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# Squeezed Variances of Smeared Boson Fields

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*Abstract.* Our previous investigations on squeezing Bogoliubov transformations in the smeared field formalism are continued. After a short introduction into the usual form of squeezing operations in quantum optics the rigorous version is formulated first in terms of certain quasifree automorphisms on the C*–Weyl algebra over a testfunction space of arbitrary dimension, and then as their dual transformations on the abstract state space. For selected classes of states the fluctuations are determined before and after such a transformation, and a general definition of a squeezed state is proposed. Especially for quasifree, classical, and coherent photon states detailed estimations for their transformed fluctuations are elaborated using a spectral theory of the general squeezing operator. Explicit criteria for non–classicality are specified and applied to squeezed white noise and other Gaussian states. It is shown, that strong squeezing of one–mode Gaussian states leads to mixed non–classical coherent states.

# 1 Introduction

There are various indicators to reveal a multi–photon state as a non–classical one, such as the negative $P$–functions, anti–bunching for the correlation functions, or sub–Poissonian counting distributions. Here we concentrate on the squeezed variances of the field operators, for which there exists an extensive experimental material in terms of phase–sensitive noise measurements (cf. e.g. [1] and references therein).

For the convenience of the reader let us recall that the theory of squeezing evolved from rather simple canonical transformations for the one–mode field (with annihilation and creation operator $a$ resp. $a^*$ for this single mode) of the following form

$$b \;:=\; \mu\, a \;+\; \nu\, a^* , \tag{1.1}$$

determining the transformed annihilation operator $b$, [2]. That the transformation (1.1) indeed is canonical, i.e., $b$ and $b^* = \overline{\mu}a^* + \overline{\nu}a$ satisfy the canonical commutation relations (CCR), requires

$$|\mu|^2 - |\nu|^2 = 1 \tag{1.2}$$

for the complex numbers $\mu$ and $\nu$. This condition yields the decomposition

$$\mu = exp\{-i\varphi\}\, cosh(s)\,, \qquad \nu = exp\{i(\varphi + \vartheta)\}\, sinh(s) \tag{1.3}$$

with a unique $s \geq 0$ and two unique phases $\varphi, \vartheta \in [0, 2\pi[$. Other but equivalent forms of one-mode squeezing have been given by [3] and [4].

A special importance was also given to two-mode transformations in the non-degenerate (different frequencies for the idler and the signal field) and in the degenerate case (equal frequencies) (cf. e.g. [5] and references therein). Since both forms can be transformed into each other (as shown in the Appendix A.1; cf. also [6]), let us present a typical multi-mode squeezing transformation in the degenerate version only, where its generator has the form

$$H_{\mathrm{sq}} = \frac{1}{2}\sum_{n=1}^{N}\Big(\zeta_n\, a^*(e_n)\, a^*(e_n) + \overline{\zeta_n}\, a(e_n)\, a(e_n)\Big)\,. \tag{1.4}$$

Here $a^*(e_n)$ is the creation and $a(e_n)$ the annihilation operator of the mode $e_n$. For convenience let us consider finitely many orthonormalized photon modes $\{e_1, \ldots, e_N\}$, which span the one-photon testfunction space $E$ with the scalar product $\langle . \mid . \rangle$. The squeezing parameters $\zeta_n$ are complex coefficients (with complex conjugates $\overline{\zeta_n}$), which in experimental realizations incorporate some classical, macroscopic pumping fields.

