

Zeitschrift: Helvetica Physica Acta
Band: 70 (1997)
Heft: 6

Artikel: Marsden-Ratiu reduction and W^2_3 algebra
Autor: Mukhopadhyay, I. / Chowdhury, A. Roy
DOI: <https://doi.org/10.5169/seals-117060>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Marsden-Ratiu Reduction and W_3^2 Algebra

By I. Mukhopadhyay and A. Roy Chowdhury

High Energy Physics Division
Department of Physics
Jadavpur University
Calcutta -700 032, India

(27.I.1997)

Abstract The W_3^2 algebra is deduced by the Marsden-Ratiu reduction in the bi-Hamiltonian framework proposed by Magri et al and compared with the usual derivations via the Drinfeld-Sokolov formalism. It is observed that the choice of A in the first Poisson tensor must be different for W_3^2 algebra.

1. Introduction

It has been known since a long time that the KdV equation $U_t = U_{xxx} + 6UU_x$ can be written as a Hamiltonian system with respect to two different Poisson structures⁽¹⁾. This property leads to a sequence of commuting Hamiltonians which can be constructed through recursion. The second hamiltonian structure in this hierarchy coincides with the canonical Lie-Poisson structure on the dual of Virasoro algebra⁽²⁾. On the other hand, in a fundamental paper, Drinfeld-Sokolov⁽³⁾presented a procedure to associate generalised KdV-type equations with any Kac-Moody algebra, which also enjoy the property of being bi-Hamiltonian. The Drinfeld-Sokolov reduction is essentially algebraic, a fundamental role being played by the idea of gauge invariance. On the other hand in the formulation of Magri et al⁽⁴⁾, a different explanation of the Hamiltonian reduction and the generation of Virasoro algebra was given using a geometrical reduction process, viz. the Marsden-Ratiu procedure. In the present paper, we utilise the idea of Marsden-Ratiu reduction and the theory of bi-Hamiltonian manifold to deduce classical W_3^2 algebra, which is associated with

the generalised DS hierarchies. We also study the co-adjoint invariance of the structure of W_3^2 .

This paper is organized as follows. In section (2) we briefly review the Marsden-Ratiu reduction⁽⁵⁾ scheme and the associated bi-Hamiltonian manifold and then apply it to derive the W_3^2 . In this context we have observed that some generalization of the formalism of ref (6) is needed for the W_3^2 case. In section (3) the co-adjoint invariance is discussed.

2. Formulation

Recall that, according to classical mechanics an integrable system is a dynamical system on a symplectic manifold M which admits a complete set of constants of motion in involution. These constants are usually constructed by means of a group of symmetry G acting symplectically on the phase space. As a first step towards developing the idea of bi-Hamiltonian manifold, we replace G by a “Poisson-action of the algebra of observables on M defined by the second Poisson structure. Manifolds endowed with a pair of “compatible Poisson brackets P_0 and P_1 , are called bi-Hamiltonian manifolds, such that one of them selects the Hamiltonians and the other selects the vector fields⁽⁷⁾.

The Marsden-Ratiu reduction scheme considers a submanifold S of M , a foliation E of S and the quotient space $N = S/E$. The foliation E is defined by the intersection with S of a distribution D in M , defined only at the points of S . The submanifold S is a symplectic leaf of the first Poisson tensor P_0 . The distribution D is the image of the kernel of P_0 with respect to P_1 . We then have the following general result:

