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# Some Schrödinger operators with power-decaying potentials and pure point spectrum, II. The discrete case

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*Abstract.* We construct deterministic potentials $V(n) = O(n^{-c})$ such that the discrete Schrödinger operator $(Hy)(n) = y(n-1) + y(n+1) + V(n)y(n)$ on $l_2(\mathbf{N})$ has dense pure point spectrum on $[-2,2]$ for almost all values of $V(1)$. As a consequence of this construction, we also obtain power-decaying potentials for which the spectrum is purely singular continuous on $(-2,2)$ for *all* values of $V(1)$.

# 1 Introduction

This work is a sequel to my paper [12]. Here, I am interested in the *discrete* one-dimensional Schrödinger operator (also known as Jacobi matrix) $H$ on $l_2(\mathbf{N})$:

$$(Hy)(n) = \begin{cases} y(n-1) + y(n+1) + V(n)y(n) & (n \geq 2) \\ y(2) + V(1)y(1) & (n = 1) \end{cases}$$

We will construct power-decaying potentials $V(n) = O(n^{-c})$ $(c > 0)$ so that $H$ has dense pure point spectrum in $[-2,2]$ (i.e. $\sigma_{ess} = [-2,2], \sigma_c = \emptyset$) for almost all values of $V(1)$. Note that by the instability of thick point spectrum [3], there must also be a dense $G_\delta$ set of $V(1)$'s for which the spectrum of $H$ is purely singular continuous in $(-2,2)$.

This is, to the best of my knowledge, even the first deterministic potential with dense pure point spectrum and $V(n) \to 0$. For *random* potentials decaying as $V(n) \sim n^{-c}$ with

$c < 1/2$, it is known that this spectral type is typical (see [7, 8, 14]), but, of course, these works do not give much information on the particular structure a fixed potential must have in order to generate pure point spectrum.

In [12], decaying potentials with pure point spectrum were constructed for continuous Schrödinger operators. Obviously, this proof does not extend to the discrete case. The aim of the present paper is to show that nevertheless the basic ideas of [12] can also be used to obtain the result stated above. The representation will closely follow that of [12]. We also refer to [12] for a more complete discussion of related works and for further references.

Some of the arguments used here are completely analogous to the corresponding considerations of [12]; in this case, we will only sketch the ideas. As in [12], a modification of our construction will yield power-decaying potentials with purely singular continuous spectrum in $(-2,2)$ for all values of $V(1)$. The only other class of decaying (deterministic) potentials which is known at present and which leads to this spectral type (namely, sparse potentials, as introduced in [11]) has a considerably slower decay (see [7]).

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# 2 Description of the model

We will investigate the following potential. Let $L_n, N_n \in \mathbf{N}, W_n \in \mathbf{R}, l_n \in \mathbf{N}_0$, and set

$$a_1 = 0, \qquad a_{n+1} = a_n + 1 + l_n + L_n N_n.$$

Define $V(m)$ as follows: For $a_n \leq m < a_{n+1}$, let $V(m) = W_n + 1/N_n$ if $m$ is one of the points $a_n + 1 + l_n + tN_n \quad (t = 0, 1, \ldots, L_n - 1)$ and $V(m) = W_n$ otherwise. The main goal of this paper is to show

**Theorem 2.1** *The parameters can be chosen such that:*
*1)* $V(n) = O(n^{-c}) \quad (c > 0)$
*2) For almost every* $E \in (-2,2)$*, the equation*

$$y(n-1) + y(n+1) + V(n)y(n) = Ey(n) \qquad (n \in \mathbf{N}) \tag{2.1}$$

*has an* $l_2$*-solution.*

The proof will be given in the following three Sections.

Note that an $l_2$-solution of (2.1) is an eigenvector of $H$ if and only if $y(0) = 0$. On the other hand, unless $y(1) = 0$, one can change the value of $V(1)$ so that $y(0) = 0$ for the new value $\tilde{V}(1)$ (namely, take $\tilde{V}(1) = V(1) + y(0)/y(1)$). Thus spectral averaging and 2) imply that $\sigma_c(H) = \emptyset$ for almost all values of $V(1)$ (see e.g. [15]). Moreover, 1) shows that the operator of multiplication by $V$ is compact, hence $\sigma_{ess} = [-2, 2]$. Putting these two statements together, we conclude that, for almost every $V(1)$, $H$ indeed has pure point spectrum which is dense in $[-2, 2]$.

As in [12], the potential is built out of periodic pieces. The basic idea is to determine these building blocks so that every energy is in (one of) the gaps sufficiently frequently. The role of the various parameters will become clear in the course of the proof. We do not aim at maximal generality; instead, we prefer to keep the discussion as transparent as possible. Therefore, we have made a definite choice for the basic periodic potential from the outset. The method applies in principle more generally, but the conditions one has to impose are very cumbersome. Also, we will not try to optimize the exponent $c$ because it seems impossible to come close to the borderline case $c = 1/2$ anyway.

