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# Mathematical Theory of N-Body Quantum Systems <sup>1</sup>

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Abstract. A short history of the subject is given at the end of the paper. The main part of the notes describes a new proof of asymptotic completeness for short-range forces, based on joint work with I.M. Sigal [21].

# 1 N-Body Systems

A system of N particles in  $\mathbb{R}^3$  with pair-interactions is described by the Hamiltonian

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2m_k} + \sum_{i < k}^{1...N} V_{ik} (x_i - x_k), \qquad (1)$$

with  $V_{ik}(x) \to 0$  as  $|x| \to \infty$ . From this standard case we extract the following basic notions:

# Configuration Space

X is a Euclidean space with salar product  $x \cdot y$ . In the case (1):

$$X = \{x = (x_1 \dots x_N) \mid x_k \in \mathbb{R}^3; \sum m_k x_k = 0\};$$

$$x \cdot y = \sum m_k (x_k \cdot y_k)_{\mathbb{R}^3}.$$
(2)

<sup>&</sup>lt;sup>1</sup>Lecture given at the Ascona Conference on Mathematical Results in Quantum Mechanics, June 1996.

 $\frac{1}{2}\dot{x}\cdot\dot{x}=\frac{1}{2}\dot{x}^2$  is the classical kinetic energy,  $p=\dot{x}$  the momentum conjugate to x. In quantum mechanics,

$$H = \frac{1}{2}p^2 + V(x)$$
 on  $L^2(X)$ , (3)

where  $p = -i\nabla$  and  $p^2 = -\Delta$  have the usual form in cartesian coordinates (not particle coordinates) of X.

### Channels

In X there is a distinguished, finite lattice L of subspaces  $a, b, \ldots$  (channels). L is closed under intersections and contains at least  $a = \{0\}$  and a = X. In the case (1) the channels correspond to all partitions of  $(1 \ldots N)$  into clusters. For example if N = 4:

partition channel  
(12)(34) 
$$\longleftrightarrow$$
  $a = \{x | x_1 = x_2; x_3 = x_4\}.$  (4)

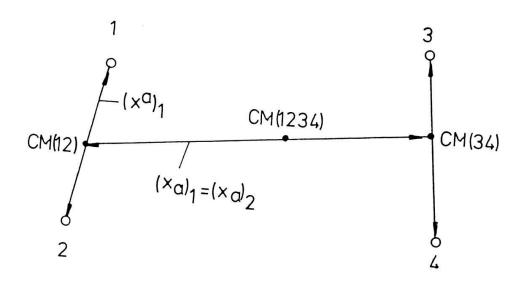
In general the partial ordering of L is defined by

$$a < b \longleftrightarrow a \subset b \; ; a \neq b \, .$$
 (5)

For each  $a \in L$  there is an orthogonal decomposition:

$$X = a \oplus a^{\perp} : x = x_a + x^a . \tag{6}$$

This corresponds to the introduction of CM (center of mass) coordinates, e.g. in the example (4):



(Fig. 1)

The relation

$$\frac{1}{2}p^2 = \frac{1}{2}(p_a)^2 + \frac{1}{2}(p^a)^2$$

expresses the familiar decomposition of the kinetic energy into CM-parts and internal parts with respect to the clusters.

#### Intercluster Distance

The basic feature of N-body systems is that they can split into widely separated, almost independent clusters. As a measure of the separation we might use the minimal distance  $d_a(x)$  in  $R^3$  of the clusters, e.g.

$$d_a(x) = \min_{i \in (12); k \in (34)} |x_i - x_k|$$
(7)

in the example (4). However, it is more consistent (and more convenient) to express the separation in terms of the geometry of X. Some reflection shows that

$$d_a(x) = 0 \longleftrightarrow x \in b; \quad b \cap a < a.$$

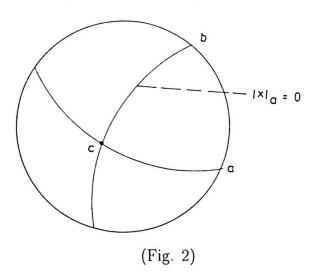


Fig. 2 shows the unit sphere in X, intersected by two channels a, b with  $b \cap a = c < a$ . This leads to the definition of the *intercluster distance* 

$$|x|_a = \min_{b \cap a < a} |x^b|; \quad a > \{0\}.$$
 (8)

For the example (4) one finds

$$|x|_a = \min_{i \in (12); k \in (34)} \left( \frac{m_i m_k}{m_i + m_k} \right)^{1/2} |x_i - x_k|.$$

