

**Zeitschrift:** Helvetica Physica Acta  
**Band:** 70 (1997)  
**Heft:** 6

**Artikel:** Pull-backs and product tests  
**Autor:** Wilce, Alexander  
**DOI:** <https://doi.org/10.5169/seals-117053>

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## Pull-backs and Product Tests

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(28.X.1996)

*Abstract.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be test spaces. We study the test space  $B(\mathcal{A}, \mathcal{B})$  consisting of graphs of bijections  $f : E \rightarrow F$  between tests  $E \in \mathcal{A}$  and  $F \in \mathcal{B}$ . Elements of  $B(\mathcal{A}, \mathcal{B})$  may be interpreted as *products*, in something like the sense of Piron, of tests in  $\mathcal{A}$  and  $\mathcal{B}$ .

### Introduction

In a long series of papers (cf [2], [3], [4] and references therein), D. J. Foulis and the late C.H. Randall developed a straightforward but versatile generalized probability theory based on what are now usually called *test spaces*. In brief: A test space  $\mathcal{A}$  is simply a non-empty collection of discrete sets  $E, F, \dots$ , each thought of as the outcome-set for some measurement or *test*. When  $\mathcal{A}$  contains only one test, one recovers (discrete) classical probability theory; when it consists of the set of maximal orthonormal bases of a Hilbert space, one recovers quantum probability theory.

This note concerns the following construction: If  $\mathcal{A}$  and  $\mathcal{B}$  are test spaces, let  $B(\mathcal{A}, \mathcal{B})$  denote the set of bijections  $f : E \rightarrow F$  between tests  $E \in \mathcal{A}$  and  $F \in \mathcal{B}$ . Identifying each such a bijection with its graph,  $B(\mathcal{A}, \mathcal{B})$  may be regarded as a test space in its own right.

We propose to interpret  $B(\mathcal{A}, \mathcal{B})$  as the test space consisting of *products*, in something close to the sense of Piron [8] and Aerts [1], of tests  $E \in \mathcal{A}$  and  $F \in \mathcal{B}$ . The construction is also of interest on purely mathematical grounds. On the one hand, it preserves various standard regularity conditions on  $\mathcal{A}$  and  $\mathcal{B}$ ; on the other hand, as soon as  $\mathcal{A}$  and  $\mathcal{B}$  contain tests with more than two outcomes, the structure of  $B(\mathcal{A}, \mathcal{B})$  becomes quite rich, *even if  $\mathcal{A}$  and  $\mathcal{B}$  are classical*. Moreover, for certain categories of “uniform” test spaces,  $B(\mathcal{A}, \mathcal{B})$  is effective as the direct product of  $\mathcal{A}$  and  $\mathcal{B}$ .

In section 1, we discuss our construction in general terms. In section 2, we discuss the stability of various regularity conditions on  $\mathcal{A}$  and  $\mathcal{B}$  under passage to  $B(\mathcal{A}, \mathcal{B})$ . In particular, we show that if  $\mathcal{A}$  and  $\mathcal{B}$  are algebraic, then  $B(\mathcal{A}, \mathcal{B})$  is algebraic as well. In section 3, we characterize the logic of  $B(\mathcal{A}, \mathcal{B})$  in the case that  $\mathcal{A}$  and  $\mathcal{B}$  are algebraic.

## 1 Questions, Products and Pull-Backs

As explained above, a **test space**<sup>1</sup> is a non-empty set  $\mathcal{A}$  of non-empty sets  $E, F, \dots$ . Elements of  $\mathcal{A}$  are called **tests** and elements of  $X := \bigcup \mathcal{A}$  are called **outcomes**. The intended interpretation is that each test  $E \in \mathcal{A}$  is an exhaustive set of mutually exclusive outcomes, as, for instance, the set of outcomes of some experiment. Borrowing terminology from classical probability theory, we refer to any subset of any test  $E \in \mathcal{A}$  as an **event** of  $\mathcal{A}$ . We write  $\mathcal{E}(\mathcal{A})$  for the set of all events of  $\mathcal{A}$ .

Test spaces provide the foundation for a very natural – and conceptually uncomplicated – generalization of elementary probability theory having both classical measure-theoretic and quantum-mechanical probability as special cases. It is worth a moment to give a sketch of this. One defines a **state** on a test space  $\mathcal{A}$  to be a map  $\omega : X \rightarrow [0, 1]$  such that  $\omega(x) \geq 0$  for each  $x \in X$  and  $\sum_{x \in E} \omega(x) = 1$  for each test  $E \in \mathcal{A}$ . In other words, a state is a real-valued function on the set of outcomes that restricts to a probability weight on each test.

Note that if  $\mathcal{A}$  consists of but a single test – i.e., if  $\mathcal{A} = \{E\}$  — then a state is simply a discrete probability distribution and we recover discrete classical probability theory. In this case, we call  $\mathcal{A}$  a *classical test space*. One can also consider the test space consisting of all countable partitions of a measurable space by measurable sets; this may be called a *Kolmogorov test space*. A *quantum test space* (or *frame manual*) is the set  $\mathcal{A}$  of all orthonormal bases of a Hilbert space  $\mathbf{H}$ . The outcomes of  $\mathcal{A}$  are the unit vectors of  $\mathbf{H}$ . Gleason's theorem [5] allows us to identify the states  $\omega$  on  $\mathcal{A}$  with density operators  $W$  on  $\mathbf{H}$  via the prescription  $\omega(x) = \langle Wx, x \rangle$  (where  $x$  is a unit vector of  $\mathbf{H}$ ).