The quotient space $N = S/E$ is a bi-Hamiltonian manifold. On N there exists a unique Poisson $\{\cdot, \cdot\}_N^\lambda$ such that

$$\{f, g\}_N^\lambda \circ \pi = \{F, G\}_M^\lambda \circ i$$

for any pair of functions F and G which extend the functions f and g of N into M , and are constant on D . Here π stands for the projection $\pi : S \mapsto N$ and i denotes the inclusion. This means that the function F satisfies the conditions,

$$\begin{aligned} F \circ i &= f \circ \pi \\ \{F, K\}_1 &= 0 \end{aligned}$$

for any function K whose differential at the point of S , belongs to the kernel of P_0 . To proceed let us consider $g = sl(3, C)$, and set

$$\begin{aligned} S = & V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_1e_{12} + V_{-1}e_{21} + \\ & V_3e_{13} + V_{-3}e_{31} + V_2e_{23} + V_{-2}e_{32} \end{aligned} \tag{1}$$

a map from the circle S^1 into the Lie algebra $sl(3, C)$. The entries of this matrix are periodic functions of the coordinate x on the circle. Let us consider this matrix as a point

on the manifold M . We then have

$$\begin{aligned}\dot{S} = & \dot{V}_{11}e_{11} + \dot{V}_{22}e_{22} + \dot{V}_{33}e_{33} + \dot{V}_1e_{12} + \\ & \dot{V}_{-1}e_{21} + \dot{V}_3e_{13} + \dot{V}_{-3}e_{31} + \dot{V}_2e_{23} + \dot{V}_{-2}e_{32},\end{aligned}\quad (2)$$

a tangent vector to M at the point S . Let

$$V = \alpha_1e_{11} + \alpha_2e_{22} + \alpha_3e_{33} + \beta_1e_{12} + \beta_2e_{21} + \delta_1e_{13} + \delta_2e_{31} + \gamma_1e_{23} + \gamma_2e_{23} \quad (3)$$

denote a covector at the point S . They are arbitrary loops from S^1 into g . To be consistent with the $sl(3, c)$ algebra, we must have

$$\sum V_{ii} = 0; \sum \alpha_i = 0, i = 1, 2, 3 \quad (4)$$

The space M is essentially an infinite dimensional Lie algebra with a canonical co-cycle

$$\omega(\dot{S}_1, \dot{S}_2) = \int_{S^1} \text{Tr} \left(\dot{S}_1 \frac{d\dot{S}_2}{dx} \right) dx \quad (5)$$

the linear map $\Omega : g \mapsto g^*$ associated with this co-cycle is

$$\Omega(V) = \frac{dV}{dx} \quad (6)$$

According to the general construction of bi-Hamiltonian manifolds, the space M is endowed with two Poisson tensors P_0 and P_1 defined by

$$P_0(V) = [A, V] \quad (7a)$$

$$P_1(V) = V_x + [V, S] \quad (7b)$$

Here V_x denotes the derivative of the loop V with respect to the co-ordinate x on S^1 , and A is a constant matrix. The crucial point is the choice of A . Specific Lie algebraic method is given in reference (6) only for the Drinfeld-Sokolov type reductions. There it was stipulated that A should belong to the centre of the Borel subalgebra. But in the case of W_3^2 we are to modify this prescription. We have observed that if we consider A to be a constant strictly lower triangular matrix belonging to $sl(3, c)$ algebra, then we can arrive at W_3^2 . But the ansatz given in ref. (6) leads only to W_3 . So we set

$$A = e_{21} + e_{31} + e_{32} \quad (8)$$

The Poisson tensor P_0 leads to

$$\begin{aligned}\dot{V}_{11} &= -\beta_1 - \delta_1 \\ \dot{V}_{22} &= \beta_1 - \gamma_1 \\ \dot{V}_{33} &= \delta_1 + \gamma_1 \\ \dot{V}_{-1} &= \alpha_1 - \alpha_2 - \gamma_1 \\ \dot{V}_{-2} &= \beta_1 + \alpha_2 - \alpha_3 \\ \dot{V}_{-3} &= \alpha_1 + \beta_2 - \gamma_2 - \alpha_3 \\ \dot{V}_1 &= -\delta_1 \\ \dot{V}_2 &= \delta_1 \\ \dot{V}_3 &= 0\end{aligned}\quad (9)$$