# 3 Preliminaries

Fix $\alpha \in [0, \pi)$, and for $E \in (-2, 2)$, let $y(n, E)$ be the solution of (2.1) with initial values $y(0, E) = \sin\alpha, y(1, E) = \cos\alpha$. We will use a Prüfer type transformation (see [7] for a comprehensive discussion). Because of statement 1) of Theorem 2.1, we clearly must have $W_n \to 0$. Thus, if $E \in (-2, 2)$, we can define $\omega_n = \omega_n(E)$ for sufficiently large $n$ by

$$2\cos\omega_n = E - W_n, \qquad \omega_n \in (0, \pi).$$

For $a_n < m \leq a_{n+1}$, let

$$Y(m, E) = \frac{1}{\sin\omega_n} \begin{pmatrix} \sin\omega_n & 0 \\ -\cos\omega_n & 1 \end{pmatrix} \begin{pmatrix} y(m-1, E) \\ y(m, E) \end{pmatrix}, \tag{3.1}$$

and write $Y(m, E) = R(m, E)(\sin\varphi(m, E), \cos\varphi(m, E))^T$ with $R > 0$. Of course, this makes sense only for large enough $m$; we will not mention such restrictions in the sequel. Moreover, since only the direction of the vector determined by $\varphi$ will matter, we need not worry about the non-uniqueness of $\varphi$. It will become clear shortly why the rather odd-looking transformation (3.1) is useful.

The transfer matrix $T(n, m; E)$ takes solution vectors $Y$ at $m$ to their value at $n$ (requiring this for two different values of $\alpha$ determines $T$ completely). So by definition we have $Y(n, E) = T(n, m; E)Y(m, E)$. Beware of the possible confusion here: the name "transfer matrix" is often used for the analogous object that updates $(y(n-1), y(n))^T$.

Because of the form of the potential $V$, there are three different types of transfer matrices we have to deal with. On the segments $\{a_n + 1, \ldots, a_n + l_n\}$, $V$ has the constant value $W_n$, and one infers easily from this that

$$T(a_n + 1 + l_n, a_n + 1; E) = \begin{pmatrix} \cos\omega_n l_n & \sin\omega_n l_n \\ -\sin\omega_n l_n & \cos\omega_n l_n \end{pmatrix} \equiv \mathrm{Rot}(\omega_n l_n), \tag{3.2}$$

i.e., $T$ is simply rotation by $\omega_n l_n$. This is the property that motivated the transformation (3.1).

On $\{a_n + 1 + l_n, \ldots, a_{n+1} - 1\}$, $V$ is periodic. Therefore, we now collect some basic facts about periodic potentials (compare also [16]). If $V(n+p) = V(n)$ for all $n$, then obviously also $T(n+p, m+p; E) = T(n, m; E)$. When applied to $V$ from above, this remark says that

$$T(a_n + 1 + l_n + mN_n, a_n + 1 + l_n + lN_n; E) = T_n(E)^{m-l}, \quad m, l \in \{0, 1, \ldots, L_n\}, \tag{3.3}$$

where we abbreviated $T(a_n + 1 + l_n + N_n, a_n + 1 + l_n; E) = T_n(E)$. A routine calculation yields

$$T_n(E) = \mathrm{Rot}(\omega_n N_n) - \frac{1}{N_n \sin\omega_n} \begin{pmatrix} \cos\omega_n \sin\omega_n(N_n - 1) & \sin\omega_n \sin\omega_n(N_n - 1) \\ \cos\omega_n \cos\omega_n(N_n - 1) & \sin\omega_n \cos\omega_n(N_n - 1) \end{pmatrix}. \tag{3.4}$$

By constancy of the Wronskian, we have $\det T_n(E) = 1$ (of course, this can also be verified directly with the aid of (3.4)). Hence, denoting the trace of $T$ by $D$ (for the time being, the dependence on $n, E$ will not be made explicit), the eigenvalues of $T$ are $\mu, \mu^{-1}$ with

$$\mu = \frac{D}{2} \pm \sqrt{\frac{D^2}{4} - 1}. \tag{3.5}$$

Here, we choose the sign so that $|\mu| \geq 1$. If $|D| \neq 2$, then $T$ is diagonalizable, i.e.

$$U^{-1}TU = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

for appropriate $U$. Note also that $\mu$ is real and $|\mu| > 1$ if $|D| > 2$ and that $|\mu| = 1$ if $|D| < 2$. This implies the following elementary estimates:

**Lemma 3.1** *a) If $|D| < 2$, then, for all $v \in \mathbf{C}^2$, $m \in \mathbf{N}$,*

$$\|T^m v\| \geq \frac{\|v\|}{\|U\|\,\|U^{-1}\|}.$$

*b) If $|D| > 2$, then there is an angle $\beta$ such that $|\varphi - \beta| \geq \epsilon \pmod{\pi}$ implies*

$$\|T^m e_\varphi\| \geq \frac{2\epsilon}{\pi} |\mu|^m$$

*(writing $e_\varphi = (\sin\varphi, \cos\varphi)^t$).*

The *proof* is straightforward and, moreover, identical with the proof of [12, Lemma 3.2]. It will therefore be omitted.