We may wish to attach numerical or other labels to the outcomes of a test. This motivates the following terminology:

**1.1 Definition:** Given a set  $V$ , we define a  **$V$ -valued question** on a test space  $\mathcal{A}$  to be a bijection<sup>2</sup>  $\alpha : E \rightarrow V$ , where  $E$  is a test belonging to  $\mathcal{A}$ . The question is *posed* by executing the test  $E$ ; its *answer* is the value  $\alpha(x) \in V$  corresponding to the secured outcomes  $x \in E$ .

Note that if  $V = \{\text{yes}, \text{no}\}$ , this corresponds to the notion of a question as defined in the work of Piron [8].

If  $\alpha : E \rightarrow V$  and  $\beta : F \rightarrow V$  are two  $V$ -valued questions, it is very natural to form their

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<sup>1</sup>called also a *manual* or *generalized sample space* in the older literature

<sup>2</sup>the condition that  $\alpha$  be bijective is benign: If not, replace  $V$  by the range of  $\alpha$  and  $E$ , by the partition  $\{\alpha^{-1}(x) \mid x \in V\}$ .

pull-back — that is, the canonical bijection  $\alpha \cdot \beta : E \times_V F \rightarrow V$  where

$$E \times_V F = \{ (x, y) \in E \times F \mid \alpha(x) = \beta(y) \}.$$

$$\begin{array}{ccc} E \times_V F & \longrightarrow & F \\ \downarrow & & \downarrow \beta \\ E & \xrightarrow{\alpha} & V \end{array}$$

More generally, given an arbitrary collection  $\{\alpha_i\}_{i \in I}$  of  $V$ -valued questions  $\alpha_i : E_i \rightarrow V$ , one can construct  $E = \{ x \in \prod_{i \in I} E_i \mid \alpha_i(x_i) = \alpha_j(x_j) \ \forall i, j \in I \}$  and set  $\Pi_i \alpha_i(x) = \Pi_j \alpha_j$  for any  $j \in I$ . (Indeed, by iterating this construction and taking a suitable direct limit, one can construct a test space that is in some sense closed under the formation of products of  $V$ -valued questions. We shall not pursue this here.)

We may interpret  $E \times_V F$  as a test, as follows: One of the tests  $E$  or  $F$  is selected. If the outcome of the selected test is, say,  $x \in E$ , then the outcome of  $E \times_V F$  is the unique pair  $(x, y) \in E \times_V F$  having  $x$  as its first component. Similarly, if the secured outcome is  $y \in F$ , the outcome of  $E \times_V F$  is the unique pair  $(x, y)$  with  $y$  as its second component. (Note that this in effect erases any record of *which* of the tests  $E$  and  $F$  was in fact selected.) To pose the question  $\alpha \cdot \beta$ , one executes  $E \times_V F$ . Upon securing, say,  $(x, y)$ , one records the value  $\alpha(x) = \beta(y)$  as answer.

As the reader familiar with [8] will have recognized, this construction is analogous to the notion of a product of yes-no questions as defined by Piron:

If  $\{\alpha_i\}$  is a family of questions, we denote by  $\Pi_i \alpha_i$  the question defined in the following manner: One measures an arbitrary one of the  $\alpha_i$  and attributes to  $\Pi_i \alpha_i$  the answer thus obtained. ([8], p. 20).

This notion makes equal sense for  $V$ -valued questions generally, and we believe our construction adequately captures it in a precise way.

The balance of this paper is devoted to a discussion of the test space consisting of tests  $E \times_V F$  arising from the formation of products of  $V$ -valued questions. This turns out to have a surprisingly rich structure. Before carrying on, it will be helpful to reformulate the definition of  $E \times_V F$  in a manner not depending explicitly upon the questions  $\alpha$  and  $\beta$ . To this end, notice that  $E \times_V F$  is simply the graph of the bijection  $\beta^{-1} \circ \alpha : E \rightarrow F$ . Conversely, given any pair of tests  $E, F \in \mathcal{A}$  and any bijection  $f : E \rightarrow F$ , we may understand  $f$  as a test corresponding to a product of  $V$ -valued questions defined on  $E$  and  $F$ , respectively. (To execute the test represented by  $f$ , one chooses  $E$  or  $F$ , executes it, and records the pair  $(x, f(x))$  or  $(f^{-1}(y), y)$  according as  $x \in E$  or  $y \in F$  is secured.)

**1.2 Definition:** For two sets  $E$  and  $F$ , we denote by  $B(E, F)$  the set of (graphs of) bijections  $f : E \rightarrow F$ , abbreviating  $B(E, E)$  to  $B(E)$ . For any two test-spaces  $\mathcal{A}, \mathcal{B}$ , we denote by  $B(\mathcal{A}, \mathcal{B})$  the collection of sets  $B(E, F)$  with  $E \in \mathcal{A}$  and  $F \in \mathcal{B}$ . We abbreviate  $B(\mathcal{A}, \mathcal{A})$  to  $B(\mathcal{A})$ .

