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Autor: Bang, Ha Huy / Chowdhury, M.A. Mansur

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Generalized Deformed Para-Bose Algebra With Complex Structure Function

By Ha Huy Bang[1] and M.A. Mansur Chowdhury[2]

International Centre for Theoretical Physics, 34100 Trieste, Italy

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Abstract. The generalized deformed algebras have been studied for para-Bose systems considering the structure function in the complex form. These algebras have been extended to $SU(2)$ and $SU(1,1)$ realizations of generalized deformed para-Bose oscillators. Interestingly, one can reproduce their form of algebras from these generalized algebras in the limiting cases.

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In the last few years there has been increasing interest in particles obeying statistics different from Bose or Fermi statistics. These generalized statistics are called para-Bose and para-Fermi statistics [1-4]. Since the advent of the theory of parastatistics there have been many attempts to generalize the canonical commutation relations. In particular, quantum deformations of the Heisenberg algebra and their possible physical applications have been widely investigated. Naturally, some properties of the deformed para-Bose systems have also been considered [5-8].

In general, it is assumed that the deformation parameter q takes only real values. Recently, in ref. [9] L. De Falco and co-workers have studied a general q-deformed Heisenberg algebra with complex deformation parameter. Using bosonization method they obtained q-boson realization of the said algebras and used these to find the energy spectrum of harmonic oscillators. Next, the authors of ref. [10] have shown that the non-Hermitian realization of

[1]On leave from Department of Physics, Vietnam National University, 90 Nguyen Trai, Hanoi, Vietnam.
[2]On leave from Research Center for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh.

a Lie-deformed Heisenberg algebra essentially amounts to the case of a Q-deformed algebra with complex q.

In this letter we have studied the generalized deformed algebras for para-Bose oscillators with the complex structure function. We have also extended these algebras for Holstein-Primakoff and Jordan-Schwinger realizations of $SU(2)$ and $SU(1,1)$. It is interesting to note here that from our generalized deformed algebras with the complex structure function one can get not only their deformed algebras but also deformed para-Bose algebras [9-15].

For a single mode para-Bose system the commutation realizations (CRs) are characterized by [16, 17]

$$[a, \mathcal{N}] = a, \quad [a^+, \mathcal{N}] = -a^+$$

where

$$\mathcal{N} = \frac{aa^+ + a^+a}{2} - \frac{p}{2}$$

and p is the order of the para-Bose system.
Also

$$aa^+ = f(\mathcal{N}+1), \quad a^+a = f(\mathcal{N})$$

with

$$f(n) = n + \frac{1}{2}\{1-(-1)^n\}(p-1).$$

Hence

$$[a, a^+] = f(\mathcal{N}+1) - f(\mathcal{N}) = 1 + (-1)^{\mathcal{N}}(p-1).$$

From these relations, an operator A^+ was constructed so that [16, 17]

$$A^+ = a^+\frac{\mathcal{N}+1}{f(\mathcal{N}+1)},$$

$$[a, A^+] = 1, \quad [A^+, \mathcal{N}] = -A^+,$$

where the number operator $\mathcal{N}$ is defined by $\mathcal{N} = A^+a$.
To construct the generalized deformed para-Bose oscillator algebras, corresponding to the annihilation and creation operators $\tilde{a}$ and $\tilde{a}^+$ respectively, we begin first with the generalized deformed Bose algebra for two operators $\tilde{a}$ and $\tilde{A}^+$ satisfy the CR [18, 19]

$$[\tilde{a}, \tilde{A}^+] = F(N+1) - F(N) \equiv [N+1] - [N], \tag{1}$$

where [8, 17]

$$\tilde{A}^+ = \frac{\tilde{a}^+(N+1)}{f(N+1)}.$$

For our system where $\tilde{a}$ and $\tilde{b}$ are not Hermitian operators the relation (1) corresponds to two relations :

$$[\tilde{a}, \tilde{B}] = [N+1] - [N], \quad [\tilde{B}^+, \tilde{a}^+] = [N+1]^* - [N]^*. \tag{2}$$

with

$$\tilde{B} = \tilde{b}\frac{N+1}{f(N+1)}, \quad \tilde{B}^+ = \frac{N+1}{f(N+1)}\tilde{b}^+. \tag{3}$$

The last relation in (2) is the adjoint one of the former. The number operator N satisfies, by definition, the CRs

$$[\tilde{a}, N] = \tilde{a}, \quad [\tilde{B}, N] = -\tilde{B}, \tag{4}$$

which implies

$$[\tilde{a}^+, N] = -\tilde{a}^+, \quad [\tilde{B}^+, N] = \tilde{B}^+. \tag{5}$$

With the help of (3), (4) and (5) we find that

$$[\tilde{b}, N] = -\tilde{b}, \quad [\tilde{b}^+, N] = \tilde{b}^+.$$

From the relations (2), (4) and (5) we obtain

$$\tilde{a}\tilde{b} = \frac{[N+1]}{N+1} f(N+1), \quad \tilde{b}\tilde{a} = \frac{[N]}{N} f(N). \tag{6}$$

Using (6) we get

$$[\tilde{a}, \tilde{b}] = \frac{[N+1]}{N+1} f(N+1) - \frac{[N]}{N} f(N) \equiv G(N), \tag{7}$$

$$[\tilde{b}^+, \tilde{a}^+] = f(N+1)\frac{[N+1]^*}{N+1} - f(N)\frac{[N]^*}{N} \equiv G^*(N). \tag{8}$$

We have just constructed the most general form of CRs (7) and (8) for non-Hermitian operators of generalized deformed para-Bose oscillators with the complex structure function. It includes as special cases the various forms of the CRs defined in the literature [5, 7, 17, 20].

