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Integral conditions for the asymptotic completeness of two-space scattering systems

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Abstract. The completeness of a two-space scattering system given by two selfadjoint operators H_1 and H_2 acting in different Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is considered. In virtue of the invariance principle it suffices to study this problem for the scattering system $\{\varphi(H_2), \varphi(H_1)\}$ where φ is an admissible function. If $\varphi(H_i), i = 1, 2$, are both integral operators an elementary L^1-condition is sufficient for the completeness. If for instance $\mathfrak{H}_1 = \mathfrak{H}_2 = L^2(\mathbb{R}^d)$ and if $\varphi(H_i)$ are integral operators with the kernels $\varphi_i(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, the scattering system $\{H_2, H_1\}$ is complete if

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \; |\varphi_2(x,y) - \varphi_1(x,y)| < \infty.$$

Keywords: mathematical scattering theory, integral operators.
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1 Introduction

Let H_1 and H_2 be two selfadjoint semibounded linear operators acting in possibly different Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, linked by a bounded identification operator J. The main objective of this article is to give a criteria ensuring the completeness of the scattering system

$\{H_2, J, H_1\}$ in terms of an integral condition for operator-valued functions of $H_i, i = 1, 2$.

It is sufficient to prove the existence and completeness of the scattering system $\{\varphi(H_2), J, \varphi(H_1)\}$ where $\varphi : \mathbb{R} \to \mathbb{R}$, is an admissible function such that the invariance principle holds in its strong form. The normal trace class condition for that is

$$\varphi(H_2)J - J\varphi(H_1) \in \mathfrak{B}_1(\mathfrak{H}_1, \mathfrak{H}_2), \tag{1.1}$$

($\mathfrak{B}_1$ - trace class operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$). The condition in (1.1) is generalized in section 2. The main remaining completeness assumption for the system $\{\varphi(H_2), J, \varphi(H_1)\}$ is

$$\varphi(H_2)\big(\varphi(H_2)J - J\varphi(H_1)\big)\varphi(H_1) \in \mathfrak{B}_1(\mathfrak{H}_1, \mathfrak{H}_2). \tag{1.2}$$

This last condition is a "sandwiched" version of (1.1). This implies an easy and smooth integral conditon if $\varphi(H_i)$ are integral operators, which occurs very often in concrete examples.

Take $\mathfrak{H}_1 = L^2(\mathbb{R}^d)$, $\mathfrak{H}_2 = L^2(\Sigma)$ with $\Sigma \subseteq \mathbb{R}^d$. Let $\varphi(H_1), \varphi(H_2)$ be integral operators in $L^2(\mathbb{R}^d)$ or $L^2(\Sigma)$, respectively. Denote their kernels by

$$\varphi_1(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},$$
$$\varphi_2(\cdot,\cdot) : \Sigma \times \Sigma \to \mathbb{R}.$$

Then the scattering system $\{H_2, J, H_1\}$ is complete (see section 3) if $|\mathbb{R}^d \backslash \Sigma| < \infty$ and if

$$\int_{\Sigma} dx \int_{\mathbb{R}^d} dy \; |\varphi_2(x,y)\chi_\Sigma(y) - \chi_\Sigma(x)\varphi_1(x,y)| < \infty. \tag{1.3}$$

The most important examples for $\varphi(H_i)$ are semigroups, e^{-2H_i}, and powers of resolvents, $(H_i + a)^{-2m}$, $m \geq 1$, $a > c > 0$. If $J = \mathbb{1}$ the only completeness condition is then for instance

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} \big|(e^{-2H_2})(x,y) - (e^{-2H_1})(x,y)\big| < \infty. \tag{1.4}$$

Applications for potential scattering, for obstacle scattering and for scattering systems with magnetic fields are explained in section 4.

Trace class conditions play an important role in abstract mathematical scattering theory. They are very useful if the information about the unperturbed system is small, i. e. if properties for the dynamics of e^{-itH_1} are unknown.

Trace class conditions go back to Kato [10], Rosenblum [13] and Birman [3]. The two space trace class criterion is due to Pearson [11].

The reader may find a long list of references on this topic in the standard textbooks on scattering theory such as Baumgärtel, Wollenberg [2], Yafaev [16] or Reed, Simon [12]. The case of trace class integral operators was studied by Deift, Simon [5] and summarized in Simon [14], chapter VII. There they used the restrictive method explained in Example 2.5. A preresult of the present article for obstacle scattering was given by Stollmann [15].

2 Generalized criterion for the completeness of scattering systems

DEFINITION 2.1. (Wave operators) Let H_1, H_2 be two selfadjoint semibounded operators acting in possibly different Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Let J be a bounded identification operator from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$.

The two-space wave operators are given by

$$\Omega_\pm(H_2, J, H_1) := \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_2} J e^{-itH_1} P_{\text{ac}}(H_1), \tag{2.1}$$

$P_{\text{ac}}(H_1)$ is the projection operator onto the absolutely continuous subspace of H_1.