The squeezing Hamiltonian $H_{\mathrm{sq}}$ leads to the canonical transformation, which is given for the $n$-th mode, more exactly, for the annihilation operator $a(e_n)$, by

$$exp\{itH_{\mathrm{sq}}\}\, a(e_n)\, exp\{-itH_{\mathrm{sq}}\} = cosh(t\,|\zeta_n|)\, a(e_n) + \tfrac{\zeta_n}{|\zeta_n|}\, sinh(t\,|\zeta_n|)\, a^*(e_n)\,, \tag{1.5}$$

which is a multi-mode version of (1.1). If $f = \sum\limits_{n=1}^{N} \langle e_n \mid f\rangle\, e_n$ decomposes the arbitrary (non-monochromatic) mode $f \in E$ into the (possibly monochromatic) modes $e_n$, then the multi-mode squeezing in the smeared field formalism[1] takes the form

$$exp\{itH_{\mathrm{sq}}\}\, a(f)\, exp\{-itH_{\mathrm{sq}}\} = a(T_l f) + a^*(T_a f)\,, \tag{1.6}$$

where here $T_l$ and $T_a$ are the (complex-) linear resp. (complex-) anti-linear operators[2]

$$T_l f := \sum_{n=1}^{N} cosh(t\,|\zeta_n|)\, \langle e_n \mid f\rangle\, e_n\,, \qquad T_a f := \sum_{n=1}^{N} \tfrac{\zeta_n}{|\zeta_n|}\, sinh(t\,|\zeta_n|)\, \langle f \mid e_n\rangle\, e_n\,. \tag{1.7}$$

[1] The creation and annihilation operators "smeared" by the testfunction $f \in E$ are given as $a^*(f) = \sum_{n=1}^{N}\langle e_n \mid f\rangle\, a^*(e_n)$ resp. $a(f) = \sum_{n=1}^{N}\overline{\langle e_n \mid f\rangle} a(e_n)$, where $f = \sum_{n=1}^{N}\langle e_n \mid f\rangle\, e_n$ decomposes according to the orthonormal basis $\{e_1, \ldots, e_N\}$ of $E$. $a(f)$ and $a^*(f)$ are adjoint to each other. The mapping $E \ni f \mapsto a^*(f)$ is complex-linear, $E \ni f \mapsto a(f)$ is complex-anti-linear, and the CCR write as $[a(f), a(g)] = [a^*(f), a^*(g)] = 0$, and $[a(f), a^*(g)] = \langle f \mid g\rangle\, \mathbb{1}$ for all testfunctions $f, g \in E$.

[2] The scalar product $\langle . \mid . \rangle$ is supposed to be right linear, i.e., the mapping $E \ni f \mapsto \langle g \mid f\rangle$ is (complex-) linear and $E \ni f \mapsto \langle f \mid g\rangle$ is (complex-) anti-linear for each $g \in E$.

Introducing the positive selfadjoint operator $S$ and the anti–linear involution $J$ on $E$ by

$$S = \sum_{n=1}^{N} |\zeta_n|\, |e_n\rangle\langle e_n|\,, \qquad Je_n = \tfrac{\zeta_n}{|\zeta_n|} e_n \quad \forall n \in \{1,\ldots,N\}\,,$$

we arrive at the operator formulation of the equations (1.7): $T_l = cosh(tS)$ and $T_a = Jsinh(tS)$. Their addition gives the real–linear operator

$$T = exp\{tJS\} = cosh(tS) + J\,sinh(tS)\,, \tag{1.8}$$

which turns out to be a symplectic transformation on $E$, since

$$Im\,\langle Tf \mid Tg\rangle = Im\,\langle f \mid g\rangle \qquad \forall f,g \in E\,. \tag{1.9}$$

Let us denote by $\alpha_T$ the canonical transformation from (1.5) resp. (1.6), that is,

$$\alpha_T(A) = exp\{itH_{\text{sq}}\}\,A\,exp\{-itH_{\text{sq}}\} \qquad \text{for every field observable } A\,. \tag{1.10}$$

Then (1.6) rewrites, resp. $\alpha_T$ acts on the field operator $\Phi(f) = 2^{-1/2}(a(f) + a^*(f))$, as

$$\alpha_T(a(f)) = a(T_l f) + a^*(T_a f)\,, \qquad \alpha_T(\Phi(f)) = \Phi(Tf)\,, \qquad f \in E\,. \tag{1.11}$$

