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Marsden-Ratiu Reduction and W_3^2 Algebra

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Abstract The W_3^2 algebra is deduced by the Marsden-Ratiu reduction in the bi-Hamiltonian framework proposed by Magri et al and compared with the usual derivations via the Drinfeld-Sokolov formalism. It is observed that the choice of A in the first Poisson tensor must be different for W_3^2 algebra.

1. Introduction

It has been known since a long time that the KdV equation $U_t = U_{xxx} + 6UU_x$ can be written as a Hamiltonian system with respect to two different Poisson structures⁽¹⁾. This property leads to a sequence of commuting Hamiltonians which can be constructed through recursion. The second hamiltonian structure in this hierarchy coincides with the canonical Lie-Poisson structure on the dual of Virasoro algebra⁽²⁾. On the other hand, in a fundamental paper, Drinfeld-Sokolov⁽³⁾ presented a procedure to associate generalised KdV-type equations with any Kac-Moody algebra, which also enjoy the property of being bi-Hamiltonian. The Drinfeld-Sokolov reduction is essentially algebraic, a fundamental role being played by the idea of gauge invariance. On the other hand in the formulation of Magri et al⁽⁴⁾, a different explanation of the Hamiltonian reduction and the generation of Virasoro algebra was given using a geometrical reduction process, viz. the Marsden-Ratiu procedure. In the present paper, we utilise the idea of Marsden-Ratiu reduction and the theory of bi-Hamiltonian manifold to deduce classical W_3^2 algebra, which is associated with

the generalised DS hierarchies. We also study the co-adjoint invariance of the structure of W_3^2 .

This paper is organized as follows. In section (2) we briefly review the Marsden-Ratiu reduction⁽⁵⁾scheme and the associated bi-Hamiltonian manifold and then apply it to derive the W_3^2 . In this context we have observed that some generalization of the formalism of ref (6) is needed for the W_3^2 case. In section (3) the co-adjoint invariance is discussed.

2. Formulation

Recall that, according to classical mechanics an integrable system is a dynamical system on a symplectic manifold M which admits a complete set of constants of motion in involution. These constants are usually constructed by means of a group of symmetry G acting symplectically on the phase space. As a first step towards developing the idea of bi-Hamiltonian manifold, we replace G by a "Poisson-action of the algebra of observables on M defined by the second Poisson structure. Manifolds endowed with a pair of "compatible Poisson brackets P_0 and P_1 , are called bi-Hamiltonian manifolds, such that one of them selects the Hamiltonians and the other selects the vector fields⁽⁷⁾.

The Marsden-Ratiu reduction scheme considers a submanifold S of M, a foliation E of S and the quotient space N = S/E. The foliation E is defined by the intersection with S of a distribution D in M, defined only at the points of S. The submanifold S is a symplectic leaf of the first Poisson tensor P_0 . The distribution D is the image of the kernel of P_0 with respect to P_1 . We then have the following general result:

The quotient space N = S/E is a bi-Hamiltonian manifold. On N there exists a unique Poisson $\{,\}_N^{\lambda}$ such that

$$\{f,g\}_N^\lambda\circ\pi=\{F,G\}_M^\lambda\circ i$$

for any pair of functions F and G which extend the functions f and g of N into M, and are constant on D. Here π stands for the projection $\pi: S \mapsto N$ and i denotes the inclusion. This means that the function F satisfies the conditions,

$$F \circ i = f \circ \pi$$
$$\{F, K\}_1 = 0$$

for any function K whose differential at the point of S, belongs to the kernel of P_0 . To proceed let us consider g = sl(3, C), and set

$$S = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_{1}e_{12} + V_{-1}e_{21} + V_{3}e_{13} + V_{-3}e_{31} + V_{2}e_{23} + V_{-2}e_{32}$$

$$(1)$$

a map from the circle S^1 into the Lie algebra sl(3,c). The entries of this matrix are periodic functions of the coordinate x on the circle. Let us consider this matrix as a point