Of course,  $B(\mathcal{A}, \mathcal{B})$  may be empty. On the other hand, as  $B(E, E) \neq \emptyset$ ,  $\mathcal{B}(\mathcal{A})$  is always rather large. Indeed, if  $\mathcal{A}$  is a totally finite test space having  $k$  operations each with  $n$  outcomes,  $B(\mathcal{A})$  has  $k^2 n!$  operations (each with  $n$  outcomes). There is a natural embedding of  $\mathcal{A}$  in  $B(\mathcal{A})$ , namely, the diagonal map  $X \rightarrow X \times X$  given by  $x \mapsto (x, x)$ . This maps each test  $E \in \mathcal{A}$  to the corresponding identity function  $\text{Id}_E$ .

In general, the set of outcomes of  $B(\mathcal{A}, \mathcal{B})$  will be smaller than  $X \times Y$  (since, e.g., there may be outcomes in the former that belong only to tests with  $n$  outcomes, and outcomes of the latter belonging only to  $k$ -outcome tests with  $k \neq n$ ). In any case, if  $(x, y)$  and  $(u, v)$  belong to  $\bigcup B(\mathcal{A}, \mathcal{B})$ , we have  $(x, y) \perp (u, v) \Rightarrow x \perp u \ \& \ y \perp v$ .

We now consider some examples.

**1.3 Example:** Suppose  $\mathcal{A}$  is a collection of pair-wise disjoint two-element sets. Then

$$B(\mathcal{A}) = \{ \{ (x, u), (y, v) \} \mid x \perp y, u \perp v \},$$

likewise a collection of pairwise-disjoint two-element sets. Notice that  $B(\mathcal{A})$  is naturally isomorphic to the set of pairs  $\{(\{x, u\}, \{y, v\}) \mid x \perp y, u \perp v\}$ , which is the model for the manual of product questions given by Foulis, Piron and Randall in [4].

Once we admit test spaces having operations with more than two outcomes, the structure of  $B(\mathcal{A})$  becomes quite involved. This is nicely illustrated even by the simplest example:

**1.4 Example:** Consider the hypergraph  $\mathcal{A} = \{E\}$  consisting of a single three-outcome experiment  $E = \{x, y, z\}$ . Then  $B(\mathcal{A}) = B(E)$  is isomorphic to the three-by-three “window” manual:

$$\begin{array}{ccccc} (x, x) & - & - & - & (y, z) & - & - & - & (z, y) \\ | & & & & | & & & & | \\ (y, y) & - & - & - & (z, x) & - & - & - & (x, z) \\ | & & & & | & & & & | \\ (z, z) & - & - & - & (x, y) & - & - & - & (y, x) \end{array}$$

As  $B(E)$  contains four-loops but no three-loops, its logic is an orthomodular poset, but not an orthomodular lattice ([7]). The state-space of  $B(E)$  is in effect the convex set of doubly stochastic  $3 \times 3$  matrices.

**1.5 Example:** Consider a Hilbert space  $\mathbf{H}$  (of any dimension, over any field) and let  $\mathcal{A}$  be the associated quantum test space, i.e., the set of all (un-ordered) orthonormal bases of  $\mathbf{H}$ . Every bijection  $f : E \rightarrow F$  between two bases  $E, F \in \mathcal{A}$  extends uniquely to a unitary operator on  $\mathbf{H}$ . If  $U$  is such an operator, its graph is a closed subspace of  $\mathbf{H} \times \mathbf{H}$ , and hence a Hilbert space in its own right. An orthonormal basis for  $U$  is simply the graph of  $U|_E$  for some  $E \in \mathcal{A}$ . Hence,  $B(\mathcal{A})$  is just the union over all unitaries  $U$ , of the frame manuals of the corresponding subspaces  $U \leq \mathbf{H} \times \mathbf{H}$ . (It is interesting to note that the set of graphs of unitaries on  $\mathbf{H}$  constitutes a *partial Hilbert space* in the sense of Gudder [6].)

We now consider a restricted class of test spaces for which the construction  $\mathcal{A}, \mathcal{B} \mapsto B(\mathcal{A}, \mathcal{B})$  behaves in a particularly satisfactory manner.

**1.6 Definition:** Let  $\kappa$  be any cardinal. A test space  $\mathcal{A}$  is  $\kappa$ -uniform iff every test  $E \in \mathcal{A}$  has cardinality  $\kappa$ .

If  $\mathcal{A}$  and  $\mathcal{B}$  are both  $\kappa$ -uniform, then  $Z = X \times Y$  and, in this case,  $(x, y) \perp (u, v)$  iff  $x \perp u$  and  $y \perp v$ . The class of  $\kappa$ -uniform test spaces is large enough to include both classical test spaces  $\mathcal{A} = \{E\}$  with  $\#(E) = \kappa$  and also the frame manual of any Hilbert space of dimension  $\kappa$ . Notice also that if  $\mathcal{A}$  and  $\mathcal{B}$  are both  $\kappa$ -uniform, then so also is  $B(\mathcal{A}, \mathcal{B})$ . In fact, as we shall now see,  $B(\mathcal{A}, \mathcal{B})$  serves as the direct product of uniform test spaces, provided we define our morphisms correctly.