Finally, we have that

$$T(a_n + 1, a_n; E) = \mathrm{Rot}(\omega_{n-1}) + \frac{1}{\sin\omega_n} \begin{pmatrix} 0 & 0 \\ \cos(\omega_{n-1} - \omega_n) - 1 & \sin(\omega_{n-1} - \omega_n) \end{pmatrix}. \tag{3.6}$$

The second term on the right-hand side comes from the change of $\omega$ at $a_n$.

As in [12], Lemma 3.1b) will be used to show that, roughly speaking, the solutions of (2.1) are increasing on the segments $\{a_n, \dots, a_{n+1}\}$, provided the Prüfer angle $\varphi$ does not lie in the small exceptional set $I_\epsilon = (\beta-\epsilon, \beta+\epsilon)$. In this case, the interval $\{a_n+1, \dots, a_n+1+l_n\}$ can be used to readjust the Prüfer angles. The following result can thus be viewed as an insurance against bad luck. Note also that this issue is more delicate here than in the continuous case.

**Lemma 3.2** *There is a constant $C$ such that for any $\epsilon > 0$ and any measurable functions $\psi, \beta$, there exists an $l \in \mathbf{N}_0$, $l \le C\epsilon^{-4}$ with*

$$|\{\omega \in S : \psi(\omega) + l\omega \in I_\epsilon(\omega) \quad (mod\ \pi)\}| \le 4\epsilon.$$

*Here, $I_\epsilon(\omega) = (\beta(\omega)-\epsilon, \beta(\omega)+\epsilon)$, $S$ is a measurable subset of $(0,\pi)$, and $|\cdot|$ denotes Lebesgue measure.*

*Proof.* It suffices to show that

$$\frac{1}{L}\sum_{l=0}^{L-1} |\{\omega \in S : \psi(\omega) + l\omega \in I_\epsilon(\omega)\}| \le 4\epsilon$$

for some $L \le C\epsilon^{-4}$ where $C$ is independent of $\epsilon, \psi, \beta, S$. This can also be written as

$$\int_S d\omega \, \frac{1}{L}\sum_{l=0}^{L-1} \chi_{I_\epsilon(\omega)}(\psi(\omega) + l\omega) \le 4\epsilon. \tag{3.7}$$

Notice that by the ergodicity of a rotation on the torus with irrational rotation number (see e.g. [2]), the limit of the left-hand side of (3.7) exists as $L \to \infty$ and equals $2|S|\epsilon/\pi \le 2\epsilon$. Therefore, we need to estimate the error term in the ergodic theorem. The method we will use is similar to [5, Proof of Prop. 5].

For $A > 0$ let

$$M(A) = \left\{\omega \in (0,\pi) : \left|\frac{\omega}{\pi} - \frac{p}{q}\right| \ge \frac{A}{q^3} \quad \forall p \in \mathbf{Z}, q \in \mathbf{N}\right\}.$$

Then

$$(0,\pi) \setminus M(A) \subset \bigcup_{q=1}^{\infty} \bigcup_{p=0}^{q} \left(\frac{p\pi}{q} - A\pi q^{-3}, \frac{p\pi}{q} + A\pi q^{-3}\right),$$

thus

$$|(0,\pi) \setminus M(A)| < 2\pi A \sum_{q=1}^{\infty} (q+1)q^{-3} < A\pi^3. \tag{3.8}$$

Now fix $\omega \in M(A)$, and denote the $m$th convergent of the continued fraction expansion of $\omega/\pi$ by $p_m/q_m$ (for the basic properties of continued fractions, consult e.g. [2, 10]). We set $q_0 = 1$, and, as usual, $[x]$ denotes the largest integer $\le x$. Any $L \in \mathbf{N}$ can be expanded in terms of the $q_n$, i.e. $L = \sum_{i=0}^{n} b_i q_i$ where $n$ is determined by $q_n \le L < q_{n+1}$. Furthermore, we have that $b_i \in \{0, 1, \dots, [q_{i+1}/q_i]\}$, $b_n \in \{0, 1, \dots, [L/q_n]\}$. (We simply let $b_n$ be the largest integer with $b_n q_n \leq L$, then we determine $b_{n-1}$ analogously as the largest integer such that $b_n q_n + b_{n-1} q_{n-1} \leq L$ etc.) Correspondingly, the sum $\sum_{l=0}^{L-1}$ in (3.7) can now be decomposed into a number of sums with exactly $q_i$ terms ($i = 0, \dots, n$). Moreover, there are precisely $b_i$ such sums with $q_i$ terms. Since $|\omega/\pi - p_i/q_i| < q_i^{-2}$ [2, Section 7.4], a standard error estimate in the ergodic theorem (compare [2, Ch. 3, Lemma 4.1], [6, Theorem 1]) shows that for each sum over $q_i$ consecutive terms we have

$$\left| \sum_{l=j}^{j+q_i-1} \chi_{I_\epsilon(\omega)}(\psi(\omega) + l\omega) - \frac{2q_i\epsilon}{\pi} \right| \leq 2.$$

Hence, using the decomposition of $L$,

$$\sum_{l=0}^{L-1} \chi_{I_\epsilon(\omega)}(\psi(\omega) + l\omega) \leq \frac{2L\epsilon}{\pi} + 2(b_0 + \dots + b_n).$$