DEFINITION 2.2. (Completeness) Assume that $\Omega_+ = \Omega_+(H_2, J, H_1)$ exists. Let $\Omega_+ = \operatorname{sgn}\Omega_+|\Omega_+|$ be its polar decomposition.

Ω_+ is called H_1-semicomplete if

$$(\operatorname{sgn}\Omega_+)^*(\operatorname{sgn}\Omega_+) = P_{\text{ac}}(H_1) \tag{2.2}$$

and H_2-semicomplete if

$$\operatorname{sgn}\Omega_+(\operatorname{sgn}\Omega_+)^* = P_{\text{ac}}(H_2). \tag{2.3}$$

Ω_+ is called complete, if it is both H_1- and H_2-semicomplete.

The scattering systems $\{H_2, J, H_1\}$ is complete if Ω_+ and Ω_- are complete, i. e. if

$$\operatorname{ran}\Omega_+ = \operatorname{ran}\Omega_-. \tag{2.4}$$

In this case $H_1 \restriction P_{\text{ac}}(H_1)\mathfrak{H}_1$ is unitarily equivalent to $H_2 \restriction P_{\text{ac}}(H_2)\mathfrak{H}_2$.

For a detailed overview on mathematical scattering theory the interested reader should consult the book by Baumgärtel, Wollenberg [2].

Sufficient conditions for a scattering system $\{H_2, J, H_1\}$ to be complete are

$$\text{(i)}\ \Omega_\pm(H_2, J, H_1),\ \Omega_\pm(H_1, J^*, H_2) \text{ exist}, \tag{2.5}$$

$$\text{(ii)}\ \underset{t\to\pm\infty}{\text{s-lim}}(J^*J - \mathbb{1}_{\mathfrak{H}_1})e^{-itH_1}P_{\text{ac}}(H_1) = 0 \tag{2.6}$$

$$\text{(iii)}\ \underset{t\to\pm\infty}{\text{s-lim}}(JJ^* - \mathbb{1}_{\mathfrak{H}_2})e^{-itH_2}P_{\text{ac}}(H_2) = 0. \tag{2.7}$$

DEFINITION 2.3. (Admissible function) Let $\psi(\cdot) : \mathbb{R} \to \mathbb{R}$. Take mutually disjoint intervalls $I_n = (a_n, b_n)$ with $\bigcup_{n=1}^N \overline{I}_n = \mathbb{R}$, N finite. ψ is called admissible on $\mathbb{R}$ if for every I_n the following conditions hold:

(i) $\psi(\cdot)$ is continuously differentiable,

(ii) $\psi' > 0$,

(iii) $\psi'(\cdot)$ is locally of bounded variation.

The property below will prove to be particularly useful in the forthcoming analysis:

$\psi^{-1}(\cdot)$ is an admissible function on $\psi(I_n)$ if ψ is admissible.

Then a possible version of the invariance principle in its strong form is (see Baumgärtel, Wollenberg [2], p. 246):
If $\Omega_+(\psi^{-1}(H_2), J, \psi^{-1}(H_1))$ exists, then $\Omega_+(H_2, J, H_1)$ exists and we have

$$\Omega_+(\psi^{-1}(H_2), J, \psi^{-1}(H_1)) = \Omega_+(H_2, J, H_1). \tag{2.8}$$

If the function satisfies (i) and (iii) but is decreasing, i. e. $\psi' < 0$ on every I_n, the invariance principle holds in the sense

$$\Omega_+(\psi^{-1}(H_2), J, \psi^{-1}(H_1)) = \Omega_-(H_2, J, H_1).$$

The invariance principal in its strong form holds for wave operators defined in Abelian sense. However we study here only trace class perturbations. In this case the invariance principle in its strong form does hold also for the wave operator definded as strong limits (2.1). A proof can be found in [2] p. 343.

EXAMPLES 2.4.

(a) Take

$$\psi(\lambda) = \begin{cases} -\frac{1}{2}\ln\lambda & \lambda \in (0,\infty) \\ -\lambda & \lambda \in (-\infty, 0) \end{cases} \tag{2.9}$$

then

$$\psi^{-1}(\mu) = e^{-2\mu}. \tag{2.10}$$

If $\Omega_+(e^{-2H_2}, J, e^{-2H_1})$ exists, then the invariance principle entails the existence of $\Omega_-(H_2, J, H_1)$.

(b) Take

$$\psi(\lambda) = \frac{1}{\sqrt{\lambda}} - a, \qquad \lambda \neq 0, a > c > 0. \tag{2.11}$$

Then

$$\psi^{-1}(\mu) = \frac{1}{(\mu + a)^2}, \tag{2.12}$$

i. e. one has to study the existence of

$$\Omega_+((H_2 + a)^{-2}, J, (H_1 + a)^{-2}).$$

The usual trace class criterion ensures the existence of the wave operators $\Omega_\pm(H_2, J, H_1), \Omega_\pm(H_1, J^*, H_2)$ if

$$\psi^{-1}(H_2)J - J\psi^{-1}(H_1) \in \mathfrak{B}_1(\mathfrak{H}_1, \mathfrak{H}_2).$$

($\mathfrak{B}_1$ - trace class operators). This condition is commonly used in the literature, but it is rather restrictive. This is illustrated in the following example.