**1.7 Definition:** By a **uniform map** between two test spaces  $\mathcal{A}$  and  $\mathcal{B}$  with outcome-sets  $X$  and  $Y$ , respectively, we mean a function  $\phi : X \rightarrow Y$  such that  $\phi(\mathcal{A}) \subseteq \mathcal{B}$  and  $x_1 \perp x_2 \Rightarrow \phi(x_1) \perp \phi(x_2)$  for all  $x_i \in X$ . (In the language of [4]: a uniform map is a *positive, outcome-preserving interpretation*.)

Note that if  $\phi$  is a uniform map, then  $\phi$  is locally bijective, in that for every  $E \in \mathcal{A}$ ,  $\phi|_E : E \rightarrow \phi(E) \in \mathcal{B}$  is a bijection.

**1.8 Theorem:** Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $\kappa$ -uniform. Then  $B(\mathcal{A}, \mathcal{B})$  is the direct product of  $\mathcal{A}$  and  $\mathcal{B}$  in the category of uniform test spaces and uniform maps.

Proof: Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $\kappa$ -uniform test spaces with  $\bigcup \mathcal{A} = X$  and  $\bigcup \mathcal{B} = Y$ . Note that  $B(\mathcal{A}, \mathcal{B})$  is again  $\kappa$ -uniform, and that  $\bigcup B(\mathcal{A}, \mathcal{B}) = X \times Y$ . Let  $\pi_1$  and  $\pi_2$  be the projections of  $X \times Y$  onto  $X$  and  $Y$ , respectively. If  $(x, y) \perp (u, v) \in \bigcup B(\mathcal{A}, \mathcal{B})$ , then  $x \perp u$  and  $y \perp v$ , so  $\pi_i(x, y) \perp \pi_i(u, v)$  for  $i = 1, 2$ . If  $f \in B(\mathcal{A}, \mathcal{B})$ , then  $\pi_1(f) = \text{dom}(f) \in \mathcal{A}$ ; similarly,  $\pi_2(f) = \text{ran}(f) \in \mathcal{B}$ . Thus, both projections are uniform maps. It now suffices to show that if  $\mathcal{C}$  is a  $\kappa$ -uniform test space with  $\bigcup \mathcal{C} = Z$ , and  $\phi : Z \rightarrow X$  and  $\psi : Z \rightarrow Y$  are uniform maps, then  $\phi \times \psi : Z \rightarrow (X \times Y)$  is an uniform map. If  $z \perp w$ , then  $\phi(z) \perp \phi(w)$  and  $\psi(z) \perp \psi(w)$ ; hence,  $(\phi \times \psi)(z) \perp (\phi \times \psi)(w)$ . Now suppose  $E \in \mathcal{C}$ . We must show that  $(\phi \times \psi)(E)$  belongs to  $B(\mathcal{A}, \mathcal{B})$ . Because  $\phi|_E$  is a bijection, we have

$$(\phi \times \psi)(E) = \{ (\phi(z), \psi(z)) \mid z \in E \} = \{ (x, \psi(\phi^{-1}(x))) \mid x \in \phi(E) \}.$$

That is,  $(\phi \times \psi)(E) = \psi \circ (\phi|_E)^{-1} : \phi(E) \rightarrow \psi(F)$ . Since  $\psi|_F$  is bijective, this last belongs to  $B(\mathcal{A}, \mathcal{B})$ .

## 2 The Structure of $B(\mathcal{A}, \mathcal{B})$

In this section, we establish (Theorems 2.2, 2.3 and 2.5) that passage from  $\mathcal{A}$  and  $\mathcal{B}$  to  $B(\mathcal{A}, \mathcal{B})$  preserves each of three standard conditions often imposed on test spaces: That of being algebraic, that of being coherent (though here we need an additional uniformity assumption), and that of being regular.

Throughout this section, let  $\mathcal{A}$  and  $\mathcal{B}$  be test spaces with outcome-sets  $X$  and  $Y$ , respectively. As noted above, the outcome-set of  $B(\mathcal{A}, \mathcal{B})$  is in general a *proper* (possibly



empty) subset of  $Z \subseteq X \times Y$ . An event for  $B(\mathcal{A}, \mathcal{B})$  is any subset of the graph of a bijection  $f : E \rightarrow F$  with  $E \in \mathcal{A}$  and  $F \in \mathcal{B}$ . Evidently, any such subset is the graph of a bijection between two events  $A \subseteq E$  and  $B \subseteq F$ . Thus,

$$\mathcal{E}(B(\mathcal{A}, \mathcal{B})) \subseteq B(\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{B})).$$

Again, the inclusion is generally proper – indeed, it is easy to see we have identity iff  $\mathcal{A}$  and  $\mathcal{B}$  are  $n$ -uniform for some finite  $n$ .

Events  $A$  and  $B$  of a test space  $\mathcal{A}$  are said to be *complementary* – the short-hand is  $A \subset B$  – iff  $A \cap B = \emptyset$  and  $A \cup B \in \mathcal{A}$ . If  $A$  and  $B$  are both complementary to a common third event, one says that  $A$  and  $B$  are *perspective*, writing  $A \sim B$ . A test space is a *algebraic* (in the older literature, a *manual*) iff, given any events  $A, B$  and  $C$ ,  $A \sim B$  and  $B \subset C$  imply  $A \subset C$ .

