Zeitschrift: Helvetica Physica Acta

Band: 70 (1997)

Heft: 6

Artikel: Exponential bounds for continuum eigenfunctions of n-body

Schrödinger operators

Autor: Griesemer, M.

DOI: https://doi.org/10.5169/seals-117057

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 24.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Exponential Bounds for Continuum Eigenfunctions of N-Body Schrödinger Operators

By M. Griesemer*

Institut für Theoretische Physik, ETH Hönggerberg, CH-8093 Zürich

(24.XII.1996)

Abstract. For any non-threshold bound state of an N-body quantum system, we give a non-isotropic exponential bound in the form of a geodesic distance associated with a suitably modified Agmon metric.

1 Introduction

Eigenfunctions of typical N-body Schrödinger operators decay exponentially in all directions of the configuration space, provided the energy is not a threshold [4]. The rate of decay depends on the direction and is not known in general. – Using the isotropic upper bound due to Froese and Herbst in Agmon's approach, we obtain an improved non-isotropic bound in the form of a geodesic distance. Our result provides a generalization of Agmon's well-known result to continuum eigenfunctions with non-threshold energy.

Consider a system of N quantum particles in \mathbf{R}^3 interacting by two-body potentials which decay pointwise to zero as the interparticle distance increases. Let H denote the Schrödinger operator of the system with center-of-mass motion removed, and suppose ψ is an eigenfunction of H with energy E. If E is discrete then a well-known theorem of Agmon tells us that

$$|\psi(x)| \le C_{\varepsilon} e^{-(1-\varepsilon)\rho_E(x)} \qquad \forall \varepsilon > 0 ,$$
 (1.1)

where $\rho_E(x)$ denotes the geodesic distance from x to the origin w.r.t. the metric $ds^2 = 2(\Sigma_x - E)dx^2$ [1]. Here $\Sigma_x \in [\inf \sigma_{ess}(H), 0]$ is a threshold and dx^2 depends on the masses.

^{*}Address as of February 1, 97: Matematisk institutt, Universitetet i Oslo, Postboks 1053, N-0316 Oslo

For E in the continuum $\rho_E(x)$ is not defined anymore because $\Sigma_x - E$ is then negative in some directions x/|x|.

In the present work we derive a bound similar to (1.1) for arbitrary eigenvalues in the case where an isotropic exponential bound is a priory given. The precise assumption is that

$$e^{(1-\epsilon)\alpha|x|}\psi \in L^2 \qquad \forall \epsilon > 0$$
 (1.2)

for some $\alpha > 0$. Using this and the method of proof for (1.1) we arrive at a non-isotropic bound $\rho_{E,\alpha}$, which, after the substitution

$$\Sigma_x \to \tilde{\Sigma}_x = \max(\Sigma_x, E + \alpha^2/2)$$
 (1.3)

in Agmon's metric, is defined in the same way as ρ_E . Our bound $\rho_{E,\alpha}$ thus improves on $\alpha|x|$ in directions where $\Sigma_x - E > \frac{1}{2}\alpha^2$ and coincides with it elsewhere. If E is a discrete eigenvalue then $\alpha = 0$ in (1.2) is admissible as well and $\rho_{E,\alpha=0} \equiv \rho_E$. To justify our assumption we recall that (1.2) for non-threshold eigenvalues follows from a well-known theorem due to Froese and Herbst, obtained under a further decay assumption on the potentials [4] (see also [7, 5, 6]). This theorem says that $E + \frac{1}{2}\alpha^2$ is a threshold (or infinite), like Σ_x by the way, if α is the largest constant for which ψ obeys (1.2).

Similar results were previously obtained by Perry and Derezinsky [8, 3]. Perry studied polynomially bounded solutions of the Schrödinger equation $H\psi = E\psi$, i.e. $\psi \in L^2_{-s}(\mathbf{R}^n)$ for some s>0 rather than $\psi \in L^2(\mathbf{R}^n)$, and he obtained that $e^{(1-\varepsilon)\rho}\psi \in L^2_{-s}$ where $\rho=\rho_{E,\alpha=0}$ in our notation. Derezinsky starts from an eigenstate which has an exponential bound g in a region bounding a cone in the configuration space. He then obtains an exponential bound for the eigenfunction in the cone which involves a geodesic distance as well as the function g.

2 Notations and Result

We work in the frame of generalized N-body quantum theory as presented for instance in [7, 5, 6].

An N-body quantum system is characterized by a triple (X, L, V), where X is a finite dimensional Euclidean space, L a finite family of subspaces of X, and V a potential in X. The family L contains $\{0\}$ and X, is closed under intersection, and the potential V has for each $a \in L$ a decomposition

$$V(x) = V^{a}(\pi^{a}x) + I_{a}(x)$$
(2.4)

into a potential V^a , depending only on the orthogonal projection $\pi^a x$ of x onto a^{\perp} , and an intercluster potential I_a which is subject to decay assumptions. For our purpose the following properties are convenient and sufficient:

- (1) $V \in L^1_{loc}(X)$ and V_- is $-\Delta/2$ form-bounded with bound smaller than 1.
- (2) $I_a(x) \to 0$ $|x|_a \to \infty$

Here Δ denotes the Laplace-Beltrami operator with respect to the metric g(x,y) = xy (inner product) in X, $V_{-}(x) := \max(-V(x), 0)$, and $|x|_a := \min_{b \not\supset a} |x^b|$. (1) and (2) ensure that the decomposition (2.4) is unique, and that $V^a \circ \pi^a$ has again property (1) in X.

