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Mathematical Theory of N-Body Quantum Systems [1]

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Abstract. A short history of the subject is given at the end of the paper. The main part of the notes describes a new proof of asymptotic completeness for short-range forces, based on joint work with I.M. Sigal [21].

1 N-Body Systems

A system of N particles in $\mathbb{R}^3$ with pair-interactions is described by the Hamiltonian

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2m_k} + \sum_{i<k}^{1\ldots N} V_{ik}\,(x_i - x_k)\,, \tag{1}$$

with $V_{ik}(x) \to 0$ as $|x| \to \infty$. From this standard case we extract the following basic notions:

Configuration Space

X is a Euclidean space with salar product $x \cdot y$. In the case (1):

$$\begin{aligned} X &= \{x = (x_1 \ldots x_N)\,|\, x_k \in \mathbb{R}^3\,; \textstyle\sum m_k x_k = 0\}\,; \\ x \cdot y &= \textstyle\sum m_k (x_k \cdot y_k)_{\mathbb{R}^3}\,. \end{aligned} \tag{2}$$

[1] Lecture given at the Ascona Conference on Mathematical Results in Quantum Mechanics, June 1996.

$\frac{1}{2}\dot{x}\cdot\dot{x} = \frac{1}{2}\dot{x}^2$ is the classical kinetic energy, $p = \dot{x}$ the momentum conjugate to x. In quantum mechanics,

$$H = \frac{1}{2}p^2 + V(x) \quad \text{on} \quad L^2(X)\,, \tag{3}$$

where $p = -i\nabla$ and $p^2 = -\Delta$ have the usual form in cartesian coordinates (not particle coordinates) of X.

Channels

In X there is a distinguished, finite lattice L of subspaces $a, b, \ldots$ (channels). L is closed under intersections and contains at least $a = \{0\}$ and $a = X$. In the case (1) the channels correspond to all partitions of $(1 \ldots N)$ into clusters. For example if $N = 4$:

$$\begin{array}{ccc} \text{partition} & & \text{channel} \\ (12)(34) & \longleftrightarrow & a = \{x | x_1 = x_2\,;\, x_3 = x_4\}\,. \end{array} \tag{4}$$

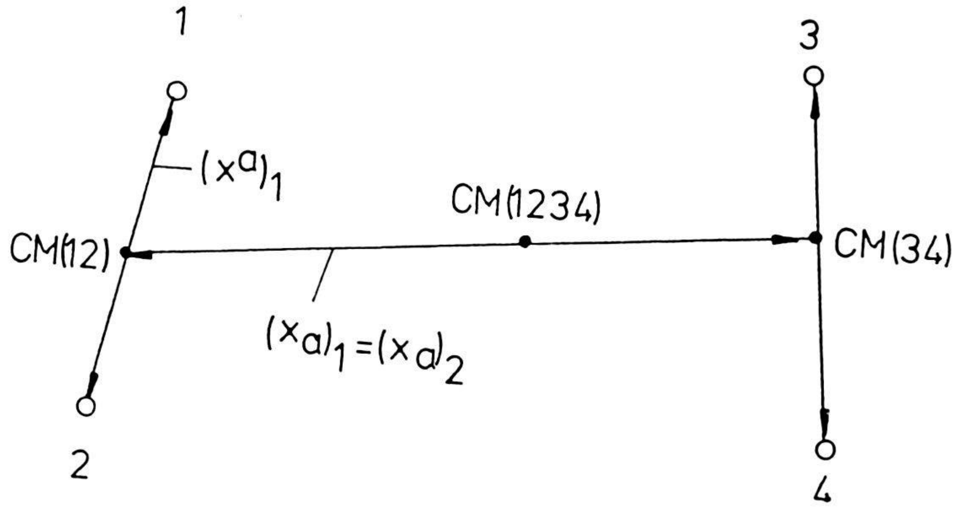
In general the partial ordering of L is defined by

$$a < b \longleftrightarrow a \subset b\;; a \neq b\,. \tag{5}$$

For each $a \in L$ there is an orthogonal decomposition:

$$X = a \oplus a^{\perp}\;:\; x = x_a + x^a\,. \tag{6}$$

This corresponds to the introduction of CM (center of mass) coordinates, e.g. in the example (4):



(Fig. 1)

The relation

$$\frac{1}{2}p^2 = \frac{1}{2}(p_a)^2 + \frac{1}{2}(p^a)^2$$

expresses the familiar decomposition of the kinetic energy into CM-parts and internal parts with respect to the clusters.