**2.1 Lemma:** Let  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  be bijections belonging to  $\mathcal{E}(B(\mathcal{A}, \mathcal{B}))$ . Then

- (1)  $f \subset g$  iff  $A \subset B$  and  $A' \subset B'$ .
- (2)  $f \sim g$  iff  $A \sim B$  and  $A' \sim B'$ .

Proof: Note that (2) is an immediate consequence of (1). To establish (1), suppose  $A \subset B$  and  $A' \subset B'$ . Then  $f \cap g = \emptyset$  and  $f \cup g \in B(A \cup B, A' \cup B') \subseteq B(\mathcal{A}, \mathcal{B})$ ; thus,  $f \subset g$ . Conversely, if  $f \subset g$ , then  $f \cap g = \emptyset$  and  $f \cup g \in B(E, F)$  for some  $E \in \mathcal{A}, F \in \mathcal{B}$ . But then  $A \cup B = E \in \mathcal{A}$  and, as  $f \cup g$  is again a bijection, we must have  $A \cap B = \emptyset$  – whence,  $A \subset B$ . Also,  $A' \cup B' = f(A) \cup g(B) = F \in \mathcal{B}$ , and, again because  $f \cup g$  is a bijection,  $A' \cap B' = \emptyset$ , so  $A' \subset B'$ .

**2.2 Theorem:** If  $\mathcal{A}$  and  $\mathcal{B}$  are algebraic, then  $B(\mathcal{A}, \mathcal{B})$  is likewise algebraic. If the test space  $B(\mathcal{A})$  is algebraic, then  $\mathcal{A}$  is algebraic.

Proof: Suppose that  $f : A \rightarrow A', g : B \rightarrow B'$  in  $\mathcal{E}(B(\mathcal{A}, \mathcal{B}))$  with  $f \sim g$  and  $g \subset h : C \rightarrow C'$ . By Lemma 1,  $A \sim B \subset C$  and  $A' \sim B' \subset C'$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are algebraic, it follows that  $A \subset C$  and  $A' \subset C'$ . But then  $f \subset h$  by Lemma 2.1. Thus,  $B(\mathcal{A}, \mathcal{B})$  is algebraic. if  $B(\mathcal{A})$  is algebraic and  $A \subset C \subset B \subset D$  in  $\mathcal{E}(\mathcal{A})$ , then  $\text{Id}_A \sim \text{Id}_B \subset \text{Id}_D$ , hence,  $\text{Id}_A \subset \text{Id}_D$ , whence,  $\text{Id}_A \cup \text{Id}_D = \text{Id}_{A \cup D}$  belongs to  $B(\mathcal{A})$  – whence,  $A \subset D$ , and it follows that  $\mathcal{A}$  is algebraic.

A test space  $\mathcal{A}$  is **coherent** [3, 4] iff for all events  $A$  and  $B$  of  $\mathcal{A}$ ,  $A \subseteq B^\perp \Rightarrow A \perp B$ .

**2.3 Theorem:** Let  $\mathcal{A}$  and  $\mathcal{B}$  be coherent and  $\kappa$ -uniform. Then  $B(\mathcal{A}, \mathcal{B})$  is also coherent.

Proof: Suppose  $f, g \in \mathcal{E}(B(\mathcal{A}, \mathcal{B}))$  with  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$ . Suppose  $f \subseteq g^\perp$ . Then for every  $x \in A$ ,  $(x, f(x)) \perp (y, g(y))$  for every  $y \in B$ ; hence,  $x \in B^\perp$  and (since  $g$  is surjective),  $f(x) \in B'^\perp$ . Thus,  $A \subseteq B^\perp$  and (since  $f$  is surjective)  $A' \subseteq B'^\perp$ . Since  $\mathcal{A}$  and  $\mathcal{B}$  are coherent,  $A \perp B$  and  $A' \perp B'$ . Thus,  $f \cap g = \emptyset$  and  $f \cup g \in B(\mathcal{E}(\mathcal{A}))$ . Since  $\mathcal{A}$  and  $\mathcal{B}$  are  $n$ -uniform,  $f \perp g$ . Thus,  $B(\mathcal{A}, \mathcal{B})$  is coherent.

A *support* of a test space  $\mathcal{A}$  is a set  $S \subseteq X = \bigcup \mathcal{A}$  such that for all  $E, F \in \mathcal{A}$ ,

$$E \cap S \subseteq F \Rightarrow F \cap S \subseteq E.$$

The usual heuristic is that  $S$  is the set of outcomes that are *possible* in some state of affairs. By way of example, if  $\omega$  is a (probabilistic) state on  $\mathcal{A}$ , then  $S_\omega = \{x \in X \mid \omega(x) > 0\}$  is a support of  $\mathcal{A}$ . Notice that  $X$  is a support, since test spaces are irredundant. It is straight-forward that the union of any collection of supports is a support; hence, the set of all supports of  $\mathcal{A}$  is a complete lattice under set inclusion. More details and motivation will be found in [4].