The $b_i$'s can be estimated as in [5], using the definition of the set $M(A)$. This leads to the inequality

$$\frac{1}{L} \sum_{l=0}^{L-1} \chi_{I_\epsilon(\omega)}(\psi(\omega) + l\omega) \leq \frac{2\epsilon}{\pi} + \frac{12 \ln L}{(AL)^{1/2}},$$

valid for $\omega \in M(A)$ and $L \geq 2$. Hence, taking (3.8) into account, we get for the integral from (3.7)

$$\int_S d\omega \, \frac{1}{L} \sum_{l=0}^{L-1} \chi_{I_\epsilon(\omega)}(\psi(\omega) + l\omega) \leq 2\epsilon + \frac{12\pi \ln L}{(AL)^{1/2}} + A\pi^3.$$

Now (3.7) is established as follows: Given any $\epsilon > 0$, first let $A = \epsilon/\pi^3$, and then take $L$ large enough so that (with this $A$) $12\pi(AL)^{-1/2} \ln L \leq \epsilon$. Then (3.7) holds, and it is easy to see that we can also achieve that $L \leq C\epsilon^{-4}$. □

# 4 Band structure

In this Section, we study $T_n(E)$ (see (3.4)) more carefully. The crucial quantity is the trace of this matrix; that is, we have to analyse the function

$$D_N(\omega) = 2\cos\omega N - \frac{\sin\omega N}{N \sin\omega}. \tag{4.1}$$

For $N \in \mathbf{N}$ and $l \in \{1, \dots, N-1\}$, let

$$\nu_l = \frac{l\pi}{N} - \frac{1}{2N^2 \sin(l\pi/N)}, \quad \delta_l = \frac{1}{2N^2 \sin(l\pi/N)}.$$

**Lemma 4.1** *Fix $\eta > 0$. Then for $N \geq N_0 = N_0(\eta)$ and $l \in \{[\eta N] + 1, \dots, [(1-\eta)N]\}$ we have:*
*a) If $\nu_{l-1} + \delta_{l-1} + 2N^{-5/2} \leq \omega \leq \nu_l - \delta_l - 2N^{-5/2}$, then $|D_N(\omega)| \leq 2 - N^{-5/2}$.*
*b) If $|\omega - \nu_l| \leq \delta_l - 2N^{-5/2}$, then $|D_N(\omega)| \geq 2 + N^{-5/2}$.*

*Proof.* For large $N$ and $\omega$ not close to 0 or $\pi$, the inequality $|D_N(\omega)| > 2$ implies that $\omega N = l\pi + \epsilon$ with $l \in \mathbf{N}$ and $\epsilon = O(N^{-1})$. For these $\omega$, a routine Taylor expansion of (4.1) shows that

$$(-1)^l D_N(\omega) = 2 - N^2(\omega - \nu_l)^2 + \frac{1}{4N^2 \sin^2(l\pi/N)} + O(N^{-4}).$$

Recall that if $I$ is an interval with $|D_N(\omega)| < 2$ for all $\omega \in I$, then $D_N(\omega)$ is monotone in $I$ (compare Theorem A.2). Now the assertions follow easily. □

For the application of Lemma 3.1, we will also need an estimate on the diagonalizing transformation $U$:

**Lemma 4.2** *In the situation of Lemma 4.1a) or b), the matrix $T$ from (3.4) (with $\omega_n = \omega, N_n = N$) can be diagonalized by a transformation $U = U(N,\omega)$ satisfying $\|U\|\,\|U^{-1}\| \le CN^{5/2}$. Here, $C$ is independent of $\omega, N$.*

The *proof* is completely analogous to the proof of [12, Lemma 4.2]. Note, however, that it may be necessary to take $N_0(\eta)$ even larger.

In the next Lemma, we show that for any compact subinterval $I \subset (-2,2)$, we can find a not too large number of small periodic potentials such that every energy $E \in I$ is in a "gap" (i.e. $|D| > 2$) of at least one of these periodic potentials. More precisely, we prove:

**Lemma 4.3** *Fix $d > 0$. Then there are $C(d) > 0$ and $N_0(d) \in \mathbf{N}$ so that for any $N \ge N_0$, there exist $W_1, \dots, W_r$ with the following properties:*
*1) $r \le CN$*
*2) $|W_i| \le 4\pi/N \quad (i = 1, \dots, r)$*
*3) For all $E \in [-2+d, 2-d]$, there is an $m \in \{1, \dots, r\}$ so that $|D_N(\omega_m(E))| \ge 2 + N^{-5/2}$.*

Recall that $\omega_m(E)$ is defined by $2\cos\omega_m = E - W_m$, $\omega_m \in (0,\pi)$.

