolution of the Schrödinger equation which is not an element of the Hilbert space, to an eigenvector in the Hilbert space. To this end we introduce the Hilbert space $\ell^2(\mathbb{Z})$ of all square-summable double sequences $(\ \eta(n)\ )_{n\in\mathbb{Z}}$ of complex numbers $\eta(n)$. As conjugation J we choose the usual complex conjugation. The canonical orthogonal basis (and JBS) of $\ell^2(\mathbb{Z})$ is give by the system $\{(\ \delta_{n,m}\ )_{m\in\mathbb{Z}}\}_{n\in\mathbb{Z}}$. Then we define the discrete Hamilton operator

$$[H\ \psi](n) \ := \ -\psi(n+1) - \psi(n-1) + V(n)\cdot\psi(n) \tag{7.14}$$

with the complex potential

$$V(n) \ := \ \left(\frac{sg(n+1)\cdot 8^{-2^{|n+1|}} - sg(n-1)\cdot 8^{-2^{|n-1|}}}{1 + sg(n)\cdot 8^{-2^{|n|}}} - 1 \right)\cdot i\ . \tag{7.15}$$

The lattice Hamilton operator H is complex symmetric, i.e. J-symmetric, and possess the formal eigenvector

$$(\ \xi_n\ )_{n\in\mathbb{Z}} \ := \ (\ i^n\cdot(1 + sg(n)\cdot 8^{-2^{|n|}})\ )_{n\in\mathbb{Z}} \tag{7.16}$$

to the eigenvalue $-i$. But obviously $(\ \xi_n\ )_{n\in\mathbb{Z}}$ is not an element of the Hilbert space $\ell^2(\mathbb{Z})$. Here two different parts in this eigenvector can be distinguished. One oscillating part $(\ i^n\ )_{n\in\mathbb{Z}}$ and a square-integrable part $(\ i^n\ 8^{-2^{|n|}}\ )_{n\in\mathbb{Z}}$. The former expression does not contain particular information about the operator H. So it is to be the aim to remove this part by a J-unitary transformation.

To this end we introduce a J-unitary operator U by the pair-componentwise transformation

$$\begin{pmatrix} \sqrt{4^{|n|}+1} & 2^{|n|}\cdot i \\ 2^{|n|}\cdot i & -\sqrt{4^{|n|}+1} \end{pmatrix} \begin{pmatrix} \xi_{2n} \\ \xi_{2n+1} \end{pmatrix} \tag{7.17}$$

The transformation U is admissible to $\ell^2(\mathbb{Z})$ and after applying it to the vector (7.16) we obtain a transformed eigenvector $U(\ \xi_n\ )_{n\in\mathbb{Z}}$ to H_U that is now square-integrable. Although the potential appears to be rather artificial the example demonstrates the essential features of the UST theory. So it is noteworthy that the construction of the operator U is completely independent from analytic continuation ideas. The idea of this transformation consists in suppressing the oscillatory part $(i^n)_{n\in\mathbb{Z}}$ of the formal eigenvector $(\xi_n)_{n\in\mathbb{Z}}$ and extracting that part that contains the actual information about the potential. It is important to note that the construction is *independent* of any complex scaling. This demonstrates that the present approach is really a generalization of the analytic continuation concept.

8 Discussion

The definition of J–isometric and J–unitary operators in a general Hilbert space $\mathcal{H}$ was given. The basic properties of these operators were stated and their similarity to isometric and unitary operators was demonstrated. Topological aspects become important since the relevant J–unitary operators are unbounded. We introduced J–biorthonormal systems which allow the representation of J–unitary operators on dense subspaces of $\mathcal{H}$. We further introduced J–projections which are analogous to orthogonal projections. These J–projections can be regarded as the simplest type of J–selfadjoint operator to which the theory can be applied. Moreover, they show the connection between J–selfadjoint operators and J–biorthonormal systems.

Although J–unitary transformations are in some (especially algebraic) aspects similar to unitary transformations the former ones have to be handled more carefully. Nice expansion properties can only be conserved on certain dense subspaces of the Hilbert space. This is necessary due to the change of the Hilbert space topology. The feature that remains is a common geometrical non–orthogonal structure provided by coordinate systems in the form of J–biorthonormal systems.

The topological considerations end up in the construction of a Fréchet space that is densely and continuously embedded in the Hilbert space $\mathcal{H}$. Its topology is determined by the J–unitary transformations. Each transformation represents an embedding of the Fréchet space in a distinct Hilbert space extension. The presented theory shows here obvious similarities to other advanced approaches in scattering theory like the rigged Hilbert space or the Hilbert subspace theory. All these theories are based on the idea of encapsulation of the Hilbert space between a locally convex vectorspace and its dual space. Differences to the latter two theories originate from the fact that in the present UST approach no generalized spectral representation of the Hamilton operator is aspired, as it is, for example, provided by the Maurin–Gel'fand theorem [43, 44]. Nevertheless the present approach yields a generalized spectral theory as the complex scaling theory shows but is independent of the analytic continuation ideas which are the basis of the complex scaling theory. Other complex Hamiltonian methods, such as the complex absorbing potential method [45, 21], emphasize this point since they do not require an analytical form of the Hamilton operator either.

The advantage of the proceeding presented by J–unitary operators consists in the simple technical performance. The UST theory offers an easy–to–use scheme for the investigation of Hamilton operators in scattering theory. Thus they allow the calculation of complex resonance eigenstates by choosing a suitable Hilbert space representation. Although the UST approach does not yield spectral representations as does the rigged Hilbert space or the Hilbert subspace theory it profits from the fact that the Hilbert space apparatus is still applicable. This is very important for concrete computations since nowadays almost every relevant quantum mechanical calculation is performed within the framework of Hilbert space theory which excels by its simple and efficient implementations.

There are still many open questions. We have seen that J–biorthonormal systems play an essential role for the representation of J–unitary operators. For this reason the construction of complete J–biorthonormal systems is to be investigated in more detail. The spectral theory of J–unitarily transformed selfadjoint operators is to be developed such that generalized spectral theory becomes available. Here a wide range of further directions of investigation presents itself.

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