Let  $\bigcup B(\mathcal{A}, \mathcal{B}) = Z \subseteq X \times Y$ . Suppose  $S$  and  $T$  are supports of  $\mathcal{A}$ . Then we define

$$S \odot T := [X \times T \cup S \times Y] \cap Z.$$

**2.4 Lemma:** *If  $S$  and  $T$  are supports of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, then  $S \odot T$  is a support of  $B(\mathcal{A}, \mathcal{B})$ .*

Proof: Suppose  $f : E \rightarrow E'$  and  $g : F \rightarrow F'$  are operations in  $B(\mathcal{A}, \mathcal{B})$ , and that

$$f \cap (S \odot T) = \{(x, f(x)) \mid x \in E \cap S \text{ or } f(x) \in E' \cap T\} \subseteq g.$$

Then  $E \cap S \subseteq F = \text{dom}(g)$  and  $E' \cap T \subseteq F' = \text{ran}(g)$ , whence, as  $S$  and  $T$  are supports,  $E \cap S = F \cap S$  and  $E' \cap T = F' \cap T$ . Moreover,  $f|_{E \cap S} = g|_{F \cap S}$  and  $f^{-1}|_{E' \cap T} = g^{-1}|_{F' \cap T}$ . Hence,  $g \cap (S \odot T) = f \cap (S \odot T)$ . Thus,  $S \odot T$  is a support of  $B(\mathcal{A})$ .

**Remark:** If  $\mu$  is a state on  $\mathcal{A}$ , then  $\mu \circ \pi_1$  is a state on  $B(\mathcal{A}, \mathcal{B})$  (provided that the latter test space exists). Hence, given a state  $\mu$  on  $\mathcal{A}$  and a state  $\nu$  on  $\mathcal{B}$ , we may form a state

$$\mu \odot \nu := \frac{1}{2}(\mu \circ \pi_1 + \nu \circ \pi_2)$$

on  $B(\mathcal{A}, \mathcal{B})$ . It is easily checked that  $S_{\mu \odot \nu} = S_\mu \odot S_\nu$ .

A test space  $\mathcal{A}$  is **regular** iff, for every  $x \in X = \bigcup \mathcal{A}$ ,  $X \setminus x^\perp$  is a support of  $\mathcal{A}$  [4]. We have:

**2.5 Theorem:** *If  $\mathcal{A}$  and  $\mathcal{B}$  are regular, so is  $B(\mathcal{A}, \mathcal{B})$ .*

Proof: For a typical outcome  $(x, y) \in Z$ , we have

$$Z \setminus (x, y)^\perp = [(X \setminus x^\perp) \times Y \cup X \times (Y \setminus y^\perp)] \cap Z = (X \setminus x^\perp) \odot (X \setminus y^\perp).$$

Since  $\mathcal{A}$  and  $\mathcal{B}$  are regular, this last is a support by Lemma 3. Hence,  $B(\mathcal{A}, \mathcal{B})$  is regular.

Let us adopt the following notation: If  $S$  is a support of a test-space  $\mathcal{A}$  and  $\alpha : E \rightarrow V$  is a  $V$ -valued observable, then we write  $\{\alpha \in A\}$  for the collection of all supports of  $\mathcal{A}$  such that  $\alpha(S \cap E) \subseteq A$ . That is:  $\{\alpha \in A\}$  is the set of all supports making the event  $\alpha^{-1}(A)$  certain to occur if the test  $E$  is made.

**2.6 Lemma:** *Let  $\alpha$  and  $\beta$  be  $V$ -valued questions and  $A \subseteq V$ . Then*

$$\{\alpha \cdot \beta \in A\} = \{\alpha \in A\} \odot \{\beta \in A\}.$$



Proof: Suppose  $\alpha : E \rightarrow V$  and  $\beta : F \rightarrow V$ . Let  $f = \beta^{-1}\alpha = \{(x, y) \in E \times F \mid \alpha(x) = \beta(y)\}$ . Then

$$(S \odot T) \cap f = \{(x, y) \mid \alpha(x) = \beta(y) \text{ \& } x \in E \cap S \text{ or } y \in F \cap T\}.$$

Hence,  $S \odot T \cap f \subseteq (\alpha \cdot \beta)^{-1}(A)$  iff  $\alpha(S \cap E) \subseteq A$  and  $\beta(T \cap F) \subseteq A$ .

As a special case of the foregoing, note that  $\alpha \cdot \beta$  is certain to take a value in  $A \subseteq V$  in a state of affairs represented by  $S \odot S$  iff both  $\alpha$  and  $\beta$  are certain to lie in  $A$  in the state of affairs represented by  $S$ .

### 3 The Logic of $B(\mathcal{A}, \mathcal{B})$

If  $\mathcal{A}$  is algebraic, the relation  $\sim$  of perspectivity is an equivalence relation on the set of events of  $\mathcal{A}$ . The set of equivalence classes of events is the **logic** of  $\mathcal{A}$ , here denoted by  $L(\mathcal{A})$ . The equivalence class  $p(A) := \{B \in \mathcal{E}(\mathcal{A}) \mid B \sim A\}$  of an event  $A$  is called the *operational proposition* corresponding to  $A$ . As is well-known,  $L(\mathcal{A})$  can be organized into an orthoalgebra via the partial binary operation  $p(A) \oplus p(B) := p(A \cup B)$ , (well)-defined for pairs of events  $A, B$  with  $A \perp B$ . (For details, see [2] and [3], or [4].)