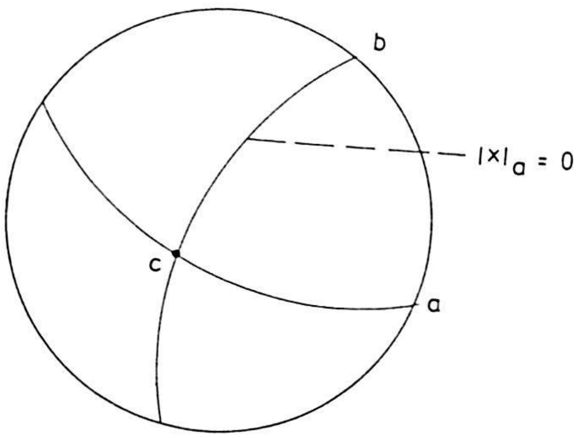
Intercluster Distance

The basic feature of N-body systems is that they can split into widely separated, almost independent clusters. As a measure of the separation we might use the minimal distance $d_a(x)$ in R^3 of the clusters, e.g.

$$d_a(x) = \min_{i \in (12);\, k \in (34)} |x_i - x_k| \tag{7}$$

in the example (4). However, it is more consistent (and more convenient) to express the separation in terms of the geometry of X. Some reflection shows that

$$d_a(x) = 0 \longleftrightarrow x \in b; \quad b \cap a < a .$$



(Fig. 2)

Fig. 2 shows the unit sphere in X, intersected by two channels a, b with $b \cap a = c < a$. This leads to the definition of the *intercluster distance*

$$|x|_a = \min_{b \cap a < a} |x^b|; \quad a > \{0\} . \tag{8}$$

For the example (4) one finds

$$|x|_a = \min_{i \in (12); k \in (34)} \left(\frac{m_i m_k}{m_i + m_k} \right)^{1/2} |x_i - x_k| .$$

Hamiltonians

For each $a > \{0\}$ the potential $V(x)$ has a unique decomposition

$$V(x) = V^a(x^a) + I_a(x); \tag{9}$$

$$I_a(x) \to 0 \quad \text{as} \quad |x|_a \to \infty , \tag{10}$$

in particular: $I_a = V$ for $a = X$. For $a = \{0\}$ we define $I_a = 0$. In the example (4):

$$V^a = V_{12} + V_{34}\,; \quad I_a = V_{13} + V_{14} + V_{23} + V_{24}\,.$$

Corresponding to $L^2(X) = L^2(a) \otimes L^2(a^\perp)$ we write:

$$\begin{aligned} H &= H_a + I_a\,; \\ H_a &= \frac{1}{2}(p_a)^2 \otimes 1 + 1 \otimes H^a\,; \\ H^a &= \frac{1}{2}(p^a)^2 + V^a(x^a) \quad \text{on} \quad L^2(a^\perp)\,. \end{aligned} \tag{11}$$

H_a describes the dynamics of a system of non-interacting clusters.

Conditions on $V(x)$

Some global conditions on $V(x)$ are required to make H (and in fact all H^a) selfadjoint on convenient domains and bounded from below. An essential postulate is that p^2 is bounded (or form-bounded) relative to H. Since this is amply covered in the literature on Schrödinger operators [30] we will not, as a rule, state such conditions in our theorems. For readers not familiar with the subject we mention that in the case (1) it suffices that $V_{ik}(\cdot) \in L^2_{loc}(\mathbb{R}^3)$; $V_{ik}(x) \to 0$ as $|x| \to \infty$ [23]. All further assumptions on $V(x)$ will only concern the behavior of $I_a(x)$ as $|x|_a \to \infty$. These conditions will be stated explicitly.

Induction Principle

As a result we now have a definition of N-body systems involving only 3 elements:

- A configuration space X
- A lattice L of channels (12)
- Conditions on $I_a(x)$.

In this sense each Hamiltonian H^a also describes a N-body system with reduced configuration space $a^\perp$, with channels $b \cap a^\perp$, $b \geq a$, and with corresponding intercluster potentials $I_b(x^a)$. Any proposition P derived from (12) can therefore be established by induction on the lattice L. To begin with, P is checked in the trivial case $a = X\, :\, H^a = 0$ on $L^2(\{0\}) = \mathbb{C}$. Then P is proved for $a = \{0\}\, :\, H^a = H$, under the induction hypothesis that P holds for any H^a with $a > \{0\}$.

2 Asymptotic Completeness

In the case of short-range potentials

$$I_a(x) = O\left(|x|_a^{-\mu}\right); \quad \mu > 1 \quad (|x|_a \to \infty) \tag{13}$$

we define outgoing scattering states ψ by the asymptotic condition

$$e^{iHt}\psi \underset{\|\,\|}{\longrightarrow} \sum_{a\in L} e^{-iH_a t}\varphi_a \quad (t\to\infty); \quad \varphi_a \in L^2(a)\otimes\mathcal{H}_B(H^a). \tag{14}$$

Here $\mathcal{H}_B(H^a)$ is the subspace spanned by the eigenvectors of H^a. Each term in the sum (14) describes a motion of non-interacting, bound clusters. We note that (14) holds trivially for $\psi \in \mathcal{H}_B(H)$ with:

$$\varphi_{\{0\}} = \psi; \quad \varphi_a = 0 \quad \text{for} \quad a > \{0\}.$$

The *existence* of scattering states for given $\{\varphi_a\}$ is well known [30]. Our task is to prove *completeness*:

Theorem 1 (Asymptotic Completeness). *Suppose, in addition to (13), that*

$$\nabla I_a(x) = O(|x|_a^{-\mu}); \quad \mu > 1. \tag{15}$$

Then every $\psi \in L^2(X)$ is a scattering state in the sense of (14).