on the manifold M. We then have

$$\dot{S} = \dot{V}_{11}e_{11} + \dot{V}_{22}e_{22} + \dot{V}_{33}e_{33} + \dot{V}_{1}e_{12} +
\dot{V}_{-1}e_{21} + \dot{V}_{3}e_{13} + \dot{V}_{-3}e_{31} + \dot{V}_{2}e_{23} + \dot{V}_{-2}e_{32},$$
(2)

a tangent vector to M at the point S. Let

$$V = \alpha_1 e_{11} + \alpha_2 e_{22} + \alpha_3 e_{33} + \beta_1 e_{12} + \beta_2 e_{21} + \delta_1 e_{13} + \delta_2 e_{31} + \gamma_1 e_{23} + \gamma_2 e_{23}$$
 (3)

denote a covector at the point S. They are arbitrary loops from S^1 into g. To be consistent with the sl(3,c) algebra, we must have

$$\sum V_{ii} = 0; \ \sum \alpha_i = 0, i = 1, 2, 3 \tag{4}$$

The space M is essentially an infinite dimensional Lie algebra with a canonical co-cycle

$$\omega(\dot{S}_1, \dot{S}_2) = \int_{S^1} \text{Tr}\left(\dot{S}_1 \frac{d\dot{S}_2}{dx}\right) dx \tag{5}$$

the linear map $\Omega: g \mapsto g^*$ associated with this co-cycle is

$$\Omega(V) = \frac{dV}{dx} \tag{6}$$

According to the general construction of bi-Hamiltonian manifolds, the space M is endowed with two Poisson tensors P_0 and P_1 defined by

$$P_0(V) = [A, V] \tag{7a}$$

$$P_1(V) = V_x + [V, S] \tag{7b}$$

Here V_x denotes the derivative of the loop V with respect to the co-ordinate x on S^1 , and A is a constant matrix. The crucial point is the choice of A. Specific Lie algebraic method is given in reference (6) only for the Drinfeld-Sokolov type reductions. There it was stipulated that A should belong to the centre of the Borel subalgebra. But in the case of W_3^2 we are to modify this prescription. We have observed that if we consider A to be a constant strictly lower triangular matrix belonging to sl(3,c) algebra, then we can arrive at W_3^2 . But the ansatz given in ref. (6) leads only to W_3 . So we set

$$A = e_{21} + e_{31} + e_{32} \tag{8}$$

The Poisson tensor P_0 leads to

$$\dot{V}_{11} = -\beta_1 - \delta_1
\dot{V}_{22} = \beta_1 - \gamma_1
\dot{V}_{33} = \delta_1 + \gamma_1
\dot{V}_{-1} = \alpha_1 - \alpha_2 - \gamma_1
\dot{V}_{-2} = \beta_1 + \alpha_2 - \alpha_3
\dot{V}_{-3} = \alpha_1 + \beta_2 - \gamma_2 - \alpha_3
\dot{V}_1 = -\delta_1
\dot{V}_2 = \delta_1
\dot{V}_3 = 0$$
(9)

Similarly from the second Poisson tensor P_1 we get

$$\dot{V}_{11} = \alpha_{1x} + \beta_1 V_{-1} + \delta_1 V_{-3} - \beta_2 V_1 - \delta_2 V_3
\dot{V}_{22} = \alpha_{2x} + \beta_2 V_1 + \gamma_1 V_{-2} - \beta_1 V_{-1} - \gamma_2 V_2
\dot{V}_{33} = \alpha_{3x} + \delta_2 V_3 + \gamma_2 V_2 - \delta_1 V_{-3} - \gamma_1 V_{-2}
\dot{V}_{-1} = \beta_{2x} + \beta_2 (V_{11} - V_{22}) + (\alpha_2 - \alpha_1) V_{-1} + \gamma_1 V_{-3} - \delta_2 V_2
\dot{V}_{-2} = \gamma_{2x} + \gamma_2 (V_{22} - V_{33}) + (\alpha_3 - \alpha_2) V_{-2} - \beta_1 V_{-3} + \delta_2 V_1
\dot{V}_{-3} = \delta_{2x} + d_2 (V_{11} - V_{33}) + (\alpha_3 - \alpha_1) V_{-3} + \gamma_2 V_{-1} - \beta_2 V_{-2}
\dot{V}_1 = \beta_{1x} + \beta_1 (V_{22} - V_{11}) + (a_1 - \alpha_2) V_1 + \delta_1 V_{-2} - V_3 \gamma_2
\dot{V}_2 = \gamma_{1x} + \gamma_1 (V_{33} - V_{22}) + (a_2 - \alpha_3) V_2 + \delta_1 V_{-1} - \beta_2 V_3
\dot{V}_3 = \delta_{1x} + \delta_1 (V_{33} - V_{11}) + (a_1 - \alpha_3) V_3 + \beta_1 V_2 - \gamma_1 V_1$$
(10)