If  $\mathcal{A}$  and  $\mathcal{B}$  are algebraic, then  $B(\mathcal{A}, \mathcal{B})$  is also algebraic, by Theorem 2.2. In this section, we characterize  $\Pi(B(\mathcal{A}, \mathcal{B}))$  in terms of  $\Pi(\mathcal{A})$  and  $\Pi(\mathcal{B})$  for a large class of algebraic test spaces.

**3.1 Definition:** Events  $A \in \mathcal{E}(\mathcal{A})$  and  $B \in \mathcal{E}(\mathcal{B})$  are **comparable** iff there exists a bijection  $f \in \mathcal{E}(B(\mathcal{A}, \mathcal{B}))$  with  $f : A \rightarrow B$ .

Note that if  $\mathcal{A}$  is  $\kappa$ -uniform, then any two proper events  $A$  and  $B$  of a given cardinality are comparable.

Let  $A$  and  $B$  be comparable events. By Lemma 2.1, the proposition  $p(f)$  corresponding to any (hence, all) bijections  $f : A \rightarrow B$  consists exactly of the union of the sets  $B(C, D)$  of bijections between  $C$  and  $D$  with  $C \sim A$  and  $D \sim B$ . Thus, the proposition  $p(f)$  is completely determined by the pair  $p(A)$  and  $p(B)$ . Let us write  $p(A, B)$  for this proposition.

**3.2 Lemma:** Let  $A, B \in \mathcal{E}(\mathcal{A})$  and  $C, D \in \mathcal{E}(\mathcal{B})$  with  $A$  and  $C$  comparable and  $B$  and  $D$  comparable. If  $p(A, B) \perp p(C, D)$ , then  $A \perp C$ ,  $B \perp D$ ,  $A \cup C$  and  $B \cup D$  are comparable, and  $p(A, B) \oplus p(C, D) = p(A \cup C, B \cup D)$ .

Proof: If  $p(A, B) \perp p(C, D)$  then for every bijection  $f : A \rightarrow B$  and every bijection  $g : C \rightarrow D$ ,  $f \cap g = \emptyset$  and  $f \cup g : A \cup C \rightarrow B \cup D$  belongs to  $\mathcal{E}(B(\mathcal{A}))$  – whence,  $A \perp C$ ,  $B \perp D$ , and  $p(A \cup C, B \cup D) = p(f \cup g) = p(f) \oplus p(g) = p(A, B) \oplus p(C, D)$ .

Note that  $A \perp C, B \perp D$  need not imply that  $p(A, B) \perp p(C, D)$  unless  $\mathcal{A}$  is uniform.

If  $\mathcal{A}$  is uniform, then any two perspective events have the same cardinality. Hence, we may define a map  $\rho : \Pi(\mathcal{A}) \rightarrow \kappa$  (where  $\kappa$  is the cardinality of a test in  $\mathcal{A}$ ) by

$$\rho(p(A)) = \#(A).$$

We call  $\rho(p)$  the **rank** of the proposition  $p \in \Pi(\mathcal{A})$ . Note also that if  $p \perp q$  then  $\rho(p \oplus q) = \rho(p) + \rho(q)$  for all  $p, q \in \Pi(\mathcal{A})$ . The proof of the following is straightforward:

**3.3 Theorem:** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $\kappa$ -uniform test spaces with logics  $L$  and  $M$ , respectively. Let*

$$L \times_{\rho} M = \{ (p, q) \in L \times M \mid \rho(p) = \rho(q) \}.$$

*For all  $(p, q), (u, v) \in L \times_{\rho} M$ , write  $(p, q) \perp (u, v)$  iff  $p \perp u$  and  $q \perp v$ , and, if this is the case, set  $(p, q) \oplus (u, v) := (p \oplus u, q \oplus v)$ . Then  $(L \times_{\rho} M, \perp, \oplus)$  is an orthoalgebra, and there is a canonical isomorphism  $L \times_{\rho} M \rightarrow \Pi(B(\mathcal{A}, \mathcal{B}))$  given by  $p(A, B) \mapsto (p(A), p(B))$ .*

**3.4 Definition:** Call an algebraic test space  $\mathcal{A}$  **saturated** iff for every  $A \in \mathcal{E}(\mathcal{A})$  there is some  $x_A \in X = \bigcup \mathcal{A}$  with  $\{x_A\} \sim A$ .

By way of example, if  $\mathcal{A}$  is any manual, the manual  $\mathcal{A}^{\#}$  of partitions of  $\mathcal{A}$ -operations by  $\mathcal{A}$ -events is saturated, with  $\{\bigcup A\} \sim A$  for any subset  $A$  of such a partition.