The boson realization of the operators $\tilde{a}$ and $\tilde{B}$ which satisfy CR (2) according to refs. [6, 14, 21] takes the form:

$$\tilde{a} = \sqrt{\frac{[N+1]}{N+1}}\, a, \tag{9}$$

$$\tilde{B} = A^+ \sqrt{\frac{[N+1]}{N+1}}, \tag{10}$$

where $N = A^+ a = \mathcal{N}$.
Combining (3) and (10) we arrived at

$$\tilde{b} = a^+ \sqrt{\frac{[N+1]}{N+1}}. \tag{11}$$

Relations (9) and (11) denote the non-linear realization of the single-mode generalized deformed para-Bose algebra in terms of a single para-boson. Further, by using the method of Jannussis [14] we may rewrite the above relations as

$$\tilde{a} = \sqrt{\frac{(\sum_{l=0}^{n} G(l))_{n=N}}{f(N+1)}}\, a, \tag{12}$$

$$\tilde{b} = a^+ \sqrt{\frac{(\sum_{l=0}^{n} G(l))_{n=N}}{f(N+1)}}. \tag{13}$$

It is possible to apply these results for any deformed para-Bose oscillator.

As a next step, the Fock space representation constructed in the following way :

$$
\begin{aligned}
N|n> &= n|n>, && (14)\\
<n|m> &= \delta_{nm}, && (15)\\
\tilde{a}|0> &= 0, && (16)\\
\tilde{a}\tilde{b}|0> &= p[1]|0> . && (17)
\end{aligned}
$$

Relation (17) generalizes the corresponding one of ref. [16] to the case of para-Bose oscillator. The orthonormalized eigenstates $|n>$ of the operator N may be obtained by repeated applications of $\tilde{b}$ on $|0>$:

$$|n>= c_n(\tilde{b})^n|0> \quad (18)$$

where numerical norm c_n can then be determined from (6) and (15)

$$|c_n|>= \sqrt{\frac{n!}{f(n)!\sqrt{[n]^*![n]!}}} \quad (19)$$

with $[n]! = [n][n-1]...[1], [0]! = 1,\ f(n)! = f(n)f(n-1)...f(1)$.
Taking into account of (6), (19) and after some manipulations the expressions for $\tilde{a}$ and $\tilde{b}$ are

$$\tilde{a}|n>= \sqrt{f(n)\frac{[n]^{3/2}}{n[n]^{*1/2}}}|n-1>,$$

$$\tilde{b}|n>= \sqrt{f(n+1)\frac{\sqrt{[n+1]^*[n+1]}}{n+1}}|n+1> .$$

For the real function F(x) it follows that

$$
\begin{aligned}
\tilde{a}|n> &= \sqrt{\frac{[n]}{n}f(n)}|n-1>, && (20)\\
\tilde{b}|n>= \tilde{a}^+|n> &= \sqrt{\frac{[n+1]}{n+1}f(n+1)}|n+1> . && (21)
\end{aligned}
$$

Furthermore, for $[n] = n$, we can deduce the results of ref. [16] from (20) and (21).

Let us turn to the case of Holstein-Primakoff realizations of $SU(2)$ and $SU(1,1)$ algebras in terms of a generalized deformed para-Bose oscillators. For $SU(2)$ we get

$$
\begin{aligned}
\tilde{J}_+ &= \tilde{b}^*\sqrt{[2\sigma - N]}, && (22)\\
\tilde{J}_- &= \sqrt{[2\sigma - N]}\tilde{a} && (23)
\end{aligned}
$$

and define the functional form of $\tilde{J}_3$ by the CR

$$[\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_3]. \quad (24)$$

Using (6) and (22)-(24) we obtain

$$[2\tilde{J}_3] = [2\sigma + 1 - N]\frac{\sqrt{[N]^*[N]}}{N}f(N) - [2\sigma - N]\frac{\sqrt{[N+1]^*[N+1]}}{N+1}f(N+1). \quad (25)$$

Hence

$$[\tilde{J}_3, \tilde{J}_+] = \{\tilde{J}_3(N) - \tilde{J}_3(N-1)\}\tilde{J}_+, \quad (26)$$

$$[\tilde{J}_3, \tilde{J}_-] = -\tilde{J}_-\{\tilde{J}_3(N) - \tilde{J}_3(N-1)\}. \quad (27)$$

In the realization of ref. [11] with $p = 1$, $F(x)$ satisfies the following relation

$$F(x)F(y+1) - F(x+1)F(y) = F(x-y),$$

then from (25) we get

$$J_3 = N - \sigma. \quad (28)$$

From (26)-(28) it follows that

$$[J_3, J_\pm] = \pm J_\pm.$$

These relations show that the CRs (24), (26) and (27) reduce to the generalized deformed bosonic $SU(2)$ algebra which was previously considered in ref. [11]. They also contain the results given in refs. [12, 13] in the limiting cases.

