A mechanistic macroscopic physical entity with a three-dimensional Hilbert space description

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Abstract It is sometimes stated that Gleason's theorem prevents the construction of hidden-variable models for quantum entities described in a more than two-dimensional Hilbert space. In this paper however we explicitly construct a classical (macroscopic) system that can be represented in a three-dimensional real Hilbert space, the probability structure appearing as the result of a lack of knowledge about the measurement context. We briefly discuss Gleason's theorem from this point of view.

1 Introduction

Even after more than 60 years there remain many problems on the 'understanding' of quantum mechanics. From the early days, a main concern of the majority of physicists reflecting on the foundations of the theory was the question of understanding the nature of the quantum probability. At the other hand, it was no problem to understand the appearance of probability in classical theories, since we all agree that it finds its origin in a lack of knowledge about a deeper deterministic reality. The archetypic example is found in thermodynamics, where the probabilities associated with macroscopic observables such as pressure, volume, temperature, energy and entropy are due to the fact that the 'real' state

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of the entity is characterized deterministically by all the microscopic variables of positions and momenta of the constituting entities, the probabilities describing *our* lack of knowledge about the microscopic state of the entity. The variables of momenta and positions of the individual entities can be considered as 'hidden variables', present in the underlying reality. This example can stand for many of the attempts that have been undertaken to explain the notion of quantum probability, and the underlying theories are called 'hidden variable' theories. In general, for a hidden variable theory, one aims at constructing a theory of an underlying deterministic reality, in such a way that the quantum observables appear as observables that do not reach this underlying 'hidden' reality and the quantum probabilities find their origin in a lack of knowledge about this underlying reality. Von Neumann [1] gave a first impossibility proof for hidden variable theories for quantum mechanics. It was remarked by Bell [2] that in the proof of this No-Go theorem, Von Neumann had made an assumption that was not necessarily justified, and Bell explicitly constructed a hidden variable model for the spin of a spin-$\frac{1}{2}$ quantum particle. Bell also criticizes the impossibility proof of Gleason [3], and he correctly points out the danger of demanding extra 'mathematical' assumptions without an exact knowledge of their physical meaning. Very specific attention was paid to this danger in the study of Kochen and Specker [4], and their impossibility proof is often considered as closing the debate. We can state that each of these impossibility proofs consists in showing that a hidden variable theory (under certain assumptions) gives rise to a certain mathematical structure for the set of observables of the physical system under consideration, while the set of observables of a quantum system does not have this mathematical structure. Therefore it is impossible to replace quantum mechanics by a hidden variable theory (satisfying the assumptions). To be more specific, if one works in the category of observables, then a hidden variable theory (under the given assumptions) gives rise to a commutative algebraic structure for the set of observables, while the set of observables of a quantum system is non-commutative. If one works in the category of properties (yes-no observables) then a hidden variable theory (satisfying the assumptions) has always a Boolean lattice structure for the set of properties while the lattice of properties of a quantum system is not Boolean. If one works in the category of probability models, then a hidden variable theory (satisfying the assumptions) has always a Kolmogorovian probability model for the set of properties while the quantum probability model is not Kolmogorovian. However, the assumptions made in these proofs are too restrictive: they only apply to 'hidden variable' theories of the state of the entity. From a physical point of view, it is possible to imagine that not only the quantum system can have a deeper underlying reality, but also the physical measurement process for each particular measurement. If this is true, then the physical origin of the quantum probabilities could be connected with a lack of knowledge about a deeper underlying reality of the measurement process. In [5,6,7] this idea was explored and it has been shown that such a lack of knowledge gives indeed rise to a quantum structure. This uncertainty about the interaction between the measurement device and the physical entity can be eliminated by introducing hidden variables that describe the fluctuations in the measurement context. However, they are not state hidden variables, they rather describe an underlying reality for each measurement process, and therefore they have been called 'hidden measurements', and the corresponding theories 'hidden measurement theories'. In [5,6,7] it is shown that a hidden measurement model can be constructed