*Proof.* In this proof, we will use several expansions and estimates which hold for large enough $N$; we will assume tacitly that $N$ is sufficiently large in this sense. First, notice that $\omega_m(E) \in [\eta, \pi - \eta]$ ($\eta = \eta(d) > 0$) for all $m$ and $E$ with $|E| \le 2 - d$, provided the $W_m$ satisfy assertion 2) of the Lemma. Thus we may apply Lemma 4.1. To this end, given (large enough) $N$, let $E_l(W) = 2\cos\nu_l + W$ (recall that $\nu_l$ depends on $N$). Then a Taylor expansion shows that

$$|\omega_m(E) - \nu_l| \le C_1(d)|E - E_l(W_m)|.$$

Hence, by Lemma 4.1b), there is a constant $C_2(d)$ so that $|E - E_l(W_m)| \le C_2 N^{-2}$ (for some $l \in \{[\eta N]+1, \dots, [(1-\eta)N]\}$) implies $|D_N(\omega_m(E))| \ge 2 + N^{-5/2}$. On the other hand, since $|2\cos\nu_l - 2\cos\nu_{l+1}| \le 4\pi N^{-1}$ (say), the interval $[-2+d, 2-d]$ can be covered with not more than $[2\pi C_2^{-1} N] + 1$ sets of the form

$$A(W) = \bigcup_{l=[\eta N]+1}^{[(1-\eta)N]} [E_l(W) - C_2 N^{-2}, E_l(W) + C_2 N^{-2}]$$

(where $0 \leq W < 4\pi N^{-1}$ for all these sets). $\square$

Lemma 4.3 suggests the following procedure for the choice of (some of) the parameters: Given a sequence of integers $N(s) \to \infty$ ($s \in \mathbf{N}$) (to be determined later), pick $d(s) > 0, d(s) \to 0$ so that $N(s) \geq N_0(d(s))$ and, say, $C(d(s)) \leq \ln N(s)$ for all $s$. For $s = 1, 2, \ldots$, choose inductively numbers $W_{j(s)+1}, \ldots, W_{j(s)+r(s)}$ which satisfy the conditions of the Lemma for $d = d(s)$ and $N = N(s)$. Here, we used the notation $j(s) = \sum_{t=1}^{s-1} r(t)$. Furthermore, we set $N_n = N(s)$ for $n = j(s)+1, \ldots, j(s)+r(s)$. The remaining parameters $N(s), l_n, L_n$ will be chosen in the next Section.

Note that the potential consists of a sequence of "blocks", indexed by $s$, and each block itself is built out of $r(s)$ periodic pieces. Moreover, any $E \in (-2,2)$ is in a gap of at least one periodic piece belonging to the block $s$ provided $s$ is large enough (this is, in a sense, the defining property of the partition into blocks). For an index $n \in \mathbf{N}$, we will write $s(n)$ for the index of the block to which the $n$th periodic potential belongs. So, $s(n)$ is determined by $j(s(n)) + 1 \leq n \leq j(s(n)) + r(s(n))$.

# 5 Asymptotic estimates

As a final preparatory step, we have to ensure that for almost all $E \in (-2,2)$, Lemma 4.1 is applicable and, moreover, that the Prüfer angle $\varphi$ is not in the exceptional set $(\beta - \epsilon, \beta + \epsilon)$ (see Lemma 3.1b)) if Lemma 4.1b) applies.

**Lemma 5.1** *Assume that $\sum_{s=1}^{\infty} N(s)^{-1/2} \ln N(s) < \infty$. Then, for almost every $E \in (-2,2)$, there is an $n_0 = n_0(E)$ such that for $n \geq n_0$, we have either*
*a) $|D_{N_n}(\omega_n(E))| \leq 2 - N_n^{-5/2}$ or*
*b) $|D_{N_n}(\omega_n(E))| \geq 2 + N_n^{-5/2}$.*
*Moreover, Lemma 4.2 applies in both cases.*

This is a statement about the band structure of the periodic pieces; in particular, this fact only depends on the form of the basic potential, i.e. the parameters $W_n$ (which enter through the functions $\omega_n(E)$) and $N_n$. Also, recall that $N_n = N(s(n))$ by construction.

*Proof.* The assertion is, of course, exactly the conclusion of Lemmas 4.1, 4.2 for $\omega_n(E)$ instead of $\omega$ and with $N = N_n$. Thus, if we let

$$A_n(d) = \{E \in [-2+d, 2-d] : |\omega_n(E) - \nu_l \pm \delta_l| < 2N_n^{-5/2} \text{ for some } l\}$$

(this notation means that the inequality is required to hold for either + or −), it suffices to show that

$$|\{E : E \in A_n(d) \text{ for infinitely many } n\}| = 0 \tag{5.1}$$

for every $d > 0$. (Notice that $A_n$ is exactly the complement of the set covered by Lemma 4.1.) By the Borel-Cantelli Lemma, (5.1) will follow from $\sum |A_n(d)| < \infty$.

