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# Exponential Bounds for Continuum Eigenfunctions of N-Body Schrödinger Operators

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*Abstract.* For any non-threshold bound state of an  $N$ -body quantum system, we give a non-isotropic exponential bound in the form of a geodesic distance associated with a suitably modified Agmon metric.

## 1 Introduction

Eigenfunctions of typical  $N$ -body Schrödinger operators decay exponentially in all directions of the configuration space, provided the energy is not a threshold [4]. The rate of decay depends on the direction and is not known in general. – Using the isotropic upper bound due to Froese and Herbst in Agmon's approach, we obtain an improved non-isotropic bound in the form of a geodesic distance. Our result provides a generalization of Agmon's well-known result to continuum eigenfunctions with non-threshold energy.

Consider a system of  $N$  quantum particles in  $\mathbf{R}^3$  interacting by two-body potentials which decay pointwise to zero as the interparticle distance increases. Let  $H$  denote the Schrödinger operator of the system with center-of-mass motion removed, and suppose  $\psi$  is an eigenfunction of  $H$  with energy  $E$ . If  $E$  is discrete then a well-known theorem of Agmon tells us that

$$|\psi(x)| \leq C_\varepsilon e^{-(1-\varepsilon)\rho_E(x)} \quad \forall \varepsilon > 0 , \quad (1.1)$$

where  $\rho_E(x)$  denotes the geodesic distance from  $x$  to the origin w.r.t. the metric  $ds^2 = 2(\Sigma_x - E)dx^2$  [1]. Here  $\Sigma_x \in [\inf \sigma_{ess}(H), 0]$  is a threshold and  $dx^2$  depends on the masses.

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For  $E$  in the continuum  $\rho_E(x)$  is not defined anymore because  $\Sigma_x - E$  is then negative in some directions  $x/|x|$ .

In the present work we derive a bound similar to (1.1) for arbitrary eigenvalues in the case where an isotropic exponential bound is *a priori* given. The precise assumption is that

$$e^{(1-\varepsilon)\alpha|x|}\psi \in L^2 \quad \forall \varepsilon > 0 \quad (1.2)$$

for some  $\alpha > 0$ . Using this and the method of proof for (1.1) we arrive at a non-isotropic bound  $\rho_{E,\alpha}$ , which, after the substitution

$$\Sigma_x \rightarrow \tilde{\Sigma}_x = \max(\Sigma_x, E + \alpha^2/2) \quad (1.3)$$

in Agmon's metric, is defined in the same way as  $\rho_E$ . Our bound  $\rho_{E,\alpha}$  thus improves on  $\alpha|x|$  in directions where  $\Sigma_x - E > \frac{1}{2}\alpha^2$  and coincides with it elsewhere. If  $E$  is a discrete eigenvalue then  $\alpha = 0$  in (1.2) is admissible as well and  $\rho_{E,\alpha=0} \equiv \rho_E$ . To justify our assumption we recall that (1.2) for non-threshold eigenvalues follows from a well-known theorem due to Froese and Herbst, obtained under a further decay assumption on the potentials [4] (see also [7, 5, 6]). This theorem says that  $E + \frac{1}{2}\alpha^2$  is a threshold (or infinite), like  $\Sigma_x$  by the way, if  $\alpha$  is the largest constant for which  $\psi$  obeys (1.2).

Similar results were previously obtained by Perry and Derezhinsky [8, 3]. Perry studied polynomially bounded solutions of the Schrödinger equation  $H\psi = E\psi$ , i.e.  $\psi \in L^2_{-s}(\mathbf{R}^n)$  for some  $s > 0$  rather than  $\psi \in L^2(\mathbf{R}^n)$ , and he obtained that  $e^{(1-\varepsilon)\rho}\psi \in L^2_{-s}$ , where  $\rho = \rho_{E,\alpha=0}$  in our notation. Derezhinsky starts from an eigenstate which has an exponential bound  $g$  in a region bounding a cone in the configuration space. He then obtains an exponential bound for the eigenfunction in the cone which involves a geodesic distance as well as the function  $g$ .

## 2 Notations and Result

We work in the frame of generalized  $N$ -body quantum theory as presented for instance in [7, 5, 6].