#### Hamiltonians

For each  $a > \{0\}$  the potential V(x) has a unique decomposition

$$V(x) = V^a(x^a) + I_a(x); (9)$$

$$I_a(x) \to 0 \text{ as } |x|_a \to \infty,$$
 (10)

in particular:  $I_a = V$  for a = X. For  $a = \{0\}$  we define  $I_a = 0$ . In the example (4):

$$V^a = V_{12} + V_{34}; \quad I_a = V_{13} + V_{14} + V_{23} + V_{24}.$$

Corresponding to  $L^2(X) = L^2(a) \otimes L^2(a^{\perp})$  we write:

$$H = H_a + I_a;$$

$$H_a = \frac{1}{2}(p_a)^2 \otimes 1 + 1 \otimes H^a;$$

$$H^a = \frac{1}{2}(p^a)^2 + V^a(x^a) \text{ on } L^2(a^{\perp}).$$
(11)

 $H_a$  describes the dynamics of a system of non-interacting clusters.

## Conditions on V(x)

Some global conditions on V(x) are required to make H (and in fact all  $H^a$ ) selfadjoint on convenient domains and bounded from below. An essential postulate is that  $p^2$  is bounded (or form-bounded) relative to H. Since this is amply covered in the literature on Schrödinger operators [30] we will not, as a rule, state such conditions in our theorems. For readers not familiar with the subject we mention that in the case (1) it suffices that  $V_{ik}(\cdot) \in L^2_{loc}(\mathbb{R}^3)$ ;  $V_{ik}(x) \to 0$  as  $|x| \to \infty$  [23]. All further assumptions on V(x) will only concern the behavior of  $I_a(x)$  as  $|x|_a \to \infty$ . These conditions will be stated explicitly.

# **Induction Principle**

As a result we now have a definition of N-body systems involving only 3 elements:

- A configuration space 
$$X$$
- A lattice  $L$  of channels
- Conditions on  $I_a(x)$ . (12)

In this sense each Hamiltonian  $H^a$  also describes a N-body system with reduced configuration space  $a^{\perp}$ , with channels  $b \cap a^{\perp}$ ,  $b \geq a$ , and with corresponding intercluster potentials  $I_b(x^a)$ . Any proposition P derived from (12) can therefore be established by induction on the lattice L. To begin with, P is checked in the trivial case  $a = X : H^a = 0$  on  $L^2(\{0\}) = \mathbb{C}$ . Then P is proved for  $a = \{0\} : H^a = H$ , under the induction hypothesis that P holds for any  $H^a$  with  $a > \{0\}$ .

# 2 Asymptotic Completeness

In the case of short-range potentials

$$I_a(x) = O(|x|_a^{-\mu}); \quad \mu > 1 \quad (|x|_a \to \infty)$$
 (13)

we define outgoing scattering states  $\psi$  by the asymptotic condition

$$e^{iHt}\psi \xrightarrow{\parallel \parallel} \sum_{a \in L} e^{-iH_a t} \varphi_a \quad (t \to \infty) \; ; \quad \varphi_a \in L^2(a) \otimes \mathcal{H}_B(H^a) \; .$$
 (14)

Here  $\mathcal{H}_B(H^a)$  is the subspace spanned by the eigenvectors of  $H^a$ . Each term in the sum (14) describes a motion of non-interacting, bound clusters. We note that (14) holds trivially for  $\psi \in \mathcal{H}_B(H)$  with:

$$\varphi_{\{0\}} = \psi; \quad \varphi_a = 0 \quad \text{for} \quad a > \{0\}.$$

The existence of scattering states for given  $\{\varphi_a\}$  is well known [30]. Our task is to prove completeness:

**Theorem 1** (Asymptotic Completeness). Suppose, in addition to (13), that

$$\nabla I_a(x) = O(|x|_a^{-\mu}); \quad \mu > 1.$$
 (15)

Then every  $\psi \in L^2(X)$  is a scattering state in the sense of (14).

A proof of this result is given in the following sections. Since the case of  $\psi \in \mathcal{H}_B(H)$  is trivial, and since the subspace of scattering states is known to be closed, it suffices to prove that

$$e^{-iHt}\psi \xrightarrow{\parallel \parallel} \sum_{a>\{0\}} e^{-iH_a t} \varphi_a \tag{16}$$

for a set of  $\psi$  which is dense in the continuous spectral subspace  $\mathcal{H}_C(H) = \mathcal{H}_B(H)^{\perp}$  of H. We will first prove the weaker statement that (16) is valid for some  $\varphi_a \in L^2(X)$ : this is called asymptotic clustering. Then we invoke the induction hypothesis that asymptotic completeness holds for the systems described by  $H^a$ ,  $a > \{0\}$ . This can be written as

$$e^{-iH_at}\varphi_a \xrightarrow{\parallel \parallel} \sum_{b \geq a} e^{-iH_bt}\varphi_{ab}; \quad \varphi_{ab} \in L^2(b) \otimes \mathcal{H}_B(H^b),$$

which is trivially satisfied for a = X. Inserting this into (16) gives

$$e^{-iHt}\psi \xrightarrow[\parallel\parallel]{} \sum_{b>0} e^{-iH_bt} \sum_{\{0\} < a < b} \varphi_{ab} ,$$

i.e. asymptotic completeness for H.