A proof of this result is given in the following sections. Since the case of $\psi \in \mathcal{H}_B(H)$ is trivial, and since the subspace of scattering states is known to be closed, it suffices to prove that

$$e^{-iHt}\psi \underset{\|\,\|}{\longrightarrow} \sum_{a>\{0\}} e^{-iH_a t}\varphi_a \tag{16}$$

for a set of ψ which is dense in the continuous spectral subspace $\mathcal{H}_C(H) = \mathcal{H}_B(H)^\perp$ of H. We will first prove the weaker statement that (16) is valid for *some* $\varphi_a \in L^2(X)$: this is called *asymptotic clustering*. Then we invoke the induction hypothesis that asymptotic completeness holds for the systems described by H^a, $a > \{0\}$. This can be written as

$$e^{-iH_a t}\varphi_a \underset{\|\,\|}{\longrightarrow} \sum_{b\geq a} e^{-iH_b t}\varphi_{ab}; \quad \varphi_{ab} \in L^2(b)\otimes\mathcal{H}_B(H^b),$$

which is trivially satisfied for $a = X$. Inserting this into (16) gives

$$e^{-iHt}\psi \underset{\|\,\|}{\longrightarrow} \sum_{b\geq 0} e^{-iH_b t} \sum_{\{0\}<a\leq b} \varphi_{ab},$$

i.e. asymptotic completeness for H.

3 Yafaev Functions and the Basic Propagation Estimate

Propagation Observables

The propagation of $\psi_t = \exp(-iHt)\psi$ in phase space can be described in terms of expectation values

$$\langle\phi_t\rangle_t = (\psi_t, \phi_t\psi_t) \tag{17}$$

of suitable (generally time dependent) observables $\phi_t(x,p)$. From

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi_t\rangle_t = \langle D_t\phi_t\rangle_t\,; \quad D_t\phi_t = i[H,\phi_t] + \partial_t\phi_t \tag{18}$$

and from estimates of $D_t\phi_t$ we can deduce growth properties of $\langle\phi_t\rangle_t$ as $t \to \infty$. Usually this analysis is restricted to finite energy shells $\Delta \subset \mathbb{R}$, i.e. to spectral subspaces

$$\mathcal{H}_\Delta(H) = \mathrm{Ran}(E_\Delta(H))\,, \tag{19}$$

where $E_\Delta(H)$ is the spectral projection of H corresponding to Δ. As a first example we discuss Mourre's inequality, which is basic for our proof of Theorem 1. Let $S \subset \mathbb{R}$ be the set of thresholds and eigenvalues of H, i.e.

$$S = \bigcup_{a\in L}\{\text{eigenvalues of } H^a\}\,. \tag{20}$$

By Mourre's Theorem [6] S is closed and countable. Since S contains the eigenvalues of $H^{\{0\}} = H$ it follows that

$$\mathcal{H}_S(H) = \mathcal{H}_B(H)\,.$$

Therefore

$$\mathcal{H}_{R\setminus S}(H) = \mathcal{H}_C(H) \tag{21}$$

is the continuous spectral subspace of H. Also part of Mourre's Theorem is the following inequality. Let $E \in R \setminus S$ be in the continuous spectrum $\sigma_C(H)$. Then there exists an open interval $\Delta \ni E$ and a strictly positive Θ such that

$$E_\Delta(H) i[H,A] E_\Delta(H) \geq \Theta E_\Delta(H)\,, \tag{22}$$

where

$$A = \frac{1}{2}D_t x^2 = \frac{1}{2}(p\cdot x + x\cdot p)\,; \tag{23}$$

$$i[H,A] = D_t A = p^2 - x\cdot\nabla V(x)\,. \tag{24}$$

Therefore Mourre's inequality (22) implies

$$\langle x^2\rangle_t \geq \Theta t^2 + O(t) \quad (t\to\infty)$$

for a dense set of states in $\mathcal{H}_\Delta(H)$. It is evident from (24) that this result rests on some global conditions on the forces $\nabla V(x)$. However, there is a variant of Mourre's Theorem [17, 36] which involves only the tails of the forces at large distances:

Lemma 2 (Mourre's inequality for x^2) *Suppose that for all* $a > \{0\}$

$$\lim_{|x|_a \to \infty} x \cdot \nabla I_a(x) = 0 . \tag{25}$$

Let $E \in \sigma_C(H)$, $E \notin S$. *Then there is an open interval* $\Delta \ni E$ *and a strictly positive* Θ *such that*

$$\langle x^2 \rangle_t \geq \Theta t^2 + O(t) \quad (t \to \infty) \tag{26}$$

for all $\psi \in \mathcal{H}_\Delta(H) \cap D(|x|)$.

Remark. The states ψ of this type (for all possible E) span a dense set in the continuous spectral subspace $\mathcal{H}_C(H)$. Therefore it will be sufficient to derive (16) from the much weaker statement (26). This requires the construction of more sophisticated propagation observables which are specially adapted to the lattice L of channels.