An  $N$ -body quantum system is characterized by a triple  $(X, L, V)$ , where  $X$  is a finite dimensional Euclidean space,  $L$  a finite family of subspaces of  $X$ , and  $V$  a potential in  $X$ . The family  $L$  contains  $\{0\}$  and  $X$ , is closed under intersection, and the potential  $V$  has for each  $a \in L$  a decomposition

$$V(x) = V^a(\pi^a x) + I_a(x) \quad (2.4)$$

into a potential  $V^a$ , depending only on the orthogonal projection  $\pi^a x$  of  $x$  onto  $a^\perp$ , and an intercluster potential  $I_a$  which is subject to decay assumptions. For our purpose the following properties are convenient and sufficient:

- (1)  $V \in L^1_{\text{loc}}(X)$  and  $V_-$  is  $-\Delta/2$  form-bounded with bound smaller than 1.
- (2)  $I_a(x) \rightarrow 0 \quad |x|_a \rightarrow \infty$

Here  $\Delta$  denotes the Laplace-Beltrami operator with respect to the metric  $g(x, y) = xy$  (inner product) in  $X$ ,  $V_-(x) := \max(-V(x), 0)$ , and  $|x|_a := \min_{b \neq a} |x^b|$ . (1) and (2) ensure that the decomposition (2.4) is unique, and that  $V^a \circ \pi^a$  has again property (1) in  $X$ .

The Hamiltonian of the system is formally given by

$$H = -\frac{1}{2}\Delta + V \quad \text{in } L^2(X) ,$$

and in this paper defined as the unique self-adjoint operator associated with the closure of the form  $\int dx \left( \frac{1}{2}|\nabla \varphi(x)|^2 + V(x)|\varphi(x)|^2 \right)$  on  $C_0^\infty(X)$ . The cluster decomposition Hamiltonians  $H_a = -\Delta/2 + V^a \circ \pi^a$  are defined analogously. We set  $\Sigma := \inf \sigma_{ess}(H)$  and  $\Sigma_a := \inf \sigma(H_a)$ . The function  $\Sigma_x$  introduced above then equals  $\Sigma_{m(x)}$  where  $m(x) := \bigcap_{b \in L: x \in b} b$ .

**Theorem 2.1** *Suppose  $H\psi = E\psi$  and  $e^{(1-\varepsilon)\alpha|x|}\psi \in L^2(X)$  for all  $\varepsilon > 0$ , where  $E < 0$  and  $\alpha > 0$ , or  $E < \Sigma$  and  $\alpha \geq 0$ . Then*

$$e^{(1-\varepsilon)\rho_{E,\alpha}}\psi \in L^2(X) \quad \forall \varepsilon > 0 ,$$

where  $\rho_{E,\alpha}$ , after the substitution  $\Sigma_a \rightarrow \tilde{\Sigma}_a := \max(\Sigma_a, E + \frac{1}{2}\alpha^2)$  in the metric, is defined in the same way as Agmon's bound  $\rho_E$ .

*Remarks.* (1) Our proof employs an approximation argument which requires a non-trivial isotropic exponential bound. This is the reason for the condition  $\alpha > 0$  in the case  $E \geq \Sigma$ . If  $E < \Sigma$  one has the bound originally due to O'Connor, which, incidentally, is also needed in proofs of Agmon's result [1, 7].

(2) A pointwise bound like the one in (1.1) immediately follows from the theorem if one has a subsolution estimate [2, 1]. To prove such an estimate slightly stronger assumptions on  $V_-$  are sufficient (see [1, Theorem 5.1]).

Here we only sketch the idea of the proof. The details may be found in [5]. We shall call  $f$  an *exponential bound (of  $\psi$ )* if  $e^{(1-\varepsilon)f}\psi \in L^2(X)$  for all  $\varepsilon > 0$ . Our main tool to obtain exponential bounds is the following lemma.

**Lemma 2.2** *Suppose  $H\psi = E\psi$ ,  $f, J \in C^\infty(X)$ ,  $J, \nabla J$  and  $\nabla f$  are bounded, and  $f \geq 0$ . Then*

$$J \left( H - \frac{1}{2}|\nabla f|^2 - E \right) J \geq \delta J^2$$

for some  $\delta > 0$  implies

$$\|Je^f\psi\| \leq \text{const} \|\chi(x \in \text{supp}(\nabla J))e^f\psi\| .$$

The constant depends on  $\delta, J, \nabla J$  and  $\nabla f$ .

Using this lemma with  $J$  being a smoothed characteristic function of the complement of cones containing the subspaces  $a \in L$  for which  $\Sigma_a \leq E + \frac{1}{2}\alpha^2$ , we show that  $f \geq 0$  is an exponential bound if

$$|\nabla f(x)|^2 \leq 2(\tilde{\Sigma}_a - E) \quad |x^a| \leq \eta|x|, |x| \geq 1 \quad (2.5)$$

for all  $a \in L$  and some  $\eta > 0$ . The condition (2.5) allows us to establish the assumption of the lemma for  $(1-\varepsilon)f$ , and furthermore it ensures that  $f(x) \leq \alpha|x| + \text{const}$  in  $\{x | J(x) \neq 1\}$  by choice of  $J$ . Therefore  $(1-J)e^{(1-\varepsilon)f}\psi \in L^2$  by assumption (1.2) and hence  $Je^{(1-\varepsilon)f}\psi \in L^2$  by the lemma. The theorem now follows by an approximation argument given in [7].

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