856 Griesemer

The Hamiltonian of the system is formally given by

$$H = -\frac{1}{2}\Delta + V \quad \text{in } L^2(X) ,$$

and in this paper defined as the unique self-adjoint operator associated with the closure of the form $\int dx \left(\frac{1}{2}|\nabla\varphi(x)|^2 + V(x)|\varphi(x)|^2\right)$ on $C_0^{\infty}(X)$. The cluster decomposition Hamiltonians $H_a = -\Delta/2 + V^a \circ \pi^a$ are defined analogously. We set $\Sigma := \inf \sigma_{ess}(H)$ and $\Sigma_a := \inf \sigma(H_a)$. The function Σ_x introduced above then equals $\Sigma_{m(x)}$ where $m(x) := \bigcap_{b \in L: x \in b} b$.

Theorem 2.1 Suppose $H\psi = E\psi$ and $e^{(1-\varepsilon)\alpha|x|}\psi \in L^2(X)$ for all $\varepsilon > 0$, where E < 0 and $\alpha > 0$, or $E < \Sigma$ and $\alpha \geq 0$. Then

$$e^{(1-\varepsilon)\rho_{E,\alpha}}\psi\in L^2(X) \qquad \forall \varepsilon>0 \ ,$$

where $\rho_{E,\alpha}$, after the substitution $\Sigma_a \to \tilde{\Sigma}_a := \max(\Sigma_a, E + \frac{1}{2}\alpha^2)$ in the metric, is defined in the same way as Agmon's bound ρ_E .

Remarks. (1) Our proof employs an approximation argument which requires a non-trivial isotropic exponential bound. This is the reason for the condition $\alpha > 0$ in the case $E \geq \Sigma$. If $E < \Sigma$ one has the bound originally due to O'Connor, which, incidentally, is also needed in proofs of Agmon's result [1, 7].

(2) A pointwise bound like the one in (1.1) immediately follows from the theorem if one has a subsolution estimate [2, 1]. To prove such an estimate slightly stronger assumptions on V_{-} are sufficient (see [1, Theorem 5.1]).

Here we only sketch the idea of the proof. The details may be found in [5]. We shall call f an exponential bound (of ψ) if $e^{(1-\varepsilon)f}\psi \in L^2(X)$ for all $\varepsilon > 0$. Our main tool to obtain exponential bounds is the following lemma.

Lemma 2.2 Suppose $H\psi = E\psi$, $f, J \in C^{\infty}(X)$, $J, \nabla J$ and ∇f are bounded, and $f \geq 0$. Then

$$J\left(H - \frac{1}{2}|\nabla f|^2 - E\right)J \ge \delta J^2$$

for some $\delta > 0$ implies

$$||Je^f\psi|| \le const ||\chi(x \in supp(\nabla J))e^f\psi||$$
.

The constant depends on δ , J, ∇J and ∇f .

Using this lemma with J being a smoothed characteristic function of the complement of cones containing the subspaces $a \in L$ for which $\Sigma_a \leq E + \frac{1}{2}\alpha^2$, we show that $f \geq 0$ is an exponential bound if

$$|\nabla f(x)|^2 \le 2(\tilde{\Sigma}_a - E) \qquad |x^a| \le \eta |x|, \ |x| \ge 1$$
(2.5)

for all $a \in L$ and some $\eta > 0$. The condition (2.5) allows us to establish the assumption of the lemma for $(1-\varepsilon)f$, and furthermore it ensures that $f(x) \leq \alpha |x| + \text{const}$ in $\{x|J(x) \neq 1\}$ by choice of J. Therefore $(1-J)e^{(1-\varepsilon)f}\psi \in L^2$ by assumption (1.2) and hence $Je^{(1-\varepsilon)f}\psi \in L^2$ by the lemma. The theorem now follows by an approximation argument given in [7].

Acknowledgment I thank W. Hunziker for drawing my attention to the problem studied here.

Griesemer 857

References

[1] S. Agmon, Lectures on exponential decay of solutions of second order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators. Mathematical Notes 29, Princeton University Press, University of Tokyo Press 1982

- [2] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, Schrödinger operators. Springer (1987)
- [3] J. Derezinsky, Exponential bounds in cones for eigenfunctions of N-body Schrödinger operators II. Ann. Soc. Math. Pol., Series I: Comment. Math. 30.2 (1991) 269-274.
- [4] R. Froese, I. Herbst, Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators. Commun. Math. Phys. 87 (1982) 429-447.
- [5] M. Griesemer, N-body quantum systems with hard-core interactions. ETH thesis No. 11644 (1996)
- [6] M. Griesemer, N-body quantum systems with singular potentials. ETH preprint: ETH-TH/96-28.
- [7] W. Hunziker, I. M. Sigal, The general theory of N-body quantum systems. in Mathematical quantum theory. II. Schrödinger operators, J. Feldmann et al. eds., AMS-publ. (1994)
- [8] P. Perry, A remark on continuum eigenfunctions of N-body Schrödinger operators. in Differential Equations, I.W. Knowles and R.T. Lewis (eds), Elsevier Science Publishers (1984)