Yafaev-Functions

Following Yafaev [38] we construct a function g on X whose properties are summarized in Lemma 3 below. Let σ be a positive, decreasing function on L:

$$\sigma_{\{0\}} > \sigma_a > \sigma_b > \sigma_X = 1 \tag{27}$$

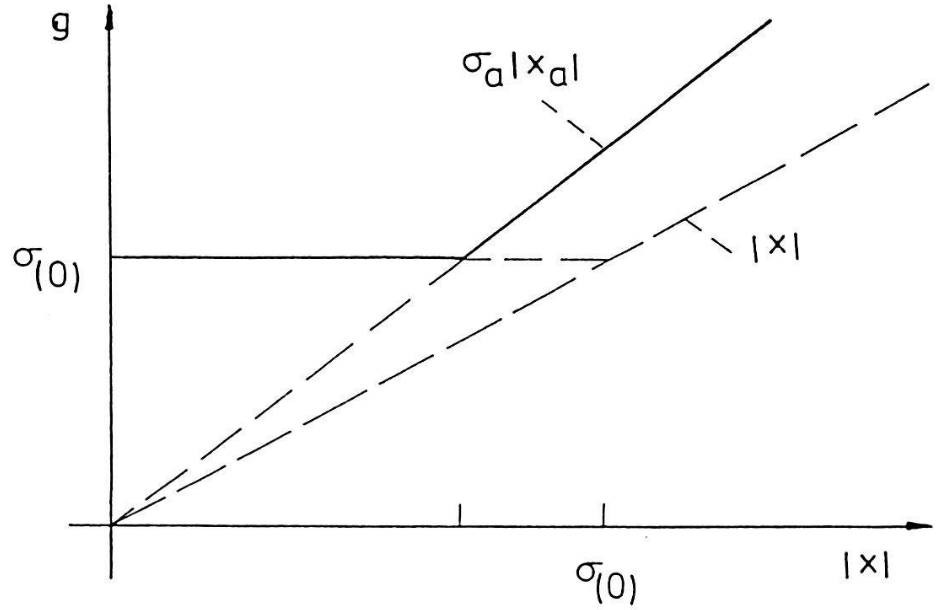
for $\{0\} < a < b < X$, to be adjusted in the course of the construction. Let

$$f_a(x) = \begin{cases} \sigma_{\{0\}} & (a = \{0\}); \\ \sigma_a |x_a| & (a > \{0\}). \end{cases}$$

Then the prototype of $g(x)$ is given by

$$g(x, \sigma) = \max_{a \in L} f_a(x) . \tag{28}$$

A radial section of $g(x, \sigma)$ is shown in Fig. 3 for a direction $x \in a$.



(Fig. 3)

$g(x,\sigma)$ is convex, constant on some compact set containing the ball $|x| < 1$, and homogeneous of degree 1 in the complement of this set. We decompose $g(x,\sigma)$ into maximal pieces:

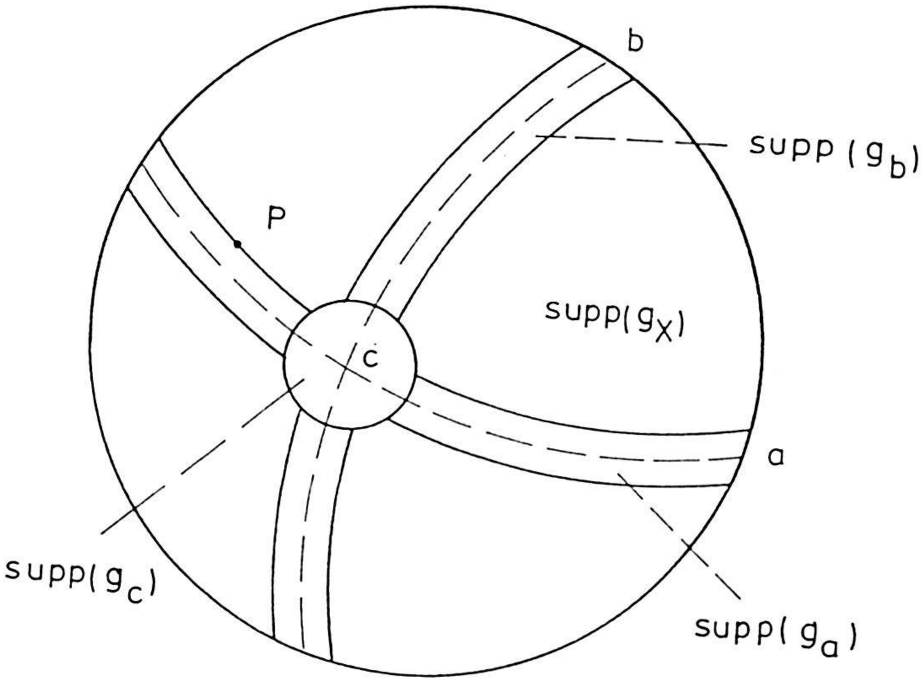
$$g(x,\sigma) = \sum_{a\in L} g_a(x,\sigma)\,; \quad g_a(x,\sigma) = \begin{cases} f_a(x) & \text{if } f_a(x) = g(x,\sigma)\,; \\ 0 & \text{otherwise}\,. \end{cases} \tag{29}$$