Let us note that the vector field defined by the first bi-vector P_0 are tangent to the affine hyperplanes $V_3 = V_{30}$ (where V_{30} is a given periodic function); so the symplectic leaves of P_0 are affine hyperplanes.

Since $\dot{V}_3 = 0$, from the Poisson tensor P_0 , let us choose $V_3 = 1$, so that

$$S = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_{1}e_{12} + V_{-1}e_{21} + e_{13} + V_{-3}e_{31} + V_{2}e_{23} + V_{-2}e_{32}$$
 (11)

The kernel of P_0 is formed by the covectors with

$$\delta_1 = \beta_1 = \gamma_1 = 0$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

$$\text{along with } \beta_2 = \gamma_2 \text{ and } V_1 + V_2 = 0$$

$$(12)$$

Now the flows given by the second Poisson tensor suggest that the distribution D is spanned by the following vector fields,

$$\dot{V}_{11} = -\beta_2 V_1 - \delta_2
\dot{V}_{22} = \beta_2 V_1 - \gamma_2 V_2
\dot{V}_{33} = \delta_2 + \gamma_2 V_2
\dot{V}_{-1} = \beta_{2x} + \beta_2 (V_{11} - V_{22}) - \delta_2 V_2
\dot{V}_{-2} = \gamma_{2x} + \gamma_2 (V_{22} - V_{33}) + \delta_2 V_1
\dot{V}_{-3} = \delta_{2x} + \delta_2 (V_{11} - V_{33}) + \gamma_2 V_{-1} - \beta_2 V_{-2}
\dot{V}_1 = -\gamma_2
\dot{V}_2 = \beta_2$$
(13)

So from these equations we obtain the elements of the matrix V,

$$\beta_2 = \dot{V}_2
\gamma_2 = -\dot{V}_1
\delta_2 = V_{33} + V_1 V_2$$
(14)

By using equation (13) in (14), we obtain

$$(V_{22} - V_2V_1) = 0$$

So we get an invariant functional of S, viz

$$U_0 = V_{22} - V_2 V_1 \tag{15}$$

Similarly we obtain, after a laborious computation, the other three invariants, viz.

$$U_{1} = V_{2}(V_{22} - V_{11}) + V_{-1} - V_{2}^{2}V_{1} - V_{2x}$$

$$U_{2} = V_{1}(V_{11} + 2V_{22}) + V_{-2} - V_{1}^{2}V_{2} + V_{1x}$$

$$U_{3} = -V_{11}V_{33} + \frac{1}{4}(V_{22} + 6V_{1}V_{2})V_{22} - \frac{3}{4}V_{1}^{2}V_{2}^{2}$$

$$+ V_{1}V_{-1} + V_{2}V_{-2} + V_{-3} + V_{11x} + \frac{1}{2}V_{22x} - \frac{1}{2}V_{2}V_{1x} - \frac{1}{2}V_{1}V_{2x}$$

$$(16)$$

These invariants closely resemble those found in ref. (9) in the discussion of the twisted version of the W_3^2 algebra. Geometrically speaking, U_0 , U_1 , U_2 , U_3 are the final variables of the quotient space N = S/E which is the space of functions on S^1 and equations (15) and (16) give the projection $\pi: S \to N$. These four invariants turn out to be the generators of the W_3^2 algebra because their Poisson brackets yield,