Similarly from the second Poisson tensor P_1 we get

$$\begin{aligned}
 \dot{V}_{11} &= \alpha_{1x} + \beta_1 V_{-1} + \delta_1 V_{-3} - \beta_2 V_1 - \delta_2 V_3 \\
 \dot{V}_{22} &= \alpha_{2x} + \beta_2 V_1 + \gamma_1 V_{-2} - \beta_1 V_{-1} - \gamma_2 V_2 \\
 \dot{V}_{33} &= \alpha_{3x} + \delta_2 V_3 + \gamma_2 V_2 - \delta_1 V_{-3} - \gamma_1 V_{-2} \\
 \dot{V}_{-1} &= \beta_{2x} + \beta_2 (V_{11} - V_{22}) + (\alpha_2 - \alpha_1) V_{-1} + \gamma_1 V_{-3} - \delta_2 V_2 \\
 \dot{V}_{-2} &= \gamma_{2x} + \gamma_2 (V_{22} - V_{33}) + (\alpha_3 - \alpha_2) V_{-2} - \beta_1 V_{-3} + \delta_2 V_1 \\
 \dot{V}_{-3} &= \delta_{2x} + d_2 (V_{11} - V_{33}) + (\alpha_3 - \alpha_1) V_{-3} + \gamma_2 V_{-1} - \beta_2 V_{-2} \\
 \dot{V}_1 &= \beta_{1x} + \beta_1 (V_{22} - V_{11}) + (a_1 - \alpha_2) V_1 + \delta_1 V_{-2} - V_3 \gamma_2 \\
 \dot{V}_2 &= \gamma_{1x} + \gamma_1 (V_{33} - V_{22}) + (a_2 - \alpha_3) V_2 + \delta_1 V_{-1} - \beta_2 V_3 \\
 \dot{V}_3 &= \delta_{1x} + \delta_1 (V_{33} - V_{11}) + (a_1 - \alpha_3) V_3 + \beta_1 V_2 - \gamma_1 V_1
 \end{aligned} \tag{10}$$

Let us note that the vector field defined by the first bi-vector P_0 are tangent to the affine hyperplanes $V_3 = V_{30}$ (where V_{30} is a given periodic function); so the symplectic leaves of P_0 are affine hyperplanes.

Since $\dot{V}_3 = 0$, from the Poisson tensor P_0 , let us choose $V_3 = 1$, so that

$$S = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_1e_{12} + V_{-1}e_{21} + e_{13} + V_{-3}e_{31} + V_2e_{23} + V_{-2}e_{32} \tag{11}$$

The kernel of P_0 is formed by the covectors with

$$\begin{aligned}
 \delta_1 &= \beta_1 = \gamma_1 = 0 \\
 \alpha_1 &= \alpha_2 = \alpha_3 = 0
 \end{aligned} \tag{12}$$

along with $\beta_2 = \gamma_2$ and $V_1 + V_2 = 0$

Now the flows given by the second Poisson tensor suggest that the distribution D is spanned by the following vector fields,

$$\begin{aligned}
 \dot{V}_{11} &= -\beta_2 V_1 - \delta_2 \\
 \dot{V}_{22} &= \beta_2 V_1 - \gamma_2 V_2 \\
 \dot{V}_{33} &= \delta_2 + \gamma_2 V_2 \\
 \dot{V}_{-1} &= \beta_{2x} + \beta_2 (V_{11} - V_{22}) - \delta_2 V_2 \\
 \dot{V}_{-2} &= \gamma_{2x} + \gamma_2 (V_{22} - V_{33}) + \delta_2 V_1 \\
 \dot{V}_{-3} &= \delta_{2x} + \delta_2 (V_{11} - V_{33}) + \gamma_2 V_{-1} - \beta_2 V_{-2} \\
 \dot{V}_1 &= -\gamma_2 \\
 \dot{V}_2 &= \beta_2
 \end{aligned} \tag{13}$$