for any arbitrary quantum mechanical system of finite dimension, and the possibility of constructing a hidden measurement model for an infinite dimensional quantum system can be found in [8,10]. Although the models presented in these papers illustrate our point about the possibility of explaining the quantum probabilities in this way, there is always the possibility to construct more concrete macroscopic models, only dealing with real macroscopic entities and real interactions between the measurement device and the entities, that give rise to quantum mechanical structures. One of the authors introduced such a real macroscopic model for the spin of a spin-$\frac{1}{2}$ quantum entity. When he presents this spin model for an audience, it was often raised that this kind of realistic macroscopic model can only be built for the case of a two-dimensional Hilbert space quantum entity, because of the theorem of Gleason and the results of Kochen and Specker. Gleason's theorem is only valid for a Hilbert space with more than two dimensions and hence not for the two-dimensional complex Hilbert space that is used in quantum mechanics to describe the spin of a spin-$\frac{1}{2}$ quantum entity. In the paper of Kochen and Specker also a spin model for the spin of a spin-$\frac{1}{2}$ quantum entity is constructed, and a real macroscopic realization of this spin model is proposed. They point out on different occasions that such a real model can only be constructed for a quantum entity with a Hilbert space of dimension not larger than two. The aim of this paper is to clarify this dimensional problem. Therefore we shall construct a real macroscopic physical entity and measurements on this entity that give rise to a quantum mechanical model for the case of a three-dimensional real Hilbert space, a situation where Gleason's theorem is already fully applicable. We remark that one of the authors [9] presented a model for a spin-1 quantum entity that allows in a rather straightforward way a hidden measurement representation. Nevertheless, since he only considered a set of coherent spin-1 states (i.e., a set of states that spans a three-dimensional Hilbert space, but that does not fill it) his model can not be considered as a satisfactory counter argument against the No-Go theorems. In section 2 we briefly give the two-dimensional examples of Aerts and Kochen-Specker and analyze their differences. In section 3 we construct a hidden measurement model with a mathematical structure for its set of states and observables that can be represented in a three-dimensional real Hilbert space.

2 The two 2-dimensional models

The physical entity that we consider is a point particle P that can move on the surface of the unit sphere S^2. Every unit vector $\mathbf{v}$ represents a state p_v of the entity. For every point $\mathbf{u}$ of S^2 we define a measurement e_u as follows: a rubber string between $\mathbf{u}$ and its antipodal point $-\mathbf{u}$ catches the particle P that falls orthogonally and sticks to it. Next, the string breaks somewhere with a uniform probability density and the particle P moves to one of the points $\mathbf{u}$ or $-\mathbf{u}$, depending on the part of the elastic it was attached to. If it arrives in $\mathbf{u}$ we will give the outcome o_1^u to the experiment, in the other case we will say that the outcome o_2^u has occurred. After the measurement the entity will be in a new

state: p_u in the case of outcome o_1^u and p_{-u} in the other case. Taking into account that the elastic breaks uniformly, it is easy to calculate the probabilities for the two results:

$$P(o_1^u|p_v) = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2}$$

$$P(o_2^u|p_v) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2}$$

with $\cos\theta = \mathbf{u}\cdot\mathbf{v}$. We have the same results for the probabilities associated with the spin measurement of a quantum entity of spin-$\frac{1}{2}$, so we can describe our macroscopic example by the ordinary quantum formalism where the set of states is given by the points of a two-dimensional complex Hilbert space. Clearly, we can also interpret this macroscopic example as a hidden variable model of the spin measurement of a quantum entity of spin-$\frac{1}{2}$. Indeed, if the point λ where the string disintegrates is known, the measurement outcome is certain. The probabilities in this model appear because of our lack of knowledge of the precise interaction between the entity and the measurement device. Every spin measurement e_u can be considered as a class of classical spin measurements e_u^λ with determined outcomes, and the probabilities are the result of an averaging process. In this example it is clear that the hidden variable λ is not a variable of the entity under study. In fact, it is a variable belonging to the measurement process as a whole.