EXAMPLE 2.5. Take $\psi^{-1}(\mu) = e^{-2\mu}$ and study

$$e^{-2H_2}J - Je^{-2H_1} = e^{-H_2}\langle x\rangle^{-\alpha}\langle x\rangle^{\alpha}(e^{-H_2}J - Je^{-H_1}) + (e^{-H_2}J - Je^{-H_1})\langle x\rangle^{\alpha}\langle x\rangle^{-\alpha}e^{-H_1}. \quad (2.13)$$

Here $\langle x\rangle^\alpha := (1+|x|^2)^{\alpha/2}$, $\alpha > 0$. Assume that e^{-H_2}, e^{-H_1} are integral operators with kernels $(e^{-H_i})(x, y), i = 1, 2$. Assume that α is large enough so that

$$\begin{aligned} e^{-H_2}\langle x\rangle^{-\alpha} &\in \mathfrak{B}_2(\mathfrak{H}_2, \mathfrak{H}_2), \\ \langle x\rangle^{-\alpha}e^{-H_1} &\in \mathfrak{B}_2(\mathfrak{H}_1, \mathfrak{H}_1) \end{aligned} \quad (2.14)$$

($\mathfrak{B}_2$ - Hilbert-Schmidt operators). Then the wave operators $\Omega_\pm(H_2, J, H_1)$ exist if

$$\int dx \int dy \, \left|\langle x\rangle^{\alpha}(e^{-H_2}J - Je^{-H_1})(x, y)\right|^2 < \infty. \quad (2.15)$$

This should be compared to the final result of this article (see Corollary 3.4), according to which the condition

$$\int dx \int dy \, \left|(e^{-H_2}J - Je^{-H_1})(x, y)\right| < \infty \quad (2.16)$$

is sufficient. In order to prove this, a modified or generalized completeness condition will be established.

CRITERION 2.6. *Let H_1, H_2, J and $\mathfrak{H}_1, \mathfrak{H}_2$ be given as above. Let $\psi(\cdot)$ be an admissible function and set $\psi^{-1} = \varphi$. Assume* $\operatorname{ran}\varphi(H_i)$ *is dense in $\mathfrak{H}_i, i = 1, 2$.*

Then the wave operators $\Omega_\pm(H_2, J, H_1)$, $\Omega_\pm(H_1, J^*, H_2)$ exist if

$$\varphi(H_2)(\varphi(H_2)J - J\varphi(H_1))\varphi(H_1) \in \mathfrak{B}_1(\mathfrak{H}_1, \mathfrak{H}_2), \quad (2.17)$$

$$\varphi(H_2)J - J\varphi(H_1) \in \mathfrak{B}_\infty(\mathfrak{H}_1, \mathfrak{H}_2) \quad (2.18)$$

($\mathfrak{B}_\infty$ - compact operators).

The wave operators are complete if

$$(J^*J - \mathbb{1}_{\mathfrak{H}_1})\varphi(H_1) \in \mathfrak{B}_\infty(\mathfrak{H}_1, \mathfrak{H}_1), \quad (2.19)$$

$$(JJ^* - \mathbb{1}_{\mathfrak{H}_2})\varphi(H_2) \in \mathfrak{B}_\infty(\mathfrak{H}_2, \mathfrak{H}_2). \quad (2.20)$$

In the one space situation with $J = \mathbb{1}$ only (2.17) and (2.18) are sufficient.

PROOF. Define $\widetilde{J} = \varphi(H_2)J\varphi(H_1)$ then (2.17) means that
$$\varphi(H_2)\widetilde{J} - \widetilde{J}\varphi(H_1) \in \mathfrak{B}_1(\mathfrak{H}_1, \mathfrak{H}_2).$$
Therefore $\Omega_\pm(\varphi(H_2), \widetilde{J}, \varphi(H_1))$ exist, i. e.
$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\varphi(H_2)}\varphi(H_2)J\varphi(H_1)e^{it\varphi(H_1)}P_{\mathrm{ac}}(\varphi(H_1))$$
exist. Now $\operatorname{ran}\varphi(H_1)$ was assumed to be dense in $\mathfrak{H}_1$. Hence
$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\varphi(H_2)}\varphi(H_2)Je^{-it\varphi(H_1)}P_{\mathrm{ac}}(\varphi(H_1))$$
exist. The Riemann-Lebesgue lemma together with (2.18) yield the existence of
$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\varphi(H_2)}J\varphi(H_1)e^{-it\varphi(H_1)}P_{\mathrm{ac}}(\varphi(H_1)).$$
The same density argument implies the existence of $\Omega_\pm(\varphi(H_2), J, \varphi(H_1))$. The invariance principle can be used for the existence of $\Omega_\pm(H_2, J, H_1)$.