So from these equations we obtain the elements of the matrix V ,

$$\begin{aligned}
 \beta_2 &= \dot{V}_2 \\
 \gamma_2 &= -\dot{V}_1 \\
 \delta_2 &= V_{33} + V_1 V_2
 \end{aligned} \tag{14}$$

By using equation (13) in (14), we obtain

$$(V_{22} - V_2 V_1) = 0$$

So we get an invariant functional of S , viz

$$U_0 = V_{22} - V_2 V_1 \quad (15)$$

Similarly we obtain, after a laborious computation, the other three invariants, viz.

$$\begin{aligned} U_1 &= V_2(V_{22} - V_{11}) + V_{-1} - V_2^2 V_1 - V_{2x} \\ U_2 &= V_1(V_{11} + 2V_{22}) + V_{-2} - V_1^2 V_2 + V_{1x} \\ U_3 &= -V_{11} V_{33} + \frac{1}{4}(V_{22} + 6V_1 V_2) V_{22} - \frac{3}{4} V_1^2 V_2^2 \\ &\quad + V_1 V_{-1} + V_2 V_{-2} + V_{-3} + V_{11x} + \frac{1}{2} V_{22x} - \frac{1}{2} V_2 V_{1x} - \frac{1}{2} V_1 V_{2x} \end{aligned} \quad (16)$$

These invariants closely resemble those found in ref. (9) in the discussion of the twisted version of the W_3^2 algebra. Geometrically speaking, U_0, U_1, U_2, U_3 are the final variables of the quotient space $N = S/E$ which is the space of functions on S^1 and equations (15) and (16) give the projection $\pi : S \mapsto N$. These four invariants turn out to be the generators of the W_3^2 algebra because their Poisson brackets yield,

$$\begin{aligned} \{U_0(x), U_0(y)\} &= -\frac{2}{3} \delta'(x - y) \\ \{U_0(x), U_1(y)\} &= U_1(x) \delta(x - y) \\ \{U_0(x), U_2(y)\} &= -U_2(x) \delta(x - y) \\ \{U_1(x), U_2(y)\} &= -\delta'(x - y) + 3U_0(x) \delta(x - y) + \{U_3(x) + \frac{3}{2} U_0'(x) - 3U_0^2(x)\} \delta(x - y) \\ \{U_3(x), U_0(y)\} &= -U_0(x) \delta'(x - y) \\ \{U_3(x), U_1(y)\} &= -\frac{3}{2} U_1(x) \delta'(x - y) - \frac{1}{2} U_1'(x) \delta(x - y) \\ \{U_3(x), U_2(y)\} &= -\frac{3}{2} U_2(x) \delta'(x - y) - \frac{1}{2} U_2'(x) \delta(x - y) \\ \{U_3(x), U_3(y)\} &= \frac{1}{2} \delta'''(x - y) - 2U_3(x) \delta'(x - y) - U_3'(x) \delta(x - y) \end{aligned} \quad (17)$$

The Poisson brackets (17) correspond to the reduction of the second Poisson tensor P_1 . To obtain these Poisson brackets we use the fact that the fundamental Poisson brackets between the different V_i 's are isomorphic to the Lie commutation relations with a central extension, and are given by

$$\{V_a(z), V_b(z')\} = f_{abc} V_c(z) \delta(z - z') - k(T^a, T^b) \delta'(z - z') \quad (18)$$

where

$$S(z) = V_a(z) T^a \quad (19)$$

and T^a denotes the generators of the Lie algebra $sl(3)$ with commutation relations

$$[T^a, T^b] = f_{abc} T^c \quad (20)$$

This fundamental Poisson bracket is, in turn, derived from the basic definition,

$$\{V_a(z), V_b(z)\} = ([dV_a, \partial + S], dV_b) \quad (21)$$

where S is the symplectic leaf containing the different V_i 's as its entries.