It is well known that each (real–linear) symplectic transformation $T$ on an arbitrary (finite resp. infinite dimensional) one–photon testfunction space $E$, lifts to a canonical transformation $\alpha_T$ (also called Bogoliubov transformation) on the photon field algebra — the CCR–algebra over $E$ — satisfying (1.11). The remarkable fact derived in [7] is that *every* symplectic transformation $T$ has a unique (polar) decomposition of the form (1.8), more exactly,

$$T = U\,(cosh(S) + J\,sinh(S)) \tag{1.12}$$

with a unitary $U$, and a selfadjoint positive $S$ commuting with the anti–linear involution $J$.

Let us, for example, identify the symplectic transformation $T$ on the one–dimensional one–photon space $E = \mathbb{C}e_1$ (spanned by the single photon mode $e_1$) corresponding to the canonical transformation (1.1) with $a = a(e_1)$. With (1.3) $T$ is given by

$$T(ze_1) = (\overline{\mu}\,z + \nu\,\overline{z})\,e_1 = exp\{i\varphi\}\,(cosh(s)\,z + exp\{i\vartheta\}\,sinh(s)\,\overline{z})\,e_1\,, \quad \forall z \in \mathbb{C}\,, \tag{1.13}$$

determining the decomposition (1.12). Condition (1.2) is equivalent to the symplectic relation (1.9).

If the canonical transformation $\alpha_T$ is shifted from the field observables (Heisenberg picture), to the state space (Schrödinger picture; cf. Subsection 2.3), then it may be applied to selected photon field states, as coherent or thermal states, and may then produce *squeezed states*. For our above finite dimensional $E$, the photon field states are given by the density operators $\rho$ on Fock space (in virtue of the Stone–von Neumann uniqueness theorem [8]). Thus the canonical transformation (1.10) writes in the Schrödinger picture as

$$\nu_T(\rho) \quad := \quad exp\{-itH_{\mathrm{sq}}\}\, \rho \, exp\{itH_{\mathrm{sq}}\} \qquad \text{for every density operator } \rho .$$

Squeezing operations $\alpha_T$ resp. $\nu_T$ could in fact be realized since 1986 by various experimental methods for situations, where some few modes dominate. For multi–mode squeezing we refer to [5] and [9]. The hitherto presented material has been rigorously derived in [6], [7]. Having so far recalled the most basic features of squeezing (Bogoliubov) transformations, let us now turn to the purpose of the present investigation.

Although there are rather simple theoretical models for states with squeezed fluctuations, the experimental developments require more general and refined theoretical methods. In general the prepared multi–photon states are non–pure and extend over (infinitely) many modes. Their fluctuations before and after a squeezing procedure are a combination of classical and quantum–mechanical variances. The effectiveness of a squeezing device can then no longer be treated by means of explicit mode–dependent analytical calculations but has to be estimated in a qualitative way. This is without doubt a challenge for mathematical physics. And the aim of the present work is to contribute to a qualitative squeezing theory, applicable to general classes of states, which are relevant in quantum optics.

In [10] we investigated under which squeezing transformations some frequently used optical Boson states, namely the quasifree, the classical, and the coherent states, obtain a non–classical generating function. With the present work we continue this analysis by calculating and comparing the field fluctuations before and after a Bogoliubov–squeezing transformation.

Our investigation is presented in terms of a rigorous smeared Boson field theory, which is based on an arbitrary one–Boson testfunction space $E$, a complex pre–Hilbert space. The choice of $E$ determines the specific Boson system and the number of modes taken into account. In this way both finitely and infinitely many field modes are covered. For massive Bosons with spin $s$, $E$ is a subspace of $\mathsf{L}^2(\Lambda) \otimes \mathbb{C}^{2s+1}$, where $\Lambda \subseteq \mathbb{R}^3$ is the quantization volume in position space. For photons the quantization procedure in the Coulomb gauge leads to a testfunction space $E$ consisting of square–integrable, divergence–free functions $f : \Lambda \to \mathbb{C}^3$ [11], [12], [13].