The piece $g_{\{0\}}(x,\sigma)$ has compact support on which it is constant. The pieces $g_a(x,\sigma)$ for $a > \{0\}$ are homogeneous of degree 1 on conical supports whose intersection with a sphere $|x| = R \geq \sigma_{\{0\}}$ is shown in Fig. 4. This figure corresponds to Fig. 2 and serves to explain the choice of σ. Suppose first that $\sigma_a = \sigma_b = \sigma_c = 1$. Then Fig. 4 reduces to Fig. 2 since $\sigma_a|x_a| = |x|$ exactly if $x \in a$, etc. We now increase σ_a, σ_b by arbitrary small amounts. Then the supports of g_a, g_b broaden into narrow belts shown in Fig. 4. Then we increase σ_c to $\sigma_c > \sigma_a, \sigma_b$, so that supp ($g_c$ grows to a disc covering the intersection of the two belts. This indicates the general construction scheme for the function σ on L which can be carried out analytically [20, 38]. Fig. 4, together with the definition (8) of the intercluster distance, suggests what can be achieved: There is a (largely arbitrary) choice of σ such that

$$|x|_a > \lambda|x| \quad \text{on} \quad \text{supp}(g_a) \tag{30}$$

for some $\lambda > 0$. Moreover, since $g_a(x,\sigma)$ is, on its support, a function of x_a,

$$\nabla g(x,\sigma) \in a \quad \text{on} \quad \text{supp}(g_a) \tag{31}$$



(Fig. 4)

except at boundary points, where $\nabla g(x,\sigma)$ is discontinious. This discontinuity is removed by a regularization $g(x,\sigma) \to g(x)$ which preserves convexity:

$$g(x) = \int g(x,\mu) \prod_{a \in L} \delta(\mu_a - \sigma_a) d\mu_a \,, \tag{32}$$

where $0 < \delta \in C_0^\infty(R)$ is a regularization of the Dirac distribution with sufficiently narrow support. The same regularization is applied to $g_a(x,\sigma)$, so that

$$g(x) = \sum_{a \in L} g_a(x) \,. \tag{33}$$

The effect of this regularization on Fig. 4 is that the boundaries are slightly smeared, but away from these strips the functional form of $g(x)$ remains the same. For further reference we list the resulting properties of g and g_a:

Lemma 3 (Properties of g)

(i) *g is smooth, convex, and homogeneous of degree 1 outside some ball: $|x| > R_2$.*

(ii) *$g(x) = g(0)$ inside some ball: $|x| < R_1$.*

(iii) *For any $x \in$ supp(∇g) there exists $a \in L$, $a > \{0\}$, such that*

$$\nabla g(x) \in a \quad \text{and} \quad |x|_a > \lambda |x| . \tag{34}$$

To explain (iii), consider the boundary point P shown in Fig. 4. There the intercluster distances with respect to a and X are both strictly positive, and after regularization we certainly have $\nabla g(P) \in X$. The functions g_a have corresponding properties *except* convexity:

Lemma 4 (Properties of g_a)

(i) *g_a is smooth, and homogeneous of degree 1 for $|x| > R_2$.*

(ii) *$g_{\{0\}}$ has compact support in $|x| < R_2$. For $a > \{0\}$, g_a is supported in $|x| > R_1$, and $|x|_a > \lambda|x|$ on* $\mathrm{supp}(g_a)$.

(iii) *∇g_a is supported in $|x| > R_1$. For any $x \in \mathrm{supp}(\nabla g_a)$ there exists $b \in L$, $b > \{0\}$ such that*

$$\nabla g_a \in b \quad \text{and} \quad |x|_b \geq \lambda |x| \,. \tag{35}$$

The Basic Propagation Estimate

All our propagation observables are derived from

$$g_t(x) = t^\delta g(t^{-\delta} x) \,, \quad 0 < \delta < 1 \tag{36}$$

for $t > 0$. By Lemma 3 g_t is smooth and convex in x,

$$g_t(x) = \begin{cases} t^\delta g(0) & (|x| < t^\delta R_1) \,; \\ g(x) & (|x| > t^\delta R_2) \,, \end{cases}$$

and, since g has bounded derivatives

$$\begin{aligned} \partial_x^k g_t(x) &= O\left(t^{\delta(1-|k|)}\right) \\ \partial_t^k g_t(x) &= O\left(t^{\delta-k}\right) \end{aligned} \tag{37}$$

as $t \to \infty$, uniformly in x. For any x it follows from (9) that

$$\nabla g_t(x) \cdot \nabla V(x) = \nabla g_t(x) \cdot \nabla V^a(x^a) + \nabla g_t(x) \cdot \nabla I_a(x)$$

for some $a > \{0\}$ depending on x. By (34) the first term vanishes since $\nabla g_t(x) \in a$. In the second term $\nabla g_t(x)$ is bounded with support in $|x| \geq t^\delta R_1$, where $|x|_a \geq t^\delta \lambda R_1$ and therefore, by (15), $\nabla I_a(x) = 0(t^{-\delta\mu})$ as $t \to \infty$, uniformly in x. As a result,

$$\|\nabla g_t \cdot \nabla V\| \leq \text{const. } t^{-\delta\mu} \tag{38}$$

for sufficiently large t. Now we compute

$$\begin{aligned} \gamma_t &= D_t g_t = \frac{1}{2}(\nabla g_t \cdot p + p \cdot \nabla g_t) + \partial_t g_t \,; \\ D_t(\gamma_t - 2\partial_t g_t) &= p g_t'' p - \frac{1}{4}\Delta^2 g_t - \nabla g_t \cdot \nabla V - \partial_t^2 g_t \,. \end{aligned} \tag{39}$$