# 3 Yafaev Functions and the Basic Propagation Estimate

## Propagation Observables

The propagation of  $\psi_t = \exp(-iHt)\psi$  in phase space can be described in terms of expectation values

$$\langle \phi_t \rangle_t = (\psi_t, \phi_t \psi_t) \tag{17}$$

of suitable (generally time dependent) observables  $\phi_t(x, p)$ . From

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \phi_t \rangle_t = \langle D_t \phi_t \rangle_t; \quad D_t \phi_t = i[H, \phi_t] + \partial_t \phi_t \tag{18}$$

and from estimates of  $D_t \phi_t$  we can deduce growth properties of  $\langle \phi_t \rangle_t$  as  $t \to \infty$ . Usually this analysis is restricted to finite energy shells  $\Delta \subset \mathbb{R}$ , i.e. to spectral subspaces

$$\mathcal{H}_{\Delta}(H) = \operatorname{Ran}(E_{\Delta}(H)), \tag{19}$$

where  $E_{\Delta}(H)$  is the spectral projection of H corresponding to  $\Delta$ . As a first example we discuss Mourre's inequality, which is basic for our proof of Theorem 1. Let  $S \subset \mathbb{R}$  be the set of thresholds and eigenvalues of H, i.e.

$$S = \bigcup_{a \in L} \{ \text{eigenvalues of } H^a \} . \tag{20}$$

By Mourre's Theorem [6] S is closed and countable. Since S contains the eigenvalues of  $H^{\{0\}} = H$  it follows that

$$\mathcal{H}_S(H) = \mathcal{H}_B(H)$$
.

Therefore

$$\mathcal{H}_{R\backslash S}(H) = \mathcal{H}_C(H) \tag{21}$$

is the continuous spectral subspace of H. Also part of Mourre's Theorem is the following inequality. Let  $E \in R \setminus S$  be in the continuous spectrum  $\sigma_C(H)$ . Then there exists an open interval  $\Delta \ni E$  and a strictly positive  $\Theta$  such that

$$E_{\Delta}(H)i[H,A]E_{\Delta}(H) \ge \Theta E_{\Delta}(H)$$
, (22)

where

$$A = \frac{1}{2}D_t x^2 = \frac{1}{2}(p \cdot x + x \cdot p);$$
 (23)

$$i[H,A] = D_t A = p^2 - x \cdot \nabla V(x). \tag{24}$$

Therefore Mourre's inequality (22) implies

$$\langle x^2 \rangle_t \ge \Theta t^2 + O(t) \quad (t \to \infty)$$

for a dense set of states in  $\mathcal{H}_{\Delta}(H)$ . It is evident from (24) that this result rests on some global conditions on the forces  $\nabla V(x)$ . However, there is a variant of Mourre's Theorem [17, 36] which involves only the tails of the forces at large distances:

**Lemma 2** (Mourre's inequality for  $x^2$ ) Suppose that for all  $a > \{0\}$ 

$$\lim_{|x|_a \to \infty} x \cdot \nabla I_a(x) = 0. \tag{25}$$

Let  $E \in \sigma_C(H)$ ,  $E \notin S$ . Then there is an open interval  $\Delta \ni E$  and a strictly positive  $\Theta$  such that

$$\langle x^2 \rangle_t \ge \Theta t^2 + O(t) \quad (t \to \infty)$$
 (26)

for all  $\psi \in \mathcal{H}_{\Delta}(H) \cap D(|x|)$ .

**Remark.** The states  $\psi$  of this type (for all possible E) span a dense set in the continuous spectral subspace  $\mathcal{H}_C(H)$ . Therefore it will be sufficient to derive (16) from the much weaker statement (26). This requires the construction of more sophisticated propagation observables which are specially adapted to the lattice L of channels.