**3.5 Lemma:** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be saturated. Then every bijection between proper events of  $\mathcal{A}$  and  $\mathcal{B}$  can be extended to an element of  $B(\mathcal{A}, \mathcal{B})$ .*

*Proof:* If  $A$  and  $B$  are proper events of the same cardinality with  $A \subseteq E \in \mathcal{A}$  and  $B \subseteq F \in \mathcal{B}$ , then there exist outcomes  $x$  and  $y$  with  $\{x\} \sim E \setminus A$  and  $\{y\} \sim F \setminus B$ . Since  $\mathcal{A}$  and  $\mathcal{B}$  are algebraic,  $\{x\} \cup A$  and  $\{y\} \cup B$  are tests in  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, to which we may extend any given bijection  $f : A \rightarrow B$  by setting  $f(x) = y$ .

**3.6 Definition:** Let  $L$  and  $M$  be two orthoalgebras. Let

$$L * M := \{ (p, q) \in L \times M \mid p = 0 \Leftrightarrow q = 0 \text{ \& } p = 1 \Leftrightarrow q = 1 \}.$$

For  $(p, q)$  and  $(r, s)$  in  $L * M$ , set  $(p, q) \perp (r, s)$  iff  $p \perp r$ ,  $q \perp s$ , and  $(p \oplus q, r \oplus s) \in L * M$ . If this is the case, define  $(p, q) \oplus (r, s) := (p \oplus r, q \oplus s)$ .

It is easily verified that  $(L * M, \perp, \oplus, (1, 1))$  is an orthoalgebra in which the orthocomplement of an element  $(p, q)$  is given by  $(p, q)' = (p', q')$ .

**3.7 Proposition:** *Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be saturated algebraic test spaces. Then*

$$L(B(\mathcal{A}_1, \mathcal{A}_2)) \simeq L(\mathcal{A}_1) * L(\mathcal{A}_2).$$

*Proof:* Let  $L = L(B(\mathcal{A}_1, \mathcal{A}_2))$  and  $L_i = L(\mathcal{A}_i)$ ,  $i = 1, 2$ . The two coordinate projections  $\pi_i : B(\mathcal{A}_1, \mathcal{A}_2) \rightarrow \mathcal{A}_i$  introduced in the proof of Theorem 1.7 lift to orthoalgebra homomorphisms  $L \rightarrow L_i$ . Since  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are saturated, these are surjections, by Lemma 3.5 above. Hence, we have a natural map  $\phi : L \rightarrow L_1 \times L_2$  given by  $\phi(p) = (\pi_1(p), \pi_2(p))$  for all  $p \in L$ . If  $\phi(p) = (1, q) \in L_1 \times L_2$ , then  $p = p(E, B)$  for some  $E \in \mathcal{A}_1$  and some event  $B \in \mathcal{A}_2$

with  $q = p(B)$ . In order for  $A$  and  $B$  to be comparable, there must exist a bijection  $f \in B(\mathcal{A}_1, \mathcal{A}_2)$  with  $B = f(A)$ . But then  $B \in \mathcal{A}_2$ , whence,  $q = 1$ . Similarly, if  $p = 0$ ,  $q = 0$  in order to preserve comparability. On the other hand, if  $p \in L(\mathcal{A}_1)$ ,  $q \in L(\mathcal{A}_2)$ , and neither  $p$  nor  $q$  is 0 or 1, then, since each manual is saturated, we may choose outcomes  $x \in X_1 = \bigcup \mathcal{A}_1$  and  $y \in X_2 = \bigcup \mathcal{A}_2$  with  $p = p(x)$  and  $q = p(y)$ . Likewise,  $p' = p(x')$  and  $q' = p(y')$  for some outcomes  $x' \in X_1$  and  $y' \in X_2$ . Thus  $\{x, x'\}$  and  $\{y, y'\}$  are two-element tests in  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively, whence,  $f = \{(x, y), (x', y')\}$  belongs to  $B(\mathcal{A}_1, \mathcal{A}_2)$ . Thus,  $\pi(p(f)) = \pi(p(x, y)) = (p, q)$  is defined. The image of  $\phi$  is therefore precisely  $L_1 * L_2$ . It remains to see that  $\phi$  is an faithful (hence, injective) orthoalgebra homomorphism. But this follows from Lemma 3.2.

A **partition of unity** in an orthoalgebra  $L$  is a finite set  $E = \{p_1, \dots, p_n\} \subseteq L \setminus \{0\}$  such that  $p_1 \oplus \dots \oplus p_n = 1$ . The collection  $\mathcal{A}_L$  of all such partitions of unity is easily seen to be a saturated manual, the logic of which is canonically isomorphic to  $L$ .

**3.8 Corollary:** *For any orthoalgebras  $L$  and  $M$ ,  $L * M \simeq L(B(\mathcal{A}_L, \mathcal{A}_M))$ .*

Call two test spaces  $\mathcal{A}$  and  $\mathcal{B}$  *uniformly compatible* iff every bijection between events of  $\mathcal{A}$  and  $\mathcal{B}$  extends to an element of  $B(\mathcal{A}, \mathcal{B})$ . (By way of example: Any two saturated algebraic test spaces, or any two uniform test spaces). The following generalization of Theorem 3.7 is straightforward. We omit the proof.

**3.9 Proposition:** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be uniformly compatible test spaces. There is a canonical embedding of  $L(B(\mathcal{A}, \mathcal{B}))$  into  $L(\mathcal{A}) * L(\mathcal{B})$  given by*

$$(p(A, B)) \mapsto (p(A), p(B))$$

*for compatible events  $A$  and  $B$ .*

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