The first term in (39) denotes the Hessian

$$pg_t''p = \sum_{ik} p_i \frac{\partial^2 g_t}{\partial x_i \partial x_k} p_k \geq 0 \tag{40}$$

since $g_t(x)$ is convex. The following 3 terms are of order $t^{-3\delta}$, $t^{-\mu\delta}$, $t^{\delta-2}$ as $t \to \infty$, uniformly in x. Since $\mu > 1$ we can now fix δ such that these terms are integrable in t, i.e.

$$0 < \delta < 1\,,\ \delta\mu > 1\,,\ 3\,\delta > 1\,.$$

Lemma 5 (Basic Propagation Estimate)

$$\int_1^\infty \mathrm{d}t \langle p\, g_t'' p\rangle_t \ \leq\ \text{const.}\ \langle H + c\rangle_0\,, \tag{41}$$

where c is some constant to make $H + c \geq 1$.

Proof. For any state ψ in the form domain of $H + c$, (39) gives

$$\int_1^T \mathrm{d}t \langle p\, g_t'' p\rangle_t \leq \langle \gamma_t - 2\partial_t g_t\rangle_t\big|_1^T + \text{const.} \leq \text{const.}\ \langle H + c\rangle_0$$

because $|\langle \gamma_t - 2\partial_t g_t\rangle| \leq$ const. $\langle H + c\rangle_t$ uniformly in t. Since the integrand $\langle p\, g_t'' p\rangle_t$ is positive, the limit $T \to \infty$ exists. □

The Asymptotic Observable γ

Corresponding to (33) we split

$$\begin{aligned} g_t &= \sum_a g_{a,t} \quad ; \quad g_{a,t}(x) = t^\delta g_a(t^{-\delta} x)\,; \\ \gamma_t &= \sum_a \gamma_{a,t} \quad ; \quad \gamma_{a,t} = D_t(g_{a,t})\,. \end{aligned} \tag{42}$$

Lemma 6

$$\gamma := s - \lim_{t\to\infty} e^{iHt} \gamma_t\, e^{-iHt} \quad \text{and} \quad \gamma_a := s - \lim_{t\to\infty} e^{iHt} \gamma_{a,t}\, e^{-iHt}$$

exist on $D(H)$. Moreover,

$$[\gamma, H] = [\gamma_a, H] = 0\,; \tag{43}$$

$$\gamma_{\{0\}} = 0 \implies \gamma = \sum_{a > \{0\}} \gamma_a\,. \tag{44}$$

Proof.

Step 1: We first discuss γ.

$$\lim_{t\to\infty} e^{iHt}\gamma_t\, e^{-iHt}(H+c)^{-1}\psi = \lim_{t\to\infty}(H+c)^{-1/2}\, e^{iHt}\gamma_t\, e^{-iHt}(H+c)^{-1/2}\psi \equiv \lim_{t\to\infty}\varphi_t \tag{45}$$

if this limit exists. This follows by expressing $(H+c)^{-1/2}$ in terms of the resolvent $(z-H)^{-1}$ (e.g. using a contour integral), and then from the fact that $[\gamma_t,(z-H)^{-1}]\to 0$ in norm as $t\to\infty$.

Step 2: In φ_t we can replace

$$\gamma_t \longrightarrow \gamma_t - 2\partial_t g_t \equiv \tilde{\gamma}_t$$

since $\partial_t g_t \sim t^{\delta-1}$ $(t\to\infty)$. Then

$$\begin{aligned}\gamma(H+c)^{-1}\psi &= \varphi_1 + \int_1^\infty \mathrm{d}t\,\partial_t\varphi_t\,;\\ \partial_t\varphi_t &= (H+c)^{-1/2}\, e^{iHt} D_t(\tilde{\gamma}_t)\, e^{-iHt}(H+c)^{-1/2}\psi\,,\end{aligned}$$

provided that $\partial_t\varphi_t$ is integrable. By (39)

$$D_t(\tilde{\gamma}_t) = p\, g_t'' p$$

up to integrable terms $O(t^{-1-\varepsilon})$ which can be dropped. Factorizing

$$p\, g_t'' p = B_t^2\,;\quad B_t = B_t^*$$

we can use the Schwarz inequality:

$$\begin{aligned}\left\|\int_{t_1}^{t_2}\mathrm{d}t\,\partial_t\varphi_t\right\|^2 &= \sup_{\|v\|=1}\left|\int_{t_1}^{t_2}(v,\partial_t\varphi_t)\right|^2\\ \le& \sup_{\|v\|=1}\left(\int_{t_1}^{t_2}\mathrm{d}t\,\|B_t\, e^{-iHt}(H+c)^{-1/2}v\|\,\|B_t\, e^{-iHt}(H+c)^{-1/2}\varphi\|\right)^2\\ \le& \left(\sup_{\|v\|=1}\int_{t_1}^{t_2}\mathrm{d}t\,\|B_t\, e^{-iHt}(H+c)^{-1/2}v\|^2\right)\times\int_{t_1}^{t_2}\mathrm{d}t\,\|B_t\, e^{-iHt}(H+c)^{-1/2}\varphi\|^2\,.\end{aligned}$$

By Lemma 5 the first factor is bounded uniformly in $t_{1,2}$, and the second factor vanishes as $t_{1,2}\to\infty$. This proves the existence of γ.