In Kochen and Specker's model, a point P on a sphere represents the quantum state of the spin-$\frac{1}{2}$ entity. However, at the same time the entity is in a hidden state which is represented by another point T_P of S_P^+, the upper half sphere with P as its north pole, determined in the following way. A disk D of the same radius as the sphere is placed perpendicular to the line OP which connects P with the center O of the sphere and centred directly above P. A particle is placed on the disk that is now shaken "randomly", i.e., in such a way that the probability that the particle will end up in a region U of the disk is proportional to the area of U. The point T_P is then the orthogonal projection of the particle. The probability density function $\mu(T_P)$ is

$$\mu(T_P) = \begin{cases} \frac{1}{\pi}\cos\theta & 0 \le \theta \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

where θ is the angle between T_P and P. If a measurement is made in the direction OQ the outcome "spin up" will be found in the case that $T_P \in S_Q^+$ and "spin down" otherwise. As a result of the measurement the new state of the entity will be Q in case of spin up and $-Q$ otherwise. The new hidden state T_Q is now determined as before, the disk being placed at Q if the new state is Q and at $-Q$ otherwise. It can be shown that the same probabilities as for the quantum spin-$\frac{1}{2}$ entity occur. It is important to remark that the hidden variable here pertains to the entity under study, as was made clear by using the expression "hidden state". But is this really the case? As we look closer we see that for every consecutive spin measurement to reveal the correct probabilities, we need each time a randomisation of the hidden state T. Thus every time a measurement occurs the hidden variable has to be reset again. In practice this means that for every measurement a new value of the variable will be needed. Thus we can make the philosophical important step to remove this "hidden

state" from the entity and absorb it within the context of the measurement itself. Once his is done, the analogy with Aerts' model is obvious. But it is also clear that a new idea nas been introduced, namely the shift of the hidden variable from the entity towards the measurement process. This is not only a new feature for a hidden variable theory, but also a natural way out of the traps of the No-Go theorems.

3 The 3-dimensional model

In this section we introduce a mechanistic macroscopic physical entity with a three-dimensional Hilbert space quantum description. Probably there exist models that are much more elegant than the one we propose, because the explicit realization would be rather non-trivial, but for our purposes it is sufficient to prove that there exists at least one. Once again we remark that the system that we present is not a representation of a quantum mechanical entity, but a macroscopic physical entity that gives rise to the same probability structure as one encounters in quantum mechanics. First we propose the model and, for reasons of readability, we present a geometrical equivalent in $\mathbf{R}^3$. In this way we can easily prove the equivalence between the model and the quantum mechanical case. In section 3.3 we shall study the probability structure of the model.

3.1 The practical realization

The entity S that we consider is a rod of length 2 which is fixed in its center point c, both sides of which have to be identified. The set of states of the entity, i.e. the set of rays in Euclidean 3-space, possibly characterized by one of the two end points of the rod (denoted by x_p), will be denoted by Σ_S. The measurement apparatus consists of three mutual orthogonal rods, parallel with rays $\hat{x}_e^1, \hat{x}_e^2, \hat{x}_e^3$, fixed in 3-space. The entity and the measurement device are coupled for a measurement in the following way (Fig. 1):

• Connection in x_e^i: for each measurement axis, an interaction rod floats in a slider which is fixed orthogonal to the rod of the measurement apparatus.

• Connection in x_p: the three interaction-rods are fixed to one slider, which floats on the "entity-rod".

• We also fix three rubber strings between the entity-rod and the three rods of the measurement apparatus.

The last ingredient that takes part in the interaction is something we call a "random gun". This is a gun, fixed on a slider that floats on and turns around the entity-rod in such a way that:

• The gun is shooting in a direction orthogonal to the entity-rod.

• The movement and the frequency of shooting are at random but such that the probability of shooting a bullet in a certain direction, and from a certain point of the entity-rod is uniformly distributed, i.e., the gun distributes the bullets uniformly in all directions and from all the points of the rod. If a bullet hits one of the connections, both the rod and string break, such that the entity can start moving, and it is clear that the two non-broken strings will tear the entity into the plane of the measurement-rods to which it is still connected.
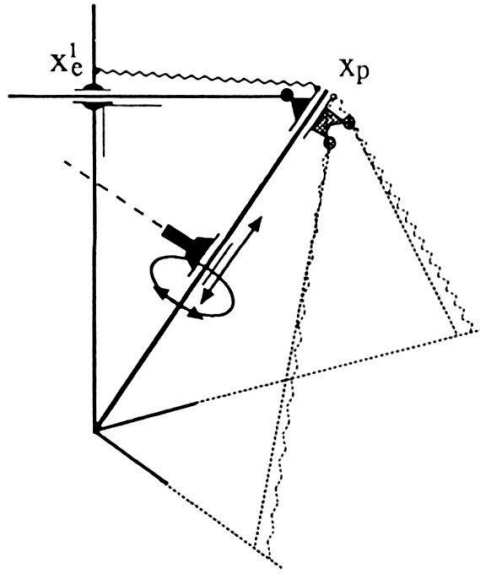

Fig. 1 : Practical realization of the model. With rods, sliders, strings and a "random gun" we construct a device with a mathematical structure equivalent to the one for a quantum entity with a three-dimensional real Hilbert state space.