### Yafaev-Functions

Following Yafaev [38] we construct a function g on X whose properties are summarized in Lemma 3 below. Let  $\sigma$  be a positive, decreasing function on L:

$$\sigma_{\{0\}} > \sigma_a > \sigma_b > \sigma_X = 1 \tag{27}$$

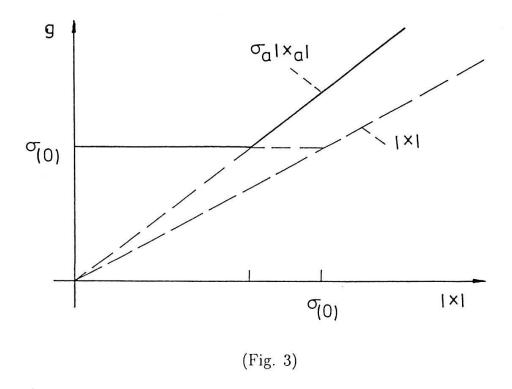
for  $\{0\} < a < b < X$ , to be adjusted in the course of the construction. Let

$$f_a(x) = \begin{cases} \sigma_{\{0\}} & (a = \{0\}); \\ \sigma_a |x_a| & (a > \{0\}). \end{cases}$$

Then the prototype of g(x) is given by

$$g(x,\sigma) = \max_{a \in L} f_a(x). \tag{28}$$

A radial section of  $g(x, \sigma)$  is shown in Fig. 3 for a direction  $x \in a$ .



 $g(x, \sigma)$  is convex, constant on some compact set containing the ball |x| < 1, and homogeneous of degree 1 in the complement of this set. We decompose  $g(x, \sigma)$  into maximal pieces:

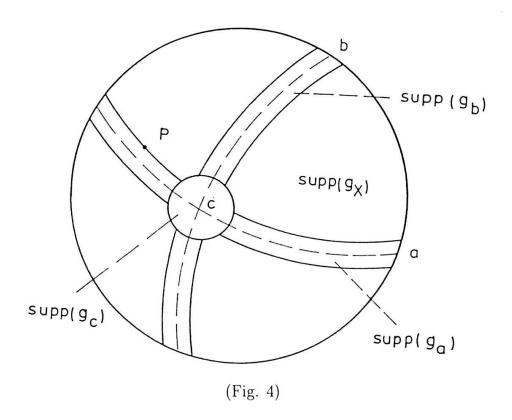
$$g(x,\sigma) = \sum_{a \in L} g_a(x,\sigma); \quad g_a(x,\sigma) = \begin{cases} f_a(x) & \text{if } f_a(x) = g(x,\sigma); \\ 0 & \text{otherwise}. \end{cases}$$
 (29)

The piece  $g_{\{0\}}(x,\sigma)$  has compact support on which it is constant. The pieces  $g_a(x,\sigma)$  for  $a > \{0\}$  are homogeneous of degree 1 on conical supports whose intersection with a sphere  $|x| = R \ge \sigma_{\{0\}}$  is shown in Fig. 4. This figure corresponds to Fig. 2 and serves to explain the choice of  $\sigma$ . Suppose first that  $\sigma_a = \sigma_b = \sigma_c = 1$ . Then Fig. 4 reduces to Fig. 2 since  $\sigma_a|x_a| = |x|$  exactly if  $x \in a$ , etc. We now increase  $\sigma_a, \sigma_b$  by arbitrary small amounts. Then the supports of  $g_a, g_b$  broaden into narrow belts shown in Fig. 4. Then we increase  $\sigma_c$  to  $\sigma_c > \sigma_a, \sigma_b$ , so that supp  $(g_c$  grows to a disc covering the intersection of the two belts. This indicates the general construction scheme for the function  $\sigma$  on L which can be carried out analytically [20, 38]. Fig. 4, together with the definition (8) of the intercluster distance, suggests what can be achieved: There is a (largely arbitrary) choice of  $\sigma$  such that

$$|x|_a > \lambda |x|$$
 on  $\operatorname{supp}(g_a)$  (30)

for some  $\lambda > 0$ . Moreover, since  $g_a(x, \sigma)$  is, on its support, a function of  $x_a$ ,

$$\nabla g(x,\sigma) \in a \quad \text{on} \quad \text{supp}(g_a)$$
 (31)



except at boundary points, where  $\nabla g(x,\sigma)$  is discontinuous. This discontinuity is removed by a regularization  $g(x,\sigma) \to g(x)$  which preserves convexity:

$$g(x) = \int g(x,\mu) \prod_{a \in L} \delta(\mu_a - \sigma_a) d\mu_a, \qquad (32)$$

where  $0 < \delta \in C_0^{\infty}(R)$  is a regularization of the Dirac distribution with sufficiently narrow support. The same regularization is applied to  $g_a(x, \sigma)$ , so that

$$g(x) = \sum_{a \in L} g_a(x). \tag{33}$$

The effect of this regularization on Fig. 4 is that the boundaries are slightly smeared, but away from these strips the functional form of g(x) remains the same. For further reference we list the resulting properties of g and  $g_a$ :