Step 3: The existence of γ_a is proved in the same way, with one essential difference: $g_{a,t}$ shares all essential properties of g_t *except convexity*, so $p\, g_{a,t}'' p$ is not positive. However, it is possible to construct a modified Yafaev function $\tilde{g}_t$ (by choosing a slightly different σ [20]) so that

$$\pm g_{a,t}'' \le \tilde{g}_t''\,.$$

Then we can split

$$p\,g''_{a,t}\,p = A_t^+ - A_t^-$$

into positive and negative parts satisfying

$$0 \le A_t^\pm \le p\,\tilde{g}''_t p\,.$$

Treating the contributions from $A_t^\pm$ separately, we factorize $A_t^\pm = (B_t^\pm)^2$ and use the propagation estimate (41) for $\tilde{g}_t$.

Step 4: $[e^{-iHs}, \gamma] = 0$ follows from

$$e^{-iHs}\gamma\, e^{iHs} - \gamma = s - \lim_{t\to\infty} e^{iHt}(\gamma_{t+s} - \gamma_t)\, e^{-iHt} = 0$$

since $(\gamma_{t+s} - \gamma_t) \to 0$ strongly on $D(H)$ for fixed s and $t \to \infty$. The same argument applies to γ_a.

Step 5: $\gamma_{\{0\}} = 0$. Since $\gamma_{\{0\}}$ exists as a strong limit,

$$\gamma_{\{0\}} = s - \lim_{t\to\infty} \frac{1}{T}\int_1^T \mathrm{d}t\, e^{iHt} D_t g_{\{0\},t}\, e^{-iHt} = \frac{1}{T}(g_{\{0\},T} - g_{\{0\},1}) = 0\,,$$

because $g_{\{0\},T} = O(T^\delta)$. □

Asymptotic Clustering

Lemma 7 (Deift-Simon Wave Operators)

$$W_a = s - \lim_{t\to\infty} e^{iH_a t}\gamma_{a,t}\, e^{-iHt} \tag{46}$$

exists on $D(H)$ for all $a \in L$.

Proof. (46) is proved as the existence of γ_a, with the following modifications. In step 1, (46) is replaced by

$$s - \lim_{t\to\infty} (H_a + c)^{1/2}\, e^{iH_a t}\gamma_{a,t}\, e^{-iHt}(H + c)^{-1/2}\,,$$

using that

$$\begin{aligned} &(H_a\gamma_{a,t} - \gamma_{a,t}H)(H + c)^{-1} \\ = \;&([H, \gamma_{a,t}] - I_a\gamma_{a,t})(H + c)^{-1} \longrightarrow 0 \end{aligned}$$

in norm as $t \to \infty$. The reason is that

$$|x|_a > t^\delta \lambda R_1 \quad \text{on supp}\ (\nabla g_{a,t}) \tag{47}$$

so that $\|I_a \gamma_{a,t}(H+c)^{-1/2}\| \to 0$ as $t \to \infty$. In step 2, $\partial_t \varphi_t$ contains the additional term

$$(H_a + c)^{-1/2}\, e^{iH_a t} I_a \widetilde{\gamma}_{a,t}\, e^{-iHt}(H+c)^{-1/2}\,.$$

Here (and only here!) we use the short-range condition (13), which together with (47) gives

$$\|I_a \widetilde{\gamma}_{a,t}(H+c)^{-1/2}\| = O(t^{-\delta\mu})$$

as $t \to \infty$. □

Lemma 8 (Asymptotic Clustering) *Let* $\psi \in \mathrm{Ran}\ (\gamma)\ :\ \psi = \gamma\varphi,\ \varphi \in D(H)$. *Then*

$$e^{-iHt}\psi \underset{\|\,\|}{\longrightarrow} \sum_{a>\{0\}} e^{-iH_a t}\, W_a \varphi\,. \tag{48}$$

Proof. We write $u(t) \approx v(t)$ for $\|u(t) - v(t)\| \to 0$ as $t \to \infty$. By Lemma 6

$$\psi = \sum_{a>\{0\}} \gamma_a \varphi \approx \sum_{a>\{0\}} e^{iHt} \gamma_{a,t}\, e^{-iHt} \varphi\,.$$

Using Lemma 7 we obtain

$$e^{-iHt}\psi \approx \sum_{a>\{0\}} e^{-iH_a t}\, e^{iH_a t} \gamma_{a,t}\, e^{-iHt} \varphi \approx \sum_{a>\{0\}} e^{-iH_a t}\, W_a \varphi\,. \qquad \square$$