3.2 A geometrical equivalent of the model

To facilitate the calculation of the probabilities we will describe what happens during the measurement from a geometrical point of view. We know that a state p of the entity is characterized by the angles $\theta_1, \theta_2, \theta_3$ between the rod and an arbitrary selected set of three orthonormal axes in Euclidean 3-space E^3. It is clear that this set of states corresponds in a one-to-one way with the states of an entity described in a three-dimensional real Hilbert space. The set of measurements to be performed on this entity S is characterized as follows. Let $\widehat{x}_e^1, \widehat{x}_e^2, \widehat{x}_e^3$ be the three mutually orthogonal rays coinciding with the rods of the measurement apparatus. Therefore, for a given state p, and a given experiment e, we take the three angles $\theta_1, \theta_2, \theta_3$ as representative parameters to characterize the state relative to the measurement apparatus. We denote by x_e^1, x_e^2, x_e^3, the orthogonal projections of x_p on the three rays $\widehat{x}_e^1, \widehat{x}_e^2, \widehat{x}_e^3$, forming a set of points representative for the couple (p, e) (Fig. 2).
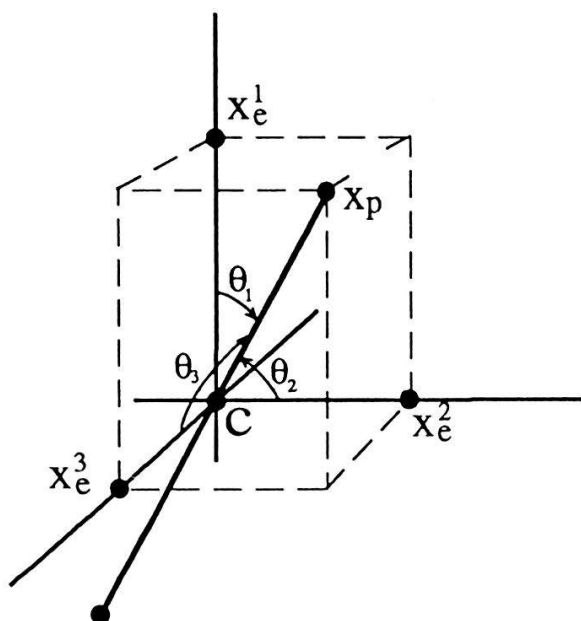
Fig. 2 : The states of the classical mechanistic entity, a rod in Euclidean 3-space, repre-
sented by x_p, one of the two end points of the rod. Thus, the different states p of the entity
are represented by the angles $\theta_1, \theta_2, \theta_3$ between the rod and three mutually orthogonal rays
$\widehat{x}_e^1, \widehat{x}_e^2, \widehat{x}_e^3$, representative for a measurement e. x_e^1, x_e^2, x_e^3, the orthogonal projections of x_p on
the three rays are thus representative for the couple (p,e).

The geometrical description of the measurement process goes as follows:

) Every point x_e^i is connected with x_p by a segment denoted by $[x_e^i, x_p]$ with length $\sin \theta_i$.
Therefore the length of the projection of $[x_e^i, x_p]$ on the rod is $\sin^2 \theta_i$ (Fig. 3).
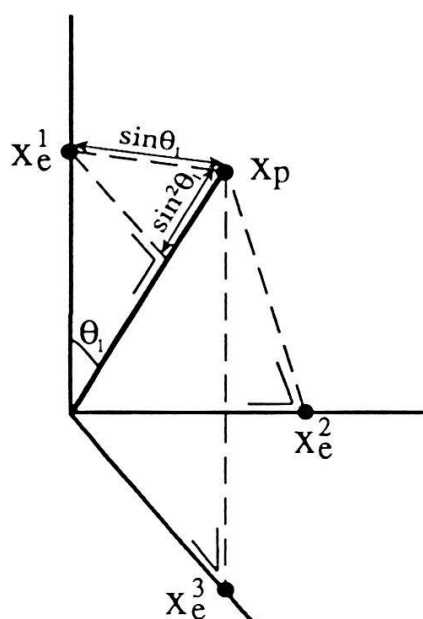