As a simple exercise, we calculate $\{V_{-1}(x), V_{-2}(y)\}$. We obtain

$$dV_{-1} = \delta V_{-1}(x)/\delta S(z) = e_{12}\delta(x - z)$$

and

$$dV_{-2} = \delta V_{-2}(z)/\delta S(y) = e_{23}\delta(z - y) \quad (22)$$

After using the expression for S given in (11), we get $\{V_{-1}(x), V_{-2}(y)\} = -V_{-3}(x)\delta(x - y)$. Exactly the same result is obtained on using (18). Finally, we calculate one Poisson bracket from the set (17) explicitly. We have

$$\begin{aligned} \{U_0(x), U_0(y)\} &= \{V_{22}(x) - V_2(x)V_1(x), V_2(y) - V_2(y)V_1(y)\} \\ &= \{V_{22}(x), V_{22}(y)\} - \{V_{22}(x), V_2(y)\}V_1(y) - \\ &\quad V_2(y)\{V_{22}(x), V_1(y)\} - V_{-2}(x)\{V_1(x), V_{22}(y)\} - \\ &\quad \{V_2(x), V_{22}(y)\}V_1(x) + V_2(x)V_1(y)\{V_1(x), V_2(y)\} + \quad (23) \\ &\quad V_1(x)V_1(y)\{V_2(x), V_2(y)\} + V_2(y)V_1(x)\{V_2(x), V_1(y)\} + \\ &\quad V_2(x)V_2(y)\{V_1(x), V_1(y)\} \\ &= \{V_{22}(x), V_{22}(y)\} \end{aligned}$$

after cancelling several terms in pairs using the antisymmetry of the Poisson brackets, whence

$$\begin{aligned} \{U_0(x), U_0(y)\} &= -k\delta'(x - y) \\ &= -\frac{2}{3}\delta'(x - y), \text{ choosing } k = \frac{2}{3} \quad (24) \end{aligned}$$

The above discussion shows how the Poisson brackets (17) are obtained and thus the classical W_3^2 algebra is derived. Thus through a rather new choice of the constant matrix A of the first Poisson tensor P_0 we have deduced the classical W_3^2 algebra. Our choice of the symplectic leaf is further justified by the discussion in ref. (10). For comparison we can mention in short the case of W_3 algebra. Here the symplectic leaf is considered to be

$$S = V_{11}(e_{11} - e_{33}) + V_1 e_{12} + V_{-1} e_{21} + V_3 e_{13} + V_{-3} e_{31} + V_2 e_{23} + V_{-2} e_{32} \quad (25)$$

where $V_1 = V_2 = 1$ and $V_3 = 0$ is the required condition. Further

$$A = e_{31} \quad (26)$$

The covector V is found to be

$$V = \frac{\alpha}{2}(e_{11} - e_{33}) + \beta_1 e_{12} + \beta_2 e_{21} + \delta_1 e_{13} + \delta_2 e_{31} + \gamma_1 e_{23} + \gamma_2 e_{32} \quad (27)$$

Proceeding as before we get two invariants, viz.

$$\begin{aligned} U_1 &= V_{11}^2 + V_{-1} + V_{-2} + 2V_{11x} \\ U_0 &= V_{11}(V_{-1} - V_{-2}) + V_{-3} + V_{11}V_{11x} + V_{11xx} + V_{-1x}, \end{aligned} \quad (28)$$

instead of four, as in the case of W_3^2 algebra. The algebra generated by U_1 and U_0 is found to be the W_3 algebra of Zamolodchikov. Finally we may mention again that the difference actually comes from the fact that in case of W_3 , "A" belongs to the centre of the strictly lower triangular matrices, while in case of W_3^2 it is itself a strictly lower triangular matrix.