To complete the proof of Theorem 1 it remains to show that $\mathrm{Ran}(\gamma)$ is dense in $\mathcal{H}_c(H)$. Since γ commutes with H, it reduces to a bounded selfadjoint operator $\mathcal{H}_\Delta(H) \to \mathcal{H}_\Delta(H)$ for any finite interval Δ. By the remark following Lemma 2 it therefore suffices to prove:

Lemma 9 (Mourre's inequality for γ) *Let* Δ *be a finite, open interval for which (26) holds. Then*

$$\gamma^2 \geq \Theta \quad \text{on} \quad \mathcal{H}_\Delta(H)\,, \tag{49}$$

so that γ *maps* $\mathcal{H}_\Delta(H)$ *onto itself.*

Proof. We consider the Heisenberg observables

$$\gamma(t) = e^{iHt} \gamma_t\, e^{-iHt}\,; \quad g(t) = e^{iHt} g_t\, e^{-iHt}\,; \quad x^2(t) = e^{iHt} x^2\, e^{-iHt}\,.$$

$\gamma(t)$ and $g(t)$ are defined as operators on the domain $D(|x|) \cap D(H)$, which is invariant under $\exp(-iHt)$; $x^2(t)$ is defined as a form on this domain. Since $\gamma(t) = D_t g(t)$,

$$\frac{1}{t} g(t) = \frac{1}{t} g(1) + \frac{1}{t} \int_1^t \mathrm{d}s\, \gamma(s) \quad \underset{s}{\longrightarrow} \quad \gamma \tag{50}$$

as $t \to \infty$ (Lemma 6). Next, we note that $g(x) \geq |x|$ implies $g_t(x) \geq |x|$ and therefore

$$g^2(t) \geq x^2(t)\,. \tag{51}$$

Now let $f \in C_0^\infty(\Delta)$. Since f is smooth, $f(H)$ maps $D(|x|)$ into itself, and Mourre's inequality (26) gives

$$f(H)x^2(t)f(H) \geq (\Theta t^2 + O(t))f^2(H) \tag{52}$$

as $t \to \infty$, in form sense on $D(|x|)$. Combining (50-52) we obtain:

$$\begin{aligned} \Theta f^2(H) &\leq \liminf_{t\to\infty} f(H)\frac{x^2(t)}{t^2}f(H) \\ &\leq \liminf_{t\to\infty} f(H)\frac{g^2(t)}{t^2}f(H) = f(H)\gamma^2 f(H) \end{aligned}$$

for all $f \in C_0^\infty(\Delta)$, which implies (49). □

A Short History

1926 **Schrödinger**: The time-dependent Schrödinger equation [32].

1932 **v. Neumann**: Hilbert space formulation of quantum mechanics [29].

1951 **Kato**: $H = H^* > -\infty$: Existence of dynamics and stability of N-body systems [23].

1959 **Hack**: Existence of scattering states (Wave operators) [18].

1960 **Zhislin**: Determination of the essential spectrum of H [39].

1963 **Faddeev**: Complete discussion of 3-body systems by stationary methods (Faddeev-equations) [14]. Later generalized to all N [19]. Limited by spectral conditions for subsystems.

1969 **Ruelle**: Ergodic space-time characterisation of bound states vs. continuum states [31, 2].

1970 **Efimov**: 3-body systems with short-range potentials can have infinitely many bound states [10]. First mathematical treatment in [37].

1971 **Lavine**: Asymptotic completeness of N-body systems with repulsive forces [24, 25]. A time-dependent proof using positive commutators.

1971 **Balslev, Combes**: Spectral analysis of N-body Hamiltonians with dilation-analytic potentials, revealing the nature of the essential spectrum and of resonances. Absence of singular continuous spectrum [3].

1972 **Iorio, O'Carrol**: Asymptotic completeness of N-body systems in the limit of weak potentials [22].

1973 **O'Connor**: Isotropic exponential bounds for N-body eigenfunctions [5]. Later extended in [4] to embedded eigenvalues in the dilation-analytic case, where positive eigenvalues are excluded.

1977 The advent of geometric (configuration space) methods of spectral analysis and scattering theory, e.g. [7, 8, 11, 35].

1978 **V. Enss**: The greatly inspiring proof of asymptotic completeness for $N = 2$, using only Ruelle's theorem and free wave packets [12]. The turning point to phase-space analysis. Later extended to $N = 3$ [13].

1981 **Mourre**: Mourre's inequality for $N = 3$ [26], soon extended to all N [27]. An infinitesimal version of dilation-analyticity with similar powers. Local decay estimates [27].

1982 **Agmon**: Anisotropic WKB-type bounds on eigenfunctions: Agmon metric [1]. The concise form of earlier results [8].

1982 **Froese, Herbst**: Exponential bounds for eigenfunctions belonging to embedded eigenvalues. Absence of positive eigenvalues [15]. Later supplemented in [28]. Fruits of Mourre's inequality.

1987 **Sigal, Soffer**: First general proof of asymptotic completeness for short-range potentials, using local decay and phase-space propagation estimates [33]. Important simplifications later in [16, 38].

1993 **Derezinski**: Asymptotic completeness for long-range potentials falling off like $r^{-\mu}$, $\mu > \sqrt{3} - 1$ [9]. Influenced by preliminary results of Sigal and Soffer who give an independent proof [34].

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