Fig. 3 : The first step of the measurement. Every point x_e^i is connected with x_p by a
segment denoted by $[x_e^i, x_p]$. The length of the projection of $[x_e^1, x_p]$ on the rod is $\sin^2 \theta_1$.

ii) Next, one of the connections $[x_e^i, x_p]$ breaks with a probability proportional to the length of the projection of $[x_e^i, x_p]$ on the rod (in Fig. 4 and Fig. 5 we suppose that $[x_e^1, x_p]$ breaks). The rod rotates into the plane of the two remaining points x_e^j, x_e^k, to which it is still connected, and such that the point x_p', the projection of x_p on the $\hat{x}_e^j \hat{x}_e^k$-plane, lies on the rod. As a consequence, the connections $[x_e^j, x_p']$ and $[x_e^k, x_p']$ are still orthogonal to the corresponding axes $\hat{x}_e^j$ and $\hat{x}_e^k$.
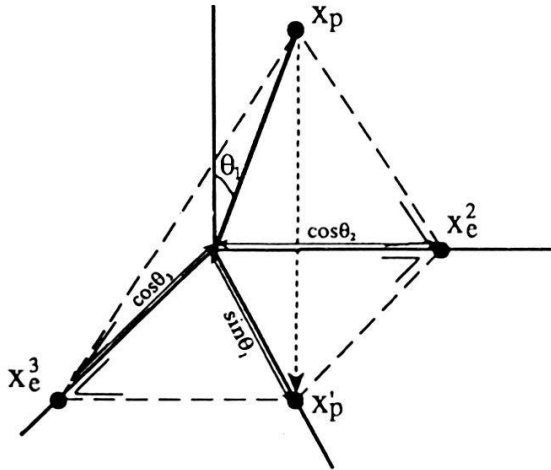

Fig. 4 : The second step of the measurement. One of the connections, e.g. $[x_e^1, x_p]$, breaks with probability proportional to the length of the projection of $[x_e^1, x_p]$ on the rod. The rod rotates into the plane of the two points x_e^2, x_e^3 in such a way that the connections $[x_e^2, x_p']$ and $[x_e^3, x_p']$ are still orthogonal to the corresponding axes $\hat{x}_e^2$ and $\hat{x}_e^3$.

iii) We proceed with this new, two-dimensional situation characterized by $\{x_p', x_e^j, x_e^k\}$ as before, denoting the angle between x_p' and x_e^j as θ_j'. One of the segments, $[x_e^j, x_p']$ or $[x_e^k, x_p']$, seizes to exist, again with a probability proportional to the length of the projection of this segment on the rod, equal to $\sin \theta_i . \sin^2 \theta_j'$. Finally, the rod rotates towards and stabilizes at the third ray, to which it is still connected.
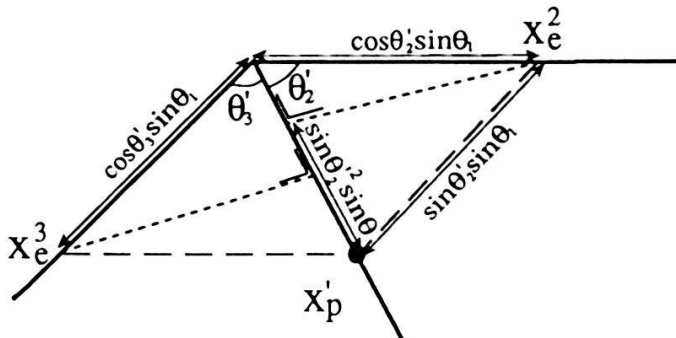

Fig. 5 : We proceed with $\{x_p', x_e^2, x_e^3\}$ as we did with $\{x_p, x_e^1, x_e^2, x_e^3\}$. One of the two existing connections breaks with probability proportional to the length of the projection of the corresponding segment on the rod.

The global process can thus be seen as a measurement e, with three possible outcomes o_e^1, o_e^2, o_e^3 on an entity S in a state p.