Clearly, we have that $|dE/d\omega_n| \leq 2$. Hence, since there are at most $N_n$ possible values for $l$ in the condition defining $A_n(d)$, we obtain $|A_n(d)| \leq CN_n^{-3/2}$. The number of indices $n$ corresponding to a fixed $s$ is $r(s) \leq N(s)\ln N(s)$, so the proof is complete. □

In order to control the set where the Prüfer angle $\varphi$ is close to its exceptional value $\beta$, fix a sequence $\epsilon(s) > 0$, and introduce (again, writing $d_n = d(s(n))$ etc.)

$$B_n = \{E \in [-2+d_n, 2-d_n] : |D_{N_n}(\omega_n(E))| > 2, |\varphi(a_n+1+l_n, E) - \beta_n(E)| < \epsilon_n\}.$$

Now Lemma 3.2 together with (3.2) show that we can find $l_n \in \{0, 1, \ldots, [C\epsilon_n^{-4}]\}$ such that $|\{\omega_n(E) : E \in B_n\}| \leq 4\epsilon_n$. As above, since $|dE/d\omega_n| \leq 2$, it follows that $|B_n| \leq 8\epsilon_n$.

There is a subtlety here that needs to be clarified: Obviously, the actual value of $l_n$ depends on the function $\varphi(a_n+1, \cdot)$, i.e. on the parameters $N_i, L_i, l_i, W_{i+1}$ for $i \leq n-1$, but we have not chosen the $L_i$ yet! So, strictly speaking, we should have fixed the $L_i$ first and only then can the $l_i$ be chosen (inductively) as described above. However, any choice of the $L_i$ would appear rather unmotivated at this point; therefore, we have reversed the logical order of the steps. Of course, this does not cause any lack of rigour, because we might as well fix all the parameters from the outset and then run through the whole proof again.

With this understanding, we obtain

**Lemma 5.2** *Assume that* $\sum_{s=1}^{\infty} \epsilon(s)N(s)\ln N(s) < \infty$. *Then there are* $l_n \leq C\epsilon_n^{-4}$ *so that*

$$|\{E \in (-2,2) : E \in B_n \text{ for infinitely many } n\}| = 0.$$

*Proof.* By the remarks above, $\sum |B_n| \leq 8 \sum \epsilon(s)N(s)\ln N(s)$ for appropriate $l_n$. □

We now fix sequences $N(s), \epsilon(s)$ satisfying the hypotheses of the last two Lemmas. Theorem 2.1 can be proved with many different choices; we take $N(s) = s^3, \epsilon(s) = s^{-5}$.

We are now ready to establish the existence of rapidly increasing solutions for almost all $E$. These concluding arguments are rather similar to the corresponding discussion in [12, Section 6]. Therefore, we will only sketch the main steps. We will look at $R_{ni}(E) := R(a_n+1+l_n+iN_n, E)$ for $n \in \mathbf{N}, i \in \{0, 1, \ldots, L_n\}$. Note that $R$ also depends on the initial condition $\alpha$, but this dependence will not be made explicit. We will use the abbreviations $R_0 = R_{n_0 i_0}, s_0 = s(n_0)$. Moreover, $L_n$ will also depend on $s$ only, so we can write $L(s)$.

**Lemma 5.3** *For almost all* $E \in (-2,2)$, *there are constants* $C_i(E) > 0, t_0(E) \in \mathbf{N}$ *such that*

$$\ln R_{ni}(E) - \ln R_0(E) \geq C_1 \sum_{t=s_0+1}^{s(n)-1} L(t)t^{-15/4} - C_2 \sum_{t=s_0}^{s(n)} t^3 \ln^2 t$$

*if* $a_n+1+l_n+iN_n \geq a_{n_0}+1+l_{n_0}+i_0N_{n_0}$ *and* $s_0 \geq t_0$.

*Proof.* Fix an $E$ which is not in the exceptional sets of Lemmas 5.1, 5.2, and let $m$ be a sufficiently large integer. According to Lemma 5.1, we have to distinguish two cases. Assume first that Lemma 5.1a) applies, i.e. $|D_{N_m}(\omega_m(E))| \leq 2 - N_m^{-5/2}$. Recall that $D_{N_m}(\omega_m(E)) = D_m(E)$ where $D_m(E)$ is the trace of $T_m(E)$ from equation (3.4). So we deduce from (3.3) and Lemmas 3.1a), 4.2 that

$$\ln R_{mj}(E) - \ln R_{mi}(E) \geq -C \ln s(m) \quad (0 \leq i \leq j \leq L_m)$$

where $C > 0$ is independent of $m, i, j$. In particular, taking $i = 0, j = L_m$ and recalling that $R$ is constant on $\{a_m+1, \ldots, a_m+1+l_m\}$ by (3.2), we see that $\ln R(a_{m+1}, E) - \ln R(a_m+1, E) \geq -C \ln s(m)$. Moreover, since $T(a_{m+1}+1, a_{m+1}; E) - \mathrm{Rot}(\omega_m)$ is of order $O(N_m^{-1})$ by (3.6) and Lemma 4.3 2), we also have that

$$\ln R(a_{m+1}+1, E) - \ln R(a_m+1, E) \geq -C \ln s(m). \tag{5.2}$$

If, on the other hand, $|D_{N_m}(\omega_m(k))| \geq 2 + N_m^{-5/2}$, then (3.2), (3.3), (3.5), (3.6) and Lemmas 3.1b), 5.2 show that

$$\ln R(a_{m+1}+1, E) - \ln R(a_m+1, E) \geq C_1 L_m s(m)^{-15/4} - C_2 \ln s(m) \tag{5.3}$$

with positive constants $C_i$ which are independent of $m$.