Similarly, Holstein-Primakoff realization of $SU(1,1)$ is

$$\tilde{K}_+ = \tilde{b}\sqrt{[2\sigma + N]},$$

$$\tilde{K}_- = \sqrt{[2\sigma + N]}\tilde{a},$$

$$-[2\tilde{K}_3] = [\tilde{K}_+, \tilde{K}_-] = [2\sigma - 1 + N]\frac{\sqrt{[N]^*[N]}}{N}f(N)$$

$$-[2\sigma + N]\frac{\sqrt{[N+1]^*[N+1]}}{N+1}f(N+1).$$

From which we can find

$$[\tilde{K}_3, \tilde{K}_+] = \{\tilde{K}_3(N) - \tilde{K}_3(N-1)\}\tilde{K}_+;$$

$$[\tilde{K}_3, \tilde{K}_-] = -\tilde{K}_-\{\tilde{K}_3(N) - \tilde{K}_3(N-1)\}.$$

Next, let us focus our attention to discuss Jordan-Schwinger realization of complex generalized deformed para-bosonic $SU(2)$ algebra. In this realization $\tilde{J}_+$ and $\tilde{J}_-$ take the form $\tilde{J}_+ = \tilde{b}_1^*\tilde{a}_2$, $\tilde{J}_- = \tilde{b}_2^*\tilde{a}_1$, and

$$[2\tilde{J}_3] = [\tilde{J}_+, \tilde{J}_-]. \quad (29)$$

It is easy to see that

$$[2\tilde{J}_3] = \frac{\sqrt{[N_1]^*[N_1]}}{N_1}\frac{\sqrt{[N_2+1]^*[N_2+1]}}{N_2+1}f(N_1)f(N_2+1) - \frac{\sqrt{[N_1+1]^*[N_1+1]}}{N_1+1}\frac{\sqrt{[N_2]^*[N_2]}}{N_2}f(N_1+1)f(N_2). \tag{30}$$

These relations can be reduced to the result of ref. [9] for $p = 1$, q is complex parameter and

$$F(x) = [x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{31}$$

Also like before from (30) we have

$$[\tilde{J}_3, \tilde{J}_+] = \{\tilde{J}_3(N_1, N_2) - \tilde{J}_3(N_1 - 1, N_2 + 1)\}\tilde{J}_+, \tag{32}$$

$$[\tilde{J}_3, \tilde{J}_-] = -\tilde{J}_-\{\tilde{J}_3(N_1, N_2) - \tilde{J}_3(N_1 - 1, N_2 + 1)\}. \tag{33}$$

It is worth mentioning here that the CRs (29), (32) and (33) coincide exactly with the corresponding CRs of the generalized deformed bosonic $SU(2)$ algebras in terms of the generalized deformed usual oscillators [19].

By the same way as mentioned above the generalized deformed bosonic algebras $SU(1,1)$ in Jordan-Schwinger realization is given by

$$[\tilde{K}_+, \tilde{K}_-] = -[2\tilde{K}_3] = \frac{\sqrt{[N_1]^*[N_1]}}{N_1}\frac{\sqrt{[N_2]^*[N_2]}}{N_2}f(N_1)f(N_2)$$

$$-\frac{\sqrt{[N_1+1]^*[N_1+1]}}{N_1+1}\frac{\sqrt{[N_2+1]^*[N_2+1]}}{N_2+1}f(N_1+1)f(N_2+1),$$

$$[\tilde{K}_3, \tilde{K}_+] = \{\tilde{K}_3(N_1, N_2) - \tilde{K}_3(N_1 - 1, N_2 - 1)\}\tilde{K}_+,$$

$$[\tilde{K}_3, \tilde{K}_-] = -\tilde{K}_-\{\tilde{K}_3(N_1, N_2) - \tilde{K}_3(N_1 - 1, N_2 - 1)\},$$

where $\tilde{K}_+ = \tilde{b}_1^*\tilde{b}_2^*$; $\tilde{K}_- = \tilde{a}_1\tilde{a}_2$.
In the case of the deformed q-harmonic oscillator for $p = 1$, q is real parameter and $F(x)$ as in (31), the above CRs yield [15]

$$[K_+, K_-] = -[2K_3], \quad [K_3, K_\pm] = \pm K_\pm,$$

with $K_3 = \frac{(N_1+N_2+1)}{2}$.

In conclusion, we can say that our result is most general because it can provide all the deformation algebras with an appropriate choice of structure function. We hope it can be applied to study the deformed quantum field theory problems.

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