$$\begin{aligned}
\{U_{0}(x), U_{0}(y)\} &= -\frac{2}{3}\delta'(x - y) \\
\{U_{0}(x), U_{1}(y)\} &= U_{1}(x)\delta(x - y) \\
\{U_{0}(x), U_{2}(y)\} &= -U_{2}(x)\delta(x - y) \\
\{U_{1}(x), U_{2}(y)\} &= -\delta'(x - y) + 3U_{0}(x)\delta(x - y) + \{U_{3}(x) + \frac{3}{2}U'_{0}(x) - 3U_{0}^{2}(x)\}\delta(x - y) \\
\{U_{3}(x), U_{0}(y)\} &= -U_{0}(x)\delta'(x - y) \\
\{U_{3}(x), U_{1}(y)\} &= -\frac{3}{2}U_{1}(x)\delta'(x - y) - \frac{1}{2}U'_{1}(x)\delta(x - y) \\
\{U_{3}(x), U_{2}(y)\} &= -\frac{3}{2}U_{2}(x)\delta'(x - y) - \frac{1}{2}U'_{2}(x)\delta(x - y) \\
\{U_{3}(x), U_{3}(y)\} &= \frac{1}{2}\delta'''(x - y) - 2U_{3}(x)\delta'(x - y) - U'_{3}(x)\delta(x - y)
\end{aligned} \tag{17}$$

The Poisson brackets (17) correspond to the reduction of the second Poisson tensor P_1 . To obtain these Poisson brackets we use the fact that the fundamental Poisson brackets between the different V_i 's are isomorphic to the Lie commutation relations with a central extension, and are given by

$$\{V_a(z), V_b(z')\} = f_{abc}V_c(z)\delta(z - z') - k(T^a, T^b)\delta'(z - z')$$
(18)

where

$$S(z) = V_a(z)T^a (19)$$

and T^a denotes the generators of the Lie algebra sl(3) with commutation relations

$$[T^a, T^b] = f_{abc}T^c (20)$$

This fundamental Poisson bracket is, in turn, derived from the basic definition,

$$\{V_a(z), V_b(z)\} = ([dV_a, \partial + S], dV_b)$$
 (21)

where S is the symplectic leaf containing the different V_i 's as its entries.

As a simple exercise, we calculate $\{V_{-1}(x), V_{-2}(y)\}$. We obtain

$$dV_{-1} = \delta V_{-1}(x)/\delta S(z) = e_{12}\delta(x-z)$$

and

$$dV_{-2} = \delta V_{-2}(z)/\delta S(y) = e_{23}\delta(z - y)$$
(22)

After using the expression for S given in (11), we get $\{V_{-1}(x), V_{-2}(y)\} = -V_{-3}(x)\delta(x-y)$. Exactly the same result is obtained on using (18). Finally, we calculate one Poisson bracket from the set (17) explicitly. We have

$$\begin{aligned}
\{U_{0}(x), U_{0}(y)\} &= \{V_{22}(x) - V_{2}(x)V_{1}(x), V_{2}(y) - V_{2}(y)V_{1}(y)\} \\
&= \{V_{22}(x), V_{22}(y)\} - \{V_{22}(x), V_{2}(y)\}V_{1}(y) - V_{2}(y)\{V_{22}(x), V_{1}(y)\} - V_{-2}(x)\{V_{1}(x), V_{22}(y) - \{V_{2}(x), V_{22}(y)\}V_{1}(x) + V_{2}(x)V_{1}(y)\{V_{1}(x), V_{2}(y)\} + V_{1}(x)V_{1}(y)\{V_{2}(x), V_{2}(y)\} + V_{2}(y)V_{1}(x)\{V_{2}(x), V_{1}(y)\} + V_{2}(x)V_{2}(y)\{V_{1}(x), V_{1}(y)\} \\
&= \{V_{22}(x), V_{22}(y)\}
\end{aligned} \tag{23}$$

after cancelling several terms in pairs using the antisymmetry of the Poisson brackets, whence

$$\{U_0(x), U_0(y)\} = -k\delta'(x - y)$$

= $-\frac{2}{3}\delta'(x - y)$, choosing $k = \frac{2}{3}$ (24)