#### **Lemma 3** (Properties of g)

- (i) g is smooth, convex, and homogeneous of degree 1 outside some ball:  $|x| > R_2$ .
- (ii) g(x) = g(0) inside some ball:  $|x| < R_1$ .
- (iii) For any  $x \in \text{supp}(\nabla g)$  there exists  $a \in L$ ,  $a > \{0\}$ , such that

$$\nabla g(x) \in a \quad \text{and} \quad |x|_a > \lambda |x|.$$
 (34)

To explain (iii), consider the boundary point P shown in Fig. 4. There the intercluster distances with respect to a and X are both strictly positive, and after regularization we certainly have  $\nabla g(P) \in X$ . The functions  $g_a$  have corresponding properties except convexity:

### **Lemma 4** (Properties of $g_a$ )

- (i)  $g_a$  is smooth, and homogeneous of degree 1 for  $|x| > R_2$ .
- (ii)  $g_{\{0\}}$  has compact support in  $|x| < R_2$ . For  $a > \{0\}$ ,  $g_a$  is supported in  $|x| > R_1$ , and  $|x|_a > \lambda |x|$  on supp $(g_a)$ .
- (iii)  $\nabla g_a$  is supported in  $|x| > R_1$ . For any  $x \in \text{supp}(\nabla g_a)$  there exists  $b \in L$ ,  $b > \{0\}$  such that

$$\nabla g_a \in b \quad \text{and} \quad |x|_b \ge \lambda |x| \,.$$
 (35)

## The Basic Propagation Estimate

All our propagation observables are derived from

$$g_t(x) = t^{\delta} g(t^{-\delta} x), \quad 0 < \delta < 1 \tag{36}$$

for t > 0. By Lemma 3  $g_t$  is smooth and convex in x,

$$g_t(x) = \begin{cases} t^{\delta} g(0) & (|x| < t^{\delta} R_1); \\ g(x) & (|x| > t^{\delta} R_2), \end{cases}$$

and, since q has bounded derivatives

$$\partial_x^k g_t(x) = O(t^{\delta(1-|k|)}) 
\partial_t^k g_t(x) = O(t^{\delta-k})$$
(37)

as  $t \to \infty$ , uniformly in x. For any x it follows from (9) that

$$\nabla g_t(x) \cdot \nabla V(x) = \nabla g_t(x) \cdot \nabla V^a(x^a) + \nabla g_t(x) \cdot \nabla I_a(x)$$

for some  $a > \{0\}$  depending on x. By (34) the first term vanishes since  $\nabla g_t(x) \in a$ . In the second term  $\nabla g_t(x)$  is bounded with support in  $|x| \geq t^{\delta} R_1$ , where  $|x|_a \geq t^{\delta} \lambda R_1$  and therefore, by (15),  $\nabla I_a(x) = 0(t^{-\delta\mu})$  as  $t \to \infty$ , uniformly in x. As a result,

$$\|\nabla g_t \cdot \nabla V\| \le \text{const. } t^{-\delta\mu}$$
 (38)

for sufficiently large t. Now we compute

$$\gamma_t = D_t g_t = \frac{1}{2} (\nabla g_t \cdot p + p \cdot \nabla g_t) + \partial_t g_t;$$

$$D_t (\gamma_t - 2\partial_t g_t) = p g_t'' p - \frac{1}{4} \Delta^2 g_t - \nabla g_t \cdot \nabla V - \partial_t^2 g_t.$$
(39)

The first term in (39) denotes the Hessian

$$pg_t''p = \sum_{ik} p_i \frac{\partial^2 g_t}{\partial x_i \partial x_k} p_k \ge 0 \tag{40}$$

since  $g_t(x)$  is convex. The following 3 terms are of order  $t^{-3\delta}$ ,  $t^{-\mu\delta}$ ,  $t^{\delta-2}$  as  $t \to \infty$ , uniformly in x. Since  $\mu > 1$  we can now fix  $\delta$  such that these terms are integrable in t, i.e.

$$0 < \delta < 1$$
,  $\delta \mu > 1$ ,  $3 \delta > 1$ .

Lemma 5 (Basic Propagation Estimate)

$$\int_{1}^{\infty} dt \langle p \, g_t'' p \rangle_t \leq \text{const. } \langle H + c \rangle_0 \,, \tag{41}$$

where c is some constant to make  $H + c \ge 1$ .