The H_1-completeness follows via (2.19) i. e.
$$\operatorname*{s-lim}_{t\to\pm\infty}(J^*J - \mathbb{1}_{\mathfrak{H}_1})\varphi(H_1)e^{-itH_1}P_{\mathrm{ac}}(H_1) = 0.$$
The same density argument yields
$$\operatorname*{s-lim}_{t\to\pm\infty}(J^*J - \mathbb{1}_{\mathfrak{H}_1})e^{-itH_1}P_{\mathrm{ac}}(H_1) = 0.$$
The assumption that $\operatorname{ran}\varphi(H_2)$ is dense in $\mathfrak{H}_2$ is used for (2.7), i. e.
$$\operatorname*{s-lim}_{t\to\pm\infty}(JJ^* - \mathbb{1}_{\mathfrak{H}_2})e^{-itH_2}P_{\mathrm{ac}}(H_2) = 0.$$

q. e. d.

In the Examples 2.4 the admissible functions have the form $\psi^{-1} = \varphi^2$ with $\varphi(\mu) = e^{-\mu}$ or $\varphi(\mu) = \frac{1}{\mu+a}$. In order to avoid square roots in the following section we work with this notation.

The next corollary is only a reformulation of Criterion 2.6.

COROLLARY 2.7. *Let H_1, H_2, J and $\mathfrak{H}_1, \mathfrak{H}_2$ given as above. Let $\psi(\cdot)$ be an admissible function and set $\psi^{-1} = \varphi^2$. Assume that* $\operatorname{ran}\varphi^2(H_i)$ *is dense in $\mathfrak{H}_i, i = 1, 2$. Set*
$$D := \varphi^2(H_2)J - J\varphi^2(H_1). \tag{2.21}$$
The wave operators $\Omega_\pm(H_2, J, H_1)$ exist and are complete if
$$\varphi^2(H_2)\,D\,\varphi^2(H_1) \in \mathfrak{B}_1(\mathfrak{H}_1, \mathfrak{H}_2) \tag{2.22}$$
$$D \in \mathfrak{B}_\infty(\mathfrak{H}_1, \mathfrak{H}_2) \tag{2.23}$$
$$(J^*J - \mathbb{1}_{\mathfrak{H}_1})\varphi^2(H_1) \in \mathfrak{B}_\infty(\mathfrak{H}_1, \mathfrak{H}_1) \tag{2.24}$$
$$(JJ^* - \mathbb{1}_{\mathfrak{H}_2})\varphi^2(H_2) \in \mathfrak{B}_\infty(\mathfrak{H}_2, \mathfrak{H}_2). \tag{2.25}$$
The main condition is (2.22). In the next section we study this generalized completeness criterion if $\varphi(H_i)$ are integral operators.

3 Integral conditions

The generalized completeness conditions in Criterion 2.6 or Corollary 2.7 are now considered for admissible functions φ^2 for which $\varphi(H_i)$ are integral operators. In this case the difference D in (2.21) is also an integral operator and the trace class condition in (2.22) is satisfied if

$$\int dx \int dy \, |D(x,y)| < \infty, \tag{3.1}$$

where $D(\cdot,\cdot)$ is the kernel of D.

CRITERION 3.1. *Let $\mathfrak{H}_1 = L^2(\mathbb{R}^d)$ and $\mathfrak{H}_2 = L^2(\Sigma)$, where $\Sigma \subseteq \mathbb{R}^d$ is an open set. The identification operator is defined by*

$$Jf := f/_\Sigma, \tag{3.2}$$

the restriction of $f \in L^2(\mathbb{R}^d)$ to Σ. H_1 is a selfadjoint semibounded operator in $L^2(\mathbb{R}^d)$, H_2 is a selfadjoint semibounded operator in $L^2(\Sigma)$.

Let ψ be an admissible function. As in the last criterion we set $\psi^{-1} = \varphi^2$. Assume now $\varphi(H_1), \varphi(H_2)$ are bounded selfadjoint integral operators in $L^2(\mathbb{R}^d)$ or $L^2(\Sigma)$, respectively. Their symmetric kernels are denoted by $\varphi_i(\cdot,\cdot)$, $i = 1,2$.

Suppose we have, that

$$\|\varphi(H_1)\| \leq \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi_1(x,y)| \, dy := a_1 \tag{3.3}$$

$$\|\varphi(H_2)\| \leq \sup_{x\in\Sigma} \int_{\Sigma} |\varphi_2(x,y)| \, dy =: a_2 \tag{3.4}$$

and assume $a = \max\{a_1, a_2\} < \infty$.

Assume additionally that $\varphi_i(\cdot,\cdot)$ are Carleman kernels, i. e.

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi_1(x,y)|^2 dy := b_1 \tag{3.5}$$

$$\sup_{x\in\Sigma} \int_{\Sigma} |\varphi_2(x,y)|^2 dy := b_2 \tag{3.6}$$

with $b := \max\{b_1, b_2\} < \infty$.

Define

$$D := \varphi^2(H_2)J - J\varphi^2(H_1); \tag{3.7}$$

this is an integral operator from $L^2(\mathbb{R}^d)$ to $L^2(\Sigma)$. Its kernel is denoted by $D(x,y)$, $x \in \Sigma$, $y \in \mathbb{R}^d$.