3.3 The probability structure of the model

After this geometrical presentation of our model it becomes very easy to calculate the probability to obtain an outcome o_e^k, equivalent with neither obtaining o_e^i nor o_e^j, so with the breaking of these two connections. Suppose that first $[x_e^i, x_p]$ breaks and then $[x_e^j, x_p']$. Since $\cos^2 \theta_i + \cos^2 \theta_j + \cos^2 \theta_k = 1$ we have $\sin^2 \theta_i + \sin^2 \theta_j + \sin^2 \theta_k = 2$ and $(\sin^2 \theta_i)/2$ is the probability for the breaking of $[x_e^i, x_p]$. In the same way $\sin^2 \theta_j' + \sin^2 \theta_k' = 1$ and $\sin^2 \theta_j'$ is the conditional probability for the breaking of $[x_e^j, x_p']$ supposing that the connection between x_e^i and the rod broke first. This yields $\frac{1}{2} \sin^2 \theta_i . \sin^2 \theta_j' = \frac{1}{2} \sin^2 \theta_i . \cos^2 \theta_k' = \frac{1}{2} \sin^2 \theta_i . (\frac{\cos \theta_k}{\sin \theta_i})^2 = \frac{1}{2} \cos^2 \theta_k$ for the requested probability. Analogously, we find the same result for the probability that first $[x_e^j, x_p]$ and then $[x_e^i, x_p']$ breaks.

Therefore we find:

$$P(o_e^k|p) = \cos^2 \theta_k$$

where θ_k is the angle between x_p and x_e^k, the eigenstate with eigen-outcome o_e^k of the measurement e on the entity S.

Now we are able to compare the probability structure associated with our model with the one encountered in quantum mechanics. For a three-dimensional real Hilbert space $\mathcal{H}_3$ we can introduce a self-adjoint operator H_e with $\{\hat{x}_e^1, \hat{x}_e^2, \hat{x}_e^3\}$ a set of mutual orthogonal eigen-rays and $\{o_e^1, o_e^2, o_e^3\}$ the corresponding eigenvalues (some of them may be equal), as $H_e = \sum_{i=1}^{i=3} o_e^i E_{\hat{x}_e^i}$, where $E_{\hat{x}_e^i}$ is the projector on the ray $\hat{x}_e^i$. Therefore, we have for every o_e^i, eigen-outcome of a measurement e and associated with an eigenstate represented by a ray $\hat{x}_e^i$, and for every state p of the system, represented by a ray $\hat{x}_p$:

$$P(o_e^i|p) = |<x_e^i| x_p>|^2 = \cos^2 \theta_i$$

where θ_i is the angle between the rays $\hat{x}_e^i$ and $\hat{x}_p$. It is therefore clear that the entity in our model corresponds in a one-to-one way with a quantum entity described in a three-dimensional real Hilbert space.

4 Discussion

In this paper we have presented a macroscopic device with a quantum-like probability structure and state space. Since one can interpret this model as a hidden variable description for a quantum entity, we can analyse the relationship with Gleason's theorem, which implies the existence of a unique probability measure for a physical entity if its state space is a more than two-dimensional separable Hilbert space $\mathcal{H}$ and if this probability measure satisfies some reasonably looking a priori assumptions. For pure states Gleason's theorem

takes the following form: if $p : \mathcal{L}(\mathcal{H}) \to [0, 1]$ is a (generalized) probability measure, there exists a unit vector $\psi \in \mathcal{H}$ such that $\forall\, P \in \mathcal{L}(\mathcal{H}) : p(P) = <\psi \mid P\psi>$, with $\mathcal{L}(\mathcal{H})$ the lattice of closed subspaces of the Hilbert space. In our case it asserts that the probability to obtain e.g. o_e^i necessarily takes the form that was given above in this paper. Therefore it is implicit in the assumptions of the theorem that the probabilities only depend on the initial and final state of the entity. However, referring to our model we see that it is easy to invent other probability measures that actually do depend on the intermediate states of the entity and therefore do not satisfy the assumptions of Gleason's theorem. For instance, one can imagine that the random gun is absent and the interaction rods break with a uniform probability density, resulting in the first probability being proportional to $\sin\theta_i$ instead of $\sin^2\theta_i$. Since the hidden measurement approach is obviously a contextual theory that keeps the Hilbert space framework for its state space, but situates the origin of the quantum probability in the measurement environment, there is no need for the existence of dispersion-free probability measures on $\mathcal{L}(\mathcal{H})$ as in the conventional non-contextual state hidden variable theories.

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