**Proof.** For any state  $\psi$  in the form domain of H+c, (39) gives

$$\int_{1}^{T} dt \langle p g_t'' p \rangle_t \leq \langle \gamma_t - 2\partial_t g_t \rangle_t \Big|_{1}^{T} + \text{const.} \leq \text{const.} \langle H + c \rangle_0$$

because  $|\langle \gamma_t - 2\partial_t g_t \rangle| \leq \text{const.} \langle H + c \rangle_t$  uniformly in t. Since the integrand  $\langle p g_t'' p \rangle_t$  is positive, the limit  $T \to \infty$  exists.  $\square$ 

# The Asymptotic Observable $\gamma$

Corresponding to (33) we split

$$g_t = \sum_{a} g_{a,t} ; \quad g_{a,t}(x) = t^{\delta} g_a(t^{-\delta} x);$$

$$\gamma_t = \sum_{a} \gamma_{a,t} ; \quad \gamma_{a,t} = D_t(g_{a,t}).$$

$$(42)$$

#### Lemma 6

$$\gamma := s - \lim_{t \to \infty} e^{iHt} \gamma_t e^{-iHt}$$
 and  $\gamma_a := s - \lim_{t \to \infty} e^{iHt} \gamma_{a,t} e^{-iHt}$ 

exist on D(H). Moreover,

$$[\gamma, H] = [\gamma_a, H] = 0; \tag{43}$$

$$\gamma_{\{0\}} = 0 \implies \gamma = \sum_{a>\{0\}} \gamma_a. \tag{44}$$

#### Proof.

Step 1: We first discuss  $\gamma$ .

$$\lim_{t \to \infty} e^{iHt} \gamma_t \, e^{-iHt} (H+c)^{-1} \psi = \lim_{t \to \infty} (H+c)^{-1/2} \, e^{iHt} \gamma_t \, e^{-iHt} (H+c)^{-1/2} \psi \equiv \lim_{t \to \infty} \varphi_t \qquad (45)$$

if this limit exists. This follows by expressing  $(H+c)^{-1/2}$  in terms of the resolvent  $(z-H)^{-1}$  (e.g. using a contour integral), and then from the fact that  $[\gamma_t, (z-H)^{-1}] \to 0$  in norm as  $t \to \infty$ .

Step 2: In  $\varphi_t$  we can replace

$$\gamma_t \longrightarrow \gamma_t - 2\partial_t g_t \equiv \widetilde{\gamma}_t$$

since  $\partial_t g_t \sim t^{\delta-1} (t \to \infty)$ . Then

$$\gamma (H+c)^{-1} \psi = \varphi_1 + \int_1^\infty dt \, \partial_t \varphi_t;$$

$$\partial_t \varphi_t = (H+c)^{-1/2} e^{iHt} D_t(\tilde{\gamma}_t) e^{-iHt} (H+c)^{-1/2} \psi,$$

provided that  $\partial_t \varphi_t$  is integrable. By (39)

$$D_t(\widetilde{\gamma}_t) = p \, g_t'' p$$

up to integrable terms  $O(t^{-1-\varepsilon})$  which can be dropped. Factorizing

$$p\,g_t''p = B_t^2\,; \quad B_t = B_t^*$$

we can use the Schwarz inequality:

$$\left\| \int_{t_1}^{t_2} dt \, \partial_t \varphi_t \right\|^2 = \sup_{\|v\|=1} \left| \int_{t_1}^{t_2} (v, \partial_t \varphi_t) \right|^2$$

$$\leq \sup_{\|v\|=1} \left( \int_{t_1}^{t_2} dt \, \|B_t \, e^{-iHt} (H+c)^{-1/2} v\| \, \|B_t \, e^{-iHt} (H+c)^{-1/2} \varphi\| \right)^2$$

$$\leq \left( \sup_{\|v\|=1} \int_{t_1}^{t_2} dt \, \|B_t \, e^{-iHt} (H+c)^{-1/2} v\|^2 \right) \times \int_{t_1}^{t_2} dt \, \|B_t \, e^{-iHt} (H+c)^{-1/2} \varphi\|^2.$$

By Lemma 5 the first factor is bounded uniformly in  $t_{1,2}$ , and the second factor vanishes as  $t_{1,2} \to \infty$ . This proves the existence of  $\gamma$ .

Step 3: The existence of  $\gamma_a$  is proved in the same way, with one essential difference:  $g_{a,t}$  shares all essential properties of  $g_t$  except convexity, so  $p g''_{a,t} p$  is not positive. However, it is possible to construct a modified Yafaev function  $\tilde{g}_t$  (by choosing a slightly different  $\sigma$  [20]) so that

$$\pm g_{a,t}'' \leq \tilde{g}_t''$$
.

Then we can split

$$p \, g_{a,t}'' \, p = A_t^+ - A_t^-$$

into positive and negative parts satisfying

$$0 \le A_t^{\pm} \le p \, \tilde{g}_t'' p$$
.