Then $\varphi^2(H_2)D\varphi^2(H_1)$ is a trace class operator, i. e.

$$\varphi^2(H_2)D\varphi^2(H_1) \in \mathfrak{B}_1(L^2(\mathbb{R}^d), L^2(\Sigma)) \tag{3.8}$$

(see (2.22)), if

$$\int_\Sigma dx \int_{\mathbb{R}^d} dy\ |D(x,y)| < \infty, \tag{3.9}$$

and the trace norm can be estimated by

$$\|\varphi^2(H_2)D\varphi^2(H_1)\|_{\mathrm{tr}} \leq a^2 \cdot b \cdot \int_\Sigma dx \int_{\mathbb{R}^d} dy\ |D(x,y)|. \tag{3.10}$$

REMARK. Concerning the situation in Example 2.5 (3.9) corresponds to (2.16) and is thus an improvement for any condition of the form

$$\int_\Sigma dx \int_{\mathbb{R}^d} \langle x\rangle^\alpha\ |D(x,y)|^2 < \infty, \tag{3.11}$$

$\alpha > 0$.

PROOF. The proof is based on a trace class lemma given by van Casteren, Demuth, Stollmann, Stolz in [4],. If A and B are integral operators with measurable kernels $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, their product AB is trace class if

$$\int \|A(\cdot,x)\|\ \|B(x,\cdot)\|\ dx < \infty,$$

and the trace norm can be estimated by

$$\|AB\|_{\mathrm{tr}} \leq \int \|A(\cdot,x)\|\ \|B(x,\cdot)\|\ dx.$$

This is used for $\varphi^2(H_2)D\varphi^2(H_1)$. We have

$$\begin{aligned}
\|\varphi^2(H_2)D\varphi^2(H_1)\|_{\mathrm{tr}} &\leq \|\varphi^2(H_2)D\varphi(H_1)\|_{\mathrm{tr}}\,\|\varphi(H_1)\| \\
&\leq a \cdot \int_\Sigma dx\Big(\int_\Sigma dy\ |\varphi_2(y,x)|^2\Big)^{\frac{1}{2}} \cdot \\
&\quad \cdot\Big(\int_{\mathbb{R}^d} dy\Big|\int_\Sigma du \int_{\mathbb{R}^d} dv\ \varphi_2(x,u)D(u,v)\varphi_1(v,y)\Big|^2\Big)^{\frac{1}{2}} \\
&\leq a\cdot b^{\frac{1}{2}} \int_\Sigma dx\Big(\int_{\mathbb{R}^d} dy \int_\Sigma du_1 \int_{\mathbb{R}^d} dv_1\ \varphi_2(x,u_1)D(u_1,v_1)\varphi_1(v_1,y)\cdot \\
&\quad \cdot \int_\Sigma du_2 \int_{\mathbb{R}^d} dv_2\ \varphi_2(x,u_2)D(u_2,v_2)\varphi_1(v_2,y)\Big)^{\frac{1}{2}} \\
&\leq a\cdot b \int_\Sigma dx \int_\Sigma du \int_{\mathbb{R}^d} dv\ \varphi_2(x,u)\,|D(u,v)| \\
&\leq a^2\cdot b \int_\Sigma du \int_{\mathbb{R}^d} dv\,|D(u,v)|.
\end{aligned}$$

q. e. d.

The next two Lemmas refer to the other conditions in Corollary 2.7.

LEMMA 3.2. *Take the assumptions of Criterion 3.1. Then D is a compact operator, i. e.*

$$D = \varphi^2(H_2)J - J\varphi^2(H_1) \in \mathfrak{B}_\infty(L^2(\mathbb{R}^d), L^2(\Sigma)) \tag{3.12}$$

(see (2.23)), if

$$\int_\Sigma dx \int_{\mathbb{R}^d} dy \; |D(x,y)| < \infty. \tag{3.13}$$

PROOF. If (3.13) is satisfied D is even a Hilbert-Schmidt operator. This follows easily by the boundness of the kernel, i. e. by

$$|D(x,y)| = \left| \int_\Sigma \varphi_2(x,u)\varphi_2(u,y)\, du\, \chi_\Sigma(y) - \chi_\Sigma(x) \int_{\mathbb{R}^d} \varphi_1(x,u)\varphi_1(u,y)\; du \right| \leq 2 \cdot b.$$

q. e. d.

Finally, in our special integral operator setting we have

$$JJ^* = \mathbb{1}_{L^2(\Sigma)} = \mathbb{1}_{\mathfrak{H}_2}$$

such that (2.7) is trivially satisfied and (2.25) is superfluous.

It remains to verify (2.24).