3. Co-adjoint Invariance

After our derivation of W_3^2 from the bi-Hamiltonian framework we can compare our results with those obtained in the gauge transformation frame-work. This method actually generates the W -algebra via the co-adjoint action invariance of certain functionals. Such an approach was used in ref. (8) to deduce the Lie-Poisson structure on the dual of the Virasoro algebra, the underlying algebra being the $sl(3, c)$ Kac-Moody algebra on S^1 . We now briefly comment on the results in case of $sl(3, c)$ leading to W_3^2 . It is now well-known that if G is an affine Lie group and g its Lie algebra then the dual space g^* of g is defined as the space of linear functionals of g . The coadjoint action is given by the formulae,

$$\text{ad}_{(Y, \mu)}^*(v, k) = ([Y, v] + kY, 0) \quad (29)$$

$$\text{Ad}_{(\phi, \mu)}^*(v, k) = (\phi v \phi^{-1} + k\phi' \phi^{-1}, k) \quad (30)$$

where $(v(x), k)$ belongs to the dual space. In the case of $sl(3, c)$ algebra, the phase space points are specified as ,

$$v(x) = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_1e_{12} + V_{-1}e_{21} + V_3e_{13} + V_{-3}e_{31} + V_2e_{23} + V_{-2}e_{32} \quad (31)$$

We put the constraint $V_3 = 1$. The maximal co-adjoint action which does not change this constraint is given by (30) with ϕ given as

$$\phi = e_{11} + e_{22} + e_{33} + Ae_{21} + Be_{31} + Ce_{32}, \text{ that is, } \text{Ad}_{(\phi, \mu)}^*(v, k) = (\bar{v}, k). \quad (32)$$

Simple algebra gives

$$A = \bar{V}_2 - V_2; \quad B = V_{11} - \bar{V}_{11} - \bar{V}_1(\bar{V}_2 - V_2); \quad C = V_1 - \bar{V}_1$$

and we also obtain that

$$\begin{aligned} V_{22} - V_2 V_1 &= \bar{V}_{22} - \bar{V}_2 \bar{V}_1 \\ V_2(V_{22} - V_{11}) - V_2^2 V_1 + V_{-1} - V_{2x} &= \bar{V}_2(\bar{V}_{22} - \bar{V}_{11}) - \bar{V}_2^2 \bar{V}_1 + \bar{V}_{-1} - \bar{V}_{2x} \end{aligned} \quad (33)$$

and so on. The upshot is that we get back the four quantities U_0, U_1, U_2 , and U_3 as the invariants of the co-adjoint action whereas the bi-Hamiltonian approach suggests that they are invariants of the flow. This can be seen to be related to the fact that we actually construct the dynamics via the co-adjoint action.

One of the authors (I. M) is grateful to CSIR for a SRF which made this work possible. He wishes to thank N. B. Manik and S. Sarkar for encouragement and cooperation. The whole work has its genesis in the excellent deliberations of Prof. F Magri at the CIMPA school held at Pondicherry, India, 1996.

4. References

1. F. Magri-J. Math. Phys. 19 (1978) 1156. In Nonlinear Evolution Equations and Dynamical Systems: Ed:M. Boiti;F. Pempinelli;G. Soliani. Lecture Notes in Physics Vol 120, Springer, 1980, p 233
2. M. Semenov-Tian-Shansky, A. G. Reyman and A. P. Fordy Lett. Math. Phys. 17 (1980)25
3. Drinfeld and Sokolov-J. Sov. Maths. 30, (1985)1975
4. F. Magri-Lectures given at the CIMPA school held at Pondicherry(1996) India (to be published)
5. J. E. Marsden , T. Ratiu-Lett. Math. Phys. 11(1986)161 G. Marmo, E. J. Saletan, A. Simoni, B. Vitale, Dynamical Systems(John Wiley, N. York)
6. P. Casati and M. Pedroni-Lett. Math. Phys. 25(1992)89
7. See ref. (4)
8. A. A. Belavin - Advanced Studies in Pure Mathematics 19(1989) 117.
9. N. Burroughs, M. DeGroot, T. Hollowood and J. L. Miramontes -Princeton preprint PUPT - 1263,
10. D. A. Depireux and P. Mathieu - Int. J. Mod. Phys. 7(1992) No. 23.