Treating the contributions from  $A_t^{\pm}$  separately, we factorize  $A_t^{\pm} = (B_t^{\pm})^2$  and use the propagation estimate (41) for  $\tilde{g}_t$ .

Step 4:  $[e^{-iHs}, \gamma] = 0$  follows from

$$e^{-iHs} \gamma e^{iHs} - \gamma = s - \lim_{t \to \infty} e^{iHt} (\gamma_{t+s} - \gamma_t) e^{-iHt} = 0$$

since  $(\gamma_{t+s} - \gamma_t) \to 0$  strongly on D(H) for fixed s and  $t \to \infty$ . The same argument applies to  $\gamma_a$ .

Step 5:  $\gamma_{\{0\}} = 0$ . Since  $\gamma_{\{0\}}$  exists as a strong limit,

$$\gamma_{\{0\}} = s - \lim_{t \to \infty} \frac{1}{T} \int_{1}^{T} dt \, e^{iHt} D_{t} g_{\{0\},t} \, e^{-iHt} = \frac{1}{T} (g_{\{0\},T} - g_{\{0\},1}) = 0 \,,$$

because  $g_{\{0\},T} = O(T^{\delta})$ .

# Asymptotic Clustering

Lemma 7 (Deift-Simon Wave Operators)

$$W_a = s - \lim_{t \to \infty} e^{iH_a t} \gamma_{a,t} e^{-iHt} \tag{46}$$

exists on D(H) for all  $a \in L$ .

**Proof.** (46) is proved as the existence of  $\gamma_a$ , with the following modifications. In step 1, (46) is replaced by

$$s - \lim_{t \to \infty} (H_a + c)^{1/2} e^{iH_a t} \gamma_{a,t} e^{-iHt} (H + c)^{-1/2},$$

using that

$$(H_a \gamma_{a,t} - \gamma_{a,t} H)(H+c)^{-1}$$

$$= ([H, \gamma_{a,t}] - I_a \gamma_{a,t})(H+c)^{-1} \longrightarrow 0$$

in norm as  $t \to \infty$ . The reason is that

$$|x|_a > t^{\delta} \lambda R_1$$
 on supp  $(\nabla g_{a,t})$  (47)

so that  $||I_a\gamma_{a,t}(H+c)^{-1/2}|| \to 0$  as  $t\to\infty$ . In step 2,  $\partial_t\varphi_t$  contains the additional term

$$(H_a+c)^{-1/2} e^{iH_at} I_a \widetilde{\gamma}_{a,t} e^{-iHt} (H+c)^{-1/2}$$
.

Here (and only here!) we use the short-range condition (13), which together with (47) gives

$$||I_a \widetilde{\gamma}_{a,t} (H+c)^{-1/2}|| = O(t^{-\delta \mu})$$

as  $t \to \infty$ .

**Lemma 8** (Asymptotic Clustering) Let  $\psi \in \text{Ran }(\gamma) : \psi = \gamma \varphi, \ \varphi \in D(H)$ . Then

$$e^{-iHt}\psi \xrightarrow{\parallel \parallel} \sum_{a>\{0\}} e^{-iH_at} W_a \varphi. \tag{48}$$

**Proof.** We write  $u(t) \approx v(t)$  for  $||u(t) - v(t)|| \to 0$  as  $t \to \infty$ . By Lemma 6

$$\psi = \sum_{a>\{0\}} \gamma_a \varphi \approx \sum_{a>\{0\}} e^{iHt} \gamma_{a,t} e^{-iHt} \varphi.$$

Using Lemma 7 we obtain

$$e^{-iHt}\psi \approx \sum_{a>\{0\}} e^{-iH_at} e^{iH_at} \gamma_{a,t} e^{-iHt} \varphi \approx \sum_{a>\{0\}} e^{-iH_at} W_a \varphi.$$

To complete the proof of Theorem 1 it remains to show that  $\operatorname{Ran}(\gamma)$  is dense in  $\mathcal{H}_c(H)$ . Since  $\gamma$  commutes with H, it reduces to a bounded selfadjoint operator  $\mathcal{H}_{\Delta}(H) \to \mathcal{H}_{\Delta}(H)$  for any finite interval  $\Delta$ . By the remark following Lemma 2 it therefore suffices to prove:

**Lemma 9** (Mourre's inequality for  $\gamma$ ) Let  $\Delta$  be a finite, open interval for which (26) holds. Then

$$\gamma^2 \ge \Theta \quad \text{on} \quad \mathcal{H}_{\Delta}(H) \,, \tag{49}$$

so that  $\gamma$  maps  $\mathcal{H}_{\Delta}(H)$  onto itself.