LEMMA 3.3. *Assume the same conditions as in Criterion 3.3. Then $(J^*J - \mathbb{1}_{\mathfrak{H}_1})\varphi^2(H_1)$ is a Hilbert-Schmidt operator if the measure $|\mathbb{R}^d \backslash \Sigma|$ is finite.*

PROOF. We habe

$$\big((J^*J - \mathbb{1}_{L^2(\mathbb{R}^d)})f\big)(x) = \chi_{\mathbb{R}^d\backslash\Sigma}(x) f(x).$$

Hence the Hilbert-Schmidt norm of $(J^*J - \mathbb{1}_{\mathfrak{H}_1})\varphi^2(H_1)$ is the square root of

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \left| \chi_{\mathbb{R}^d\backslash\Sigma}(x) \int_{\mathbb{R}^d} \varphi_1(x,u)\varphi_1(u,y)\; du \right|^2$$

$$\leq \int_{\mathbb{R}^d\backslash\Sigma} dx \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} du \int_{\mathbb{R}^d} dv \; |\varphi_1(x,u)\varphi_1(u,y)\varphi_1(x,v)\varphi_1(v,y)|$$

$$\leq \int_{\mathbb{R}^d\backslash\Sigma} dx \int_{\mathbb{R}^d} du \int_{\mathbb{R}^d} dv \; |\varphi_1(x,u)|\,|\varphi_1(x,v)| \Big(\int dy \; |\varphi_1(u,y)|^2 \Big)^{\frac{1}{2}} \Big(\int dy \; |\varphi_1(v,y)|^2 \Big)^{\frac{1}{2}}$$

$$\leq b \cdot a^2 \, |\mathbb{R}^d \backslash \Sigma|. \qquad (|\cdot| \text{ - Lebesgue-measure})$$

q. e. d.

The last two Lemmas and Criterion 3.1 may be summarized.

Corollary 3.4. *Assume the same conditions as in Criterion 3.1. In brief these were: H_1, H_2 selfadjoint semibounded operators in $L^2(\mathbb{R}^d), L^2(\Sigma)$. $Jf = f/_\Sigma$. $\varphi^2(\cdot)$ the inverse of an admissible function. $\varphi(H_i)$ selfadjoint bounded, Carleman integral operators. $D = \varphi^2(H_2)J - J\varphi^2(H_1)$. Then*

(a) *One-space formulation.*
If $\Sigma = \mathbb{R}^d$, i. e. $\mathfrak{H}_1 = \mathfrak{H}_2 = L^2(\mathbb{R}^d)$, i. e. $J = \mathbb{1}$, the scattering system $\{H_2, H_1\}$ is complete if

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, |D(x,y)| < \infty.$$

(b) *Two-space formulation.*
The wave operators for the scattering system $\{H_2, J, H_1\}$ exist, if

$$\int_{\Sigma} dx \int_{\mathbb{R}^d} dy\, |D(x,y)| < \infty.$$

And they are complete if additionally

$$|\mathbb{R}^d \backslash \Sigma| < \infty.$$

PROBLEM. The completeness of scattering systems implies the invariance of the absolutely continuous spectra. Here $\sigma_{\mathrm{ac}}(H_1) = \sigma_{\mathrm{ac}}(H_2)$ if

$$\int dx \int dy\, |D(x,y)| < \infty.$$

On the other hand one knows that the essential spectra are stable, i. e. $\sigma_{\mathrm{ess}}(H_1) = \sigma_{\mathrm{ess}}(H_2)$, if

$$\int dx \int dy\, |D(x,y)|^2 < \infty.$$

A possible problem is to find similar integral conditions to study the behaviour of the singularly continuous spectra. This may be possible, for instance, if the absolutely continuous spectra are empty.

4 Applications

In the following applications our results are used for semigroups and resolvents, i. e. for $\varphi(\mu) = e^{-\mu}$ or $\varphi(\mu) = \frac{1}{\mu+a}$, $\mu > -c, c > 0, a > c$. In such cases one has to verify

$$\int_{\Sigma} dx \int_{\mathbb{R}^d} dy\, \left|(e^{-2H_2}J - Je^{-2H_1})(x,y)\right| < \infty \tag{4.1}$$

or

$$\int\limits_{\Sigma} dx \int\limits_{\mathbb{R}^d} dy \left|((H_2+a)^{-2}J - J(H_1+a)^{-2})(x,y)\right| < \infty. \tag{4.2}$$

Via the Laplace transform (4.2) is related to (4.1).

If we assume that e^{-tH_1} and e^{-tH_2} are L_1-L_∞ smoothing, then the semigroups possess a kernel, denoted by $(e^{-tH_i})(x,y)$, satisfying

$$\sup_{x,y}(e^{-tH_i})(x,y) \le c_t \tag{4.3}$$

where c_t has typically the form $t^{-\alpha}e^{\omega t}$ with positive constants α, ω.

The semigroup kernels have to satisfy (3.3) and (3.4), i. e.

$$\sup_x \int\limits_{\mathbb{R}^d \text{ or } \Sigma} \left|e^{-tH_i}(x,y)\right| dy \le a < \infty. \tag{4.4}$$

If (4.3) and (4.4) are given these are Carleman kernels such that (3.5) and (3.6) are fulfilled.