Now the assertion is obtained by summing up the contributions from (5.2), (5.3) and by noting that for every $s$, there is at least one $m$ with $s(m) = s$ for which (5.3) is true (by Lemma 4.3 3)). An additional argument is needed to handle the case when $|D_{N_{n_0}}(\omega_{n_0}(E))| \geq 2 + N_{n_0}^{-5/2}$ because Lemma 5.2 does not exclude the possibility that the Prüfer angle at $a_{n_0} + 1 + l_{n_0} + i_0 N_{n_0}$ is close to the exceptional value $\beta$. We refer to [12] for details on this (technical) issue. □

Thus, if $L(s)$ grows sufficiently fast, the $R_{ni}$ are also rapidly increasing. We can take, for instance, $L(s) = [s^{31/4}]$; then, using some elementary (and crude) estimates, we get from Lemma 5.3 for almost all $E$ and sufficiently large $s_0$

$$\ln R_{ni}(E) - \ln R_0(E) \geq \begin{cases} -C_1(E) s_0^3 \ln^2 s_0 & s(n) = s_0, s_0+1 \\ C_2(E) s(n)^4 & s(n) \geq s_0 + 2 \end{cases} \tag{5.4}$$

where, as usual, the $C_i$ are positive constants which are independent of $n_0, i_0, n, i$.

The proof of the existence of $l_2$-solutions (given the existence of rapidly increasing solutions) also follows the lines of [12]. The basic tool is the following reformulation of [9, Theorem 8.1] (this Theorem, in turn, is modelled on results of [13]). In fact, we need a slightly more general version here because the transfer matrix from (3.6) is not unimodular.

**Lemma 5.4** ([9]) *Assume that there is an increasing sequence $x_n \to \infty$ such that:*
*1)* $\|T(x_{n+1}, x_n; E)\| \leq C_1(E)$ *for all* $n$
*2)* $\det T(x_n, x_0; E) \to C_2(E) \neq 0 \quad (n \to \infty)$
*3) There is a vector* $v \in \mathbf{R}^2$ *with* $\sum \|T(x_n, x_0; E)v\|^{-2} < \infty$.

*Then either*
*a) There exists another vector* $u \neq 0$ *such that (writing* $\rho_n = \|T(x_n, x_0; E)u\|$, $R_n = \|T(x_n, x_0; E)v\|$)

$$\rho_n^2 \leq C(E)\left[R_n^{-2}\left(\sum_{i=n}^{\infty}(R_n/R_i)^2\right)^2 + R_n^{-2}\right].$$

*or*
*b)* $\rho_n/R_n \to \infty$ *if* $u$ *is not a constant multiple of* $v$.

A step-by-step check shows that the proof of [9, Theorem 8.1] still works in this slightly more general setting (compare also [12]).

We can now proceed as in [12, Section 6] to prove the following facts: The hypotheses of Lemma 5.4 hold for almost all $E$ if $x_n$ is the sequence of the points $a_n + 1 + l_n + iN_n$. As for assumption 2), note that $\det T = 1$ for the transfer matrices from (3.2), (3.4) and $\det T(a_n+1, a_n) = \sin\omega_{n-1}/\sin\omega_n$ by (3.6). Conclusion b) of the Lemma can be true only for an $E$-set of measure zero (by an application of Theorem A.3). However, if Lemma 5.4a) applies, then this estimate together with (5.4) ensure that there is an $l_2$-solution for these $E$. Since it is easy to see that, say, $V(n) = O(n^{-1/9})$ (with the parameters specified above), the proof of Theorem 2.1 is, finally, complete.

# 6 Singular continuous spectrum

In complete analogy to [12, Theorem 7.1], a "sparse" variant of the potential constructed above leads to purely singular continuous spectrum in $(-2, 2)$. More precisely, for an increasing sequence of integers $s_m$, introduce an auxiliary potential as follows: For $a_n \leq m < a_{n+1}$, define

$$V_{aux}(m) = \begin{cases} V(m) & s(m) \in \{s_i : i \in \mathbb{N}\} \\ 0 & \text{otherwise} \end{cases}.$$

To obtain the new potential $\tilde{V}$, readjust the $l_n$ so that (an analogue of) Lemma 5.2 holds (use Lemma 3.2, of course). Of course, on the segments with $V \equiv 0$, we define $R(m, E) = \|Y(m, E)\|$ where now

$$Y(m, E) = \frac{1}{\sin\omega}\begin{pmatrix} \sin\omega & 0 \\ -\cos\omega & 1 \end{pmatrix}\begin{pmatrix} y(m-1, E) \\ y(m, E) \end{pmatrix}$$

and $2\cos\omega = E$ (compare this definition with (3.1)). Then $R$ is constant on the segments where $V \equiv 0$, so the above reasoning also applies to $\tilde{V}$, i.e., for almost all $E \in (-2, 2)$, $R$ increases. Using Theorem A.3 with an appropriate test function $f$ (see [12] for details), we see that $H$ has no absolutely continuous spectrum. Moreover, if the segments separating the blocks $s_m$ are rapidly increasing, then no solution is in $l_2$ (again, details can be found in [12]). Thus we get

**Theorem 6.1** *If $s_m$ increases sufficiently rapidly, then the modified potential $\tilde{V}$ satisfies*
*1) $\tilde{V}(n) = O(n^{-c}) \quad (c > 0)$*
*2) For arbitrary $V(1)$, the spectrum is purely singular continuous in $(-2, 2)$.*

# A Appendix

In this Appendix, we compile some basic facts about Jacobi matrices for the reader's (and author's) convenience. These statements are, of course, known, but it is difficult to find suitable references for them.