The above discussion shows how the Poisson brackets (17) are obtained and thus the classical W_3^2 algebra is derived. Thus through a rather new choice of the constant matrix A of the first Poisson tensor P_0 we have deduced the classical W_3^2 algebra. Our choice of the symplectic leaf is further justified by the discussion in ref. (10). For comparison we can mention in short the case of W_3 algebra. Here the symplectic leaf is considered to be

$$S = V_{11}(e_{11} - e_{33}) + V_1e_{12} + V_{-1}e_{21} + V_3e_{13} + V_{-3}e_{31} + V_2e_{23} + V_{-2}e_{32}$$
 (25)

where $V_1 = V_2 = 1$ and $V_3 = 0$ is the required condition. Further

$$A = e_{31} \tag{26}$$

The covector V is found to be

$$V = \frac{\alpha}{2}(e_{11} - e_{33}) + \beta_1 e_{12} + \beta_2 e_{21} + \delta_1 e_{13} + \delta_2 e_{31} + \gamma_1 e_{23} + \gamma_2 e_{32}$$
 (27)

Proceeding as before we get two invariants, viz.

$$U_{1} = V_{11}^{2} + V_{-1} + V_{-2} + 2V_{11x}$$

$$U_{0} = V_{11}(V_{-1} - V_{-2}) + V_{-3} + V_{11}V_{11x} + V_{11xx} + V_{-1x},$$
(28)

instead of four, as in the case of W_3^2 algebra. The algebra generated by U_1 and U_0 is found to be the W_3 algebra of Zamolodchikov. Finally we may mention again that the difference actually comes from the fact that in case of W_3 , "A" belongs to the centre of the strictly lower triangular matrices, while in case of W_3^2 it is itself a strictly lower triangular matrix.

3. Co-adjoint Invariance

After our derivation of W_3^2 from the bi-Hamiltonian framework we can compare our results with those obtained in the gauge transformation frame-work. This method actually generates the W-algebra via the co-adjoint action invariance of certain functionals. Such an approach was used in ref. (8) to deduce the Lie-Poisson structure on the dual of the Virasoro algebra, the underlying algebra being the sl(3,c) Kac-Moody algebra on S^1 . We now briefly comment on the results in case of sl(3,c) leading to W_3^2 . It is now well-known that if G is an affine Lie group and g its Lie algebra then the dual space g^* of g is defined as the space of linear functionals of g. The coadjoint action is given by the formulae,

$$ad_{(Y,\mu)}^*(v,k) = ([Y,v] + kY,0)$$
(29)

$$Ad_{(\phi,\mu)}^*(v,k) = (\phi v \phi^{-1} + k \phi' \phi^{-1}, k)$$
(30)

where (v(x), k) belongs to the dual space. In the case of sl(3, c) algebra, the phase space points are specified as,

$$v(x) = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_{1}e_{12} + V_{-1}e_{21} + V_{3}e_{13} + V_{-3}e_{31} + V_{2}e_{23} + V_{-2}e_{32}$$
 (31)

We put the constraint $V_3 = 1$. The maximal co-adjoint action which does not change this constraint is given by (30) with ϕ given as

$$\phi = e_{11} + e_{22} + e_{33} + Ae_{21} + Be_{31} + Ce_{32}, \text{ that is, } Ad^*_{(\phi,\mu)}(v,k) = (\bar{v},k).$$
 (32)

Simple algebra gives

$$A = \bar{V}_2 - V_2$$
; $B = V_{11} - \bar{V}_{11} - \bar{V}_1(\bar{V}_2 - V_2)$; $C = V_1 - \bar{V}_1(\bar{V}_2 - V_2)$

and we also obtain that

$$V_{22} - V_2 V_1 = \bar{V}_{22} - \bar{V}_2 \bar{V}_1 V_2 (V_{22} - V_{11}) - V_2^2 V_1 + V_{-1} - V_{2x} = \bar{V}_2 (\bar{V}_{22} - \bar{V}_{11}) - \bar{V}_2^2 \bar{V}_1 + \bar{V}_{-1} - \bar{V}_{2x}$$
(33)

and so on. The upshot is that we get back the four quantities U_0 U_1,U_2 , and U_3 as the invariants of the co-adjoint action whereas the bi-Hamiltonian approach suggests that they are invariants of the flow. This can be seen to be related to the fact that we actually construct the dynamics via the co-adjoint action.

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