This means that for L^1-L^∞ smoothing semigroups the generators form a complete scattering system if (4.4) and (4.1) are true. Till now we have not assumed any relationship between H_1 and H_2, except the condition in (4.1). H_2 can be a perturbartion of H_1 in zero order, i. e. a potential perturbation or a perturbation arising from the imposition of boundary conditions, or H_2 can also be a perturbation of H_1 of first or second order. We give here four examples for each of these situations. They are known in the literature. Hence we shall largely omit the proofs. Our purpose is simply to illustrate the results of Corollary 3.4.

4.1 Potential perturbations

Let V be a realvalued function such that $H_2 = H_1 + V$ is a selfadjoint operator in $L^2(\mathbb{R}^d)$. Here $\mathfrak{H}_1 = \mathfrak{H}_2 = L^2(\mathbb{R}^d)$. Assume that H_1 and H_2 generate L^1-L^∞ smoothing strongly continuous semigroups. Let

$$\sup_x \int \left|(e^{-tH_1})(x,y)\right| dy \le a \tag{4.5}$$

where a is independent of t. Assume

$$\left|(e^{-t(H_1+V)})(x,y)\right| \le ce^{ct}\left|(e^{-tH_1})(x,y)\right|^\beta \tag{4.6}$$

with $c > 0, \beta > 0$. Using the formula of Duhamel the condition in (4.1) is

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \left|(e^{-2(H_1+V)})(x,y) - (e^{-2H_1})(x,y)\right| \\
&\leq \int_0^2 ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} du \left|(e^{-(2-s)(H_1+V)})(x,u)\right| |V(u)| \left|(e^{-sH_1})(u,y)\right| \\
&\leq a^2 \cdot c \cdot e^{2c} \int_{\mathbb{R}^d} |V(u)| du. \qquad (4.7)
\end{aligned}$$

This means that $\{H_1, H_1 + V\}$ forms a complete scattering system if $V \in L^1(\mathbb{R}^d)$ and if $H_1 + V$ generates a selfadjoint semigroup the kernel of which satisfies (4.6). In order to verify (4.6) one can use the Feynman-Kac-formula (see Demuth, van Casteren [7]) or in a more abstract operator theoretical way the Trotter-product formula (see Arendt, Demuth [1]).

4.2 Higher order perturbations

In a similar way one can study perturbations of higher order (see Demuth, Ouhabaz [9] for details). Let H_1 be the selfadjoint realization of the Laplacian in $L^2(\mathbb{R}^d)$. Let H_2 be an operator of a system with a magnetic field. H_2 is given via sesquilinear forms, i. e. H_2 is the operator associated to the form

$$-\sum_{k,j=1}^{d} \left(a_{kj}(D_k - ib_k)u, (D_j - ib_j)v\right) \qquad (4.9)$$

with u, v from the corresponding form domain, $D_k := \frac{\partial}{\partial x_k}$. The coefficients have to satisfy

(i) $a_{kj} = a_{jk}$, a_{kj} real valued, bounded,
$\sum_{k,j=1}^{d} a_{kj}(x)\xi_k\overline{\xi}_j \geq \alpha|\xi|^2, \alpha > 0$ for all $x \in \mathbb{R}^d$, $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{C}^d$.
$a_{kj}(\cdot) - \delta_{kj} \in L^1(\mathbb{R}^d)$ for all $k, j \in \{1, 2, \ldots, d\}$.

(ii) b_j are realvalued, bounded, $b_j \in L^1(\mathbb{R}^d)$, $j = 1, 2, \ldots, d$.

In this case $\{H_1, H_2\}$ forms a complete scattering system. In particular, this is true for non-regular magnetic potentials. The proof uses Corollary 3.4 for $\varphi(\mu) = (\mu + a)^{-m}$, i. e. for powers of resolvents. The details are given in [9]. Typical are the L^1-conditions for $a_{kj} - \delta_{kj}$ and b_j, which have the same background as the L^1-condition for the potential in section 4.1.

4.3 Perturbations by boundary conditions

First let us consider pure obstacle scattering where no potentials are present. Let K_1 be a selfadjoint, positive generator of a strongly continuous semigroup in $L^2(\mathbb{R}^d)$. Assume that this semigroup is an integral operator with the kernel $(e^{-tK_1})(x,y) = \varphi_1(t,x,y)$. Let φ_1 be a continuous function mapping $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ to $[0,\infty)$. Assume

$$\int_{\mathbb{R}^d} \varphi_1(t,x,y)\, dy = 1 \tag{4.10}$$

and

$$\sup_{x,y\in\mathbb{R}^d} \varphi_1(t,x,y) < \infty. \tag{4.11}$$

Denote by $C_\infty(\mathbb{R}^d)$ the set of continuous functions vanishing at infinity. For $f \in C_\infty(\mathbb{R}^d)$ we suppose

$$\lim_{t\to 0}(e^{-tK_1}f)(x) = f(x) \tag{4.12}$$

and

$$e^{-tK_1}C_\infty \subseteq C_\infty. \tag{4.13}$$

(4.12) and (4.13), the Feller property, ensures that φ_1 generates a strong Markov process, denoted by $((\Omega,\mathcal{F},P_x),X(\cdot),(\mathbb{R}^d,\mathfrak{B}^d))$, having the generator K_1. The interested reader may find details in [7] and [8].