We begin with the variation of constants formula. Write

$$y(n-1) + y(n+1) + V(n)y(n) =: (\tau y)(n) \qquad (n \in \mathbf{N})$$

for functions $y : \mathbf{N}_0 \to \mathbf{C}$, and denote the Wronskian of two such functions by $W(u,v) = u(n)v(n+1) - u(n+1)v(n)$. If $u, v$ both solve $\tau y = zy$, then $W(u,v)$ is independent of $n$, as is easily verified.

**Lemma A.1** *Let $u, v$ be solutions of $(\tau - z)y = 0$ with $W(u,v) = 1$. Then the solutions of $(\tau - z)y = f$ are precisely given by*

$$y(n) = c_1 u(n) + c_2 v(n) + v(n) \sum_{k=1}^{n} u(k)f(k) - u(n) \sum_{k=1}^{n} v(k)f(k)$$

*with $c_i \in \mathbf{C}$.*

*Proof.* This is checked by a straightforward calculation. □

Clearly, this formula also holds with summation up to $n-1$ (instead of $n$) in both sums.

**Theorem A.2** *Let $D_N(E)$ be the trace of the transfer matrix $T(N+1, 1; E)$, associated with the equation $\tau y = Ey$, and suppose that $|D_N(E)| < 2$ for all $E$ in some interval $I$. Then $D_N$ is a strictly monotone function of $E \in I$.*

*Proof.* This is merely an adaption of the corresponding proof for the continuous Schrödinger operator (see [4, Theorem 2.3.1]) to the discrete case.

Since the trace is invariant under similarity transformations, we may work with the transfer matrix for the vectors $(y(n-1), y(n))^T$ where $y$ solves $\tau y = Ey$. Thus $T$ is given by

$$T(N+1, 1; E) = \begin{pmatrix} u(N,E) & v(N,E) \\ u(N+1,E) & v(N+1,E) \end{pmatrix}$$

where $u, v$ are the solutions of $\tau y = Ey$ with $u(0,E) = v(1,E) = 1$, $v(0,E) = u(1,E) = 0$. In particular, $D_N(E) = u(n,E) + v(N+1,E)$. Differentiating the difference equation with respect to $E$ (since the solutions are polynomials in $E$ for fixed $n$, they are certainly differentiable) and using Lemma A.1 yields (dropping $E$ on the right-hand side)

$$D_N'(E) = (v(N+1) - u(N)) \sum_{k=1}^{N} u(k)v(k) + v(N) \sum_{k=1}^{N} u(k)^2 - u(N+1) \sum_{k=1}^{N} v(k)^2. \quad \text{(A.1)}$$

Since $W(u,v) = 1$, we see that

$$D_N^2(E) = (u(N) - v(N+1))^2 + 4 + 4u(N+1)v(N),$$

and combining this with (A.1) shows

$$\begin{aligned} 4u(N+1)D_N'(E) &= -\sum_{k=1}^{N} [2u(N+1)v(k) + (u(N) - v(N+1))u(k)]^2 \\ &\quad -(4 - D_N^2) \sum_{k=1}^{N} u(k)^2. \end{aligned}$$

Hence $D_N'$ does not vanish in $I$ and, therefore, has constant sign there. □

We conclude with an eigenfunction estimate often referred to as Schnol's Theorem. In principle, it can be proved with the methods of [1], but the following argument is simpler. It is closely related to the ideas used in [9] (where considerably deeper results are obtained). We consider the operator $H$ defined in Section 1; as above, $v(n,E)$ denotes the solution of the equation $\tau v = Ev$ satisfying $v(0,E) = 0, v(1,E) = 1$.

**Theorem A.3** *Fix $f(n) \in l_2(\mathbf{N})$. Then, for spectrally almost every $E$, we also have $f(\cdot)v(\cdot,E) \in l_2$.*

*Proof.* Let $d\rho(t) = d\|E(t)\delta_1\|^2$ be the spectral measure of $H$ for the vector $\delta_1(n) = \delta_{1n}$. Then the usual spectral representation shows that $\int v(n,E)^2\, d\rho(E) = 1$ (cf. [9, Theorem 2.1D]). Hence

$$\int \left( \sum_{n=1}^{\infty} |f(n)v(n,E)|^2 \right) d\rho(E) = \sum_{n=1}^{\infty} |f(n)|^2 < \infty,$$

proving the assertion, since $\delta_1$ is a cyclic vector. □

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