**Proof.** We consider the Heisenberg observables

$$\gamma(t) = e^{iHt} \gamma_t e^{-iHt}; \quad g(t) = e^{iHt} g_t e^{-iHt}; \quad x^2(t) = e^{iHt} x^2 e^{-iHt}.$$

 $\gamma(t)$  and g(t) are defined as operators on the domain  $D(|x|) \cap D(H)$ , which is invariant under  $\exp(-iHt)$ ;  $x^2(t)$  is defined as a form on this domain. Since  $\gamma(t) = D_t g(t)$ ,

$$\frac{1}{t}g(t) = \frac{1}{t}g(1) + \frac{1}{t} \int_{1}^{t} ds \, \gamma(s) \quad \xrightarrow{s} \quad \gamma \tag{50}$$

as  $t \to \infty$  (Lemma 6). Next, we note that  $g(x) \ge |x|$  implies  $g_t(x) \ge |x|$  and therefore

$$g^2(t) \ge x^2(t) \,. \tag{51}$$

Now let  $f \in C_0^{\infty}(\Delta)$ . Since f is smooth, f(H) maps D(|x|) into itself, and Mourre's inequality (26) gives

$$f(H)x^{2}(t)f(H) \ge (\Theta t^{2} + O(t))f^{2}(H)$$
 (52)

as  $t \to \infty$ , in form sense on D(|x|). Combining (50-52) we obtain:

$$\Theta f^{2}(H) \leq \lim_{t \to \infty} \inf f(H) \frac{x^{2}(t)}{t^{2}} f(H) 
\leq \lim_{t \to \infty} \inf f(H) \frac{g^{2}(t)}{t^{2}} f(H) = f(H) \gamma^{2} f(H)$$

for all  $f \in C_0^{\infty}(\Delta)$ , which implies (49).

## A Short History

- 1926 Schrödinger: The time-dependent Schrödinger equation [32].
- 1932 v. Neumann: Hilbert space formulation of quantum mechanics [29].
- 1951 Kato:  $H = H^* > -\infty$ : Existence of dynamics and stability of N-body systems [23].
- 1959 Hack: Existence of scattering states (Wave operators) [18].
- 1960 **Zhislin**: Determination of the essential spectrum of H [39].
- 1963 **Faddeev**: Complete discussion of 3-body systems by stationary methods (Faddeev-equations) [14]. Later generalized to all N [19]. Limited by spectral conditions for subsystems.
- 1969 **Ruelle**: Ergodic space-time characterisation of bound states vs. continuum states [31, 2].
- 1970 **Efimov**: 3-body systems with short-range potentials can have infinitely many bound states [10]. First mathematical treatment in [37].
- 1971 Lavine: Asymptotic completeness of N-body systems with repulsive forces [24, 25]. A time-dependent proof using positive commutators.
- 1971 **Balslev, Combes**: Spectral analysis of N-body Hamiltonians with dilation-analytic potentials, revealing the nature of the essential spectrum and of resonances. Absence of singular continuous spectrum [3].
- 1972 Iorio, O'Carrol: Asymptotic completeness of N-body systems in the limit of weak potentials [22].

1973 **O'Connor**: Isotropic exponential bounds for N-body eigenfunctions [5]. Later extended in [4] to embedded eigenvalues in the dilation-analytic case, where positive eigenvalues are excluded.

- 1977 The advent of geometric (configuration space) methods of spectral analysis and scattering theory, e.g. [7, 8, 11, 35].
- 1978 V. Enss: The greatly inspiring proof of asymptotic completeness for N=2, using only Ruelle's theorem and free wave packets [12]. The turning point to phase-space analysis. Later extended to N=3 [13].
- 1981 **Mourre**: Mourre's inequality for N=3 [26], soon extended to all N [27]. An infinitesimal version of dilation-analyticity with similar powers. Local decay estimates [27].
- 1982 **Agmon**: Anisotropic WKB-type bounds on eigenfunctions: Agmon metric [1]. The concise form of earlier results [8].
- 1982 Froese, Herbst: Exponential bounds for eigenfunctions belonging to embedded eigenvalues. Absence of positive eigenvalues [15]. Later supplemented in [28]. Fruits of Mourre's inequality.
- 1987 **Sigal, Soffer**: First general proof of asymptotic completeness for short-range potentials, using local decay and phase-space propagation estimates [33]. Important simplifications later in [16, 38].
- 1993 **Derezinski**: Asymptotic completeness for long-range potentials falling off like  $r^{-\mu}$ ,  $\mu > \sqrt{3} 1$  [9]. Influenced by preliminary results of Sigal and Soffer who give an independent proof [34].

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