$E_x\{\cdot\}$ denotes the exspectations with respect to the Feller measure P_x. The conditional Feller measure $E_x^{y,t}(\cdot)$ pins the motion $\{X(t), t \geq 0\}$ at x at time 0 and at y at time t. It can be defined by

$$E_x^{y,t}(A) := E_x\{\varphi_1(t-s,X(s),y)\mathbb{1}_A\} \tag{4.14}$$

whenever A is an event in the field $\mathcal{F}_s$, $s < t$. Clearly, we have that

$$E_x^{y,t}(\mathbb{R}^d) = \varphi_1(t,x,y). \tag{4.15}$$

Obstacles can be introduced stochastically. Let Γ be a closed subset of $\mathbb{R}^d$ with positive and finite Lebesgue measure. Its first hitting time is given by

$$T_\Gamma := \inf\{s > 0, X(s) \in \Gamma\}. \tag{4.16}$$

Setting $\Sigma = \mathbb{R}^d\backslash\Gamma$ we can introduce

$$(U(t)f)(x) := E_x\{f(X(t)), T_\Gamma \geq t\}. \tag{4.17}$$

$U(t)/_{L^2(\Sigma)}$ forms a strongly continuous semigroup in $L^2(\Sigma)$. Its generator is now K_2. K_2 corresponds to K_1 together with Dirichlet boundary conditions on $\partial\Sigma = \partial\Gamma$. K_2 is selfadjoint in $L^2(\Sigma)$. Its semigroup has the kernel

$$(e^{-tK_2})(x,y) = \varphi_2(t,x,y) = E_x^{y,t}\{T_\Gamma > t\}. \tag{4.18}$$

The kernels φ_1 and φ_2 satisfy all the conditions of Corollary 3.4(b). Note that $|\Gamma| < \infty$ was already assumed.

Hence $\{K_2, J, K_1\}$ forms a complete scattering system if the following integral is finite:

$$\int_\Sigma dx \int_{\mathbb{R}^d} dy\, \left|E_x^{y,t}(\mathbb{R}^d) - E_x^{y,t}\{T_\Gamma \geq t\}\right| = \int_\Sigma dx\; E_x\{T_\Gamma < t\}$$

This can be estimated by:

$$\begin{aligned}\int_{\Sigma} dx\; E_x\{T_\Gamma < 1\} &\le e^t \int_{\mathbb{R}^d} dx E_x\{e^{-T_\Gamma}, T_\Gamma < \infty\} \\ &\le e^t \operatorname{cap}(\Gamma). \end{aligned} \tag{4.19}$$

In (4.19) cap(Γ) means the capacity of Γ which is defined as

$$\operatorname{cap}(\Gamma) := \inf_{f,O} \Big\{ (K_1^{1/2} f, K_1^{1/2} f) + (f,f), f \in \operatorname{dom} K_1^{1/2}, f \ge \chi_O, \\ O \text{ open}, \; \Gamma \subset O \Big\}. \tag{4.20}$$

The conclusion is that $\{K_1, J, K_2\}$ forms a complete scattering system if the capacity of Γ is finite.

A similar result will be obtained in obstacle scattering if the potentials are not neglected. Let K_1 be a free Feller operator as described above such that (4.10)-(4.13) is satisfied. Assume a Kato-Feller potential V, such that the form sum $K_1 \dot{+} V$ is a well-defined selfadjoint operator generating a L^1-L^∞ smoothing semigroup. Assume that

$$\big(e^{-t(K_1 \dot{+} V)}\big)(x,y) \le c e^{ct} \varphi_1(\beta t, x, y) \tag{4.21}$$

for $c > 0, \beta > 0$, (compare this with (4.6)). Now, the family

$$\big(W(t)f\big)(x) := E_x\Big\{ e^{-\int_0^t V(X(s))\,ds} f(X(t)), T_\Gamma > t \Big\}. \tag{4.22}$$

restricted to $L^2(\Sigma)$ forms again a strongly continuous semigroup in $L^2(\Sigma)$ as in the unperturbed case (see (4.17)). Its generator is know K_2. One can give conditions such that K_2 is exactly the Friedrichs extension of $K_1 \dot{+} V$ restricted to $\operatorname{dom}(K_1 \dot{+} V) \cap L^2(\Sigma)$.

The kernel of e^{-tK_2} is here

$$\varphi_2(t,x,y) = E_x^{y,t}\Big\{ e^{-\int_0^t V(X(s))\,ds}, T_\Gamma > t \Big\}. \tag{4.23}$$

Define the $(\lambda + V)$-harmonic function by

$$h_{\lambda+V}(x) := E_x\Big\{ e^{-\int_0^{T_\Gamma} (\lambda + V(X(s))\,ds}, T_\Gamma < \infty \Big\} \tag{4.24}$$

the scattering system with potential V and obstacle on Γ is then complete if

$$\int_{\mathbb{R}^d} dx\; h_{\lambda+V}(x) < \infty. \tag{4.25}$$

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