For the description of the Boson field states the technique of generating functions is used (cf. Subsection 2.1). These characteristic functions (in symmetric ordering) are independent of any Hilbert space representation of the CCR, and are (for an finite dimensional $E$) closely related to the $W$– (or Wigner–), $P$–, and $Q$–representations of photon states $\omega$ [12], [14], [6], which is exhibited in the Appendix A.2. The characteristic function $C_\omega(f) = \langle W(f)\rangle_\omega$, $f \in E$, of the Boson field state $\omega$ is the expectation of the smeared Weyl or displacement operator $W(f) = exp\{i\Phi(f)\}$ and contains all statistical informations about the distribution of the field observables $\Phi(f) = 2^{-1/2}(a(f) + a^*(f))$. Especially in QED the observables of the magnetic and electric field are summarized into the smeared field expressions $\Phi(f)$ with (complex) testfunctions $f \in E$, and the expectation values of the products of the field operator $\Phi(f)$ for the photon state $\omega$ are obtained by differentiating $C_\omega(tf)$ with respect to the real parameter $t$ in a similar way as in classical probability theory. Most importantly, the field variances $\mathsf{Var}(\omega; f) := \langle \Delta\Phi(f)^2 \rangle_\omega$ for each testfunction $f \in E$ are obtained as

$$\mathsf{Var}(\omega; f) = \langle \Phi(f)^2 \rangle_\omega - \langle \Phi(f) \rangle_\omega^2 = \left( \frac{dC_\omega(tf)}{dt}\Big|_{t=0} \right)^2 - \frac{d^2 C_\omega(tf)}{dt^2}\Big|_{t=0} .$$

Bogoliubov transformations are in one-to-one correspondence with the (real-linear) symplectic transformations on the testfunction space $E$ (cf. the considerations above and the Subsections 2.2 and 2.3). Since for the symplectic $T$ the associated Bogoliubov transformation in the Schrödinger picture is the affine bijection $\nu_T$ on the state space of the Bosonic C*-Weyl algebra $\mathcal{W}(E)$, one has $C_{\nu_T(\omega)}(f) = C_\omega(Tf)\ \forall f \in E$ for the characteristic function of every Boson state $\omega$ and its Bogoliubov transform $\nu_T(\omega)$. This relation between the characteristic functions allows for the calculation of the field variances of the Bogoliubov transformed $\nu_T(\omega)$ from those of the original state $\omega$ (cf. Subsection 2.4).

The squeezing concept in the smeared field formalism is introduced in Subsection 2.5. Let us emphasize here that the smearing is not only indispensable for a mathematical definition of the field operators, but provides us in this connection via the testfunctions also with those notions, which express the relevant fluctuation aspects of a squeezing procedure. In physical experiments and theoretical applications only a limited range of testmodes is taken into account, which is assumed (without restriction of generality) to be a real- or complex-linear subspace $F$ of $E$. That is, every manifestation of the field fluctuations is realized through an $F$-*window*, which is given by the experimental or theoretical possibilities. Thus, the altered fluctuations by a squeezing device are also realized through this $F$-window, which in general is wider than a one-mode test space and smaller than the entire mode space $E$. Our general definition of squeezing is adapted to this $F$-dependence and compares the minimal fluctuations before and after a change $\nu_T$ in the state preparation as they appear through this $F$-window. In the definition of $F$-*squeezing* nothing is said about the origin of a possible diminishing of fluctuations, and the concept may of course be used also for purely classical fluctuations. But also for genuine quantum fluctuations $F$-squeezing may come about by a mere *rotation* in the testmode space $E$ (i.e., $T$ is a unitary transformation on $E$), as will be demonstrated by some examples and then has nothing to do with a typical squeezing operation. Only a refined analysis is capable to identify those squeezed fluctuations which characterize a non-classical state. A theoretically important special case is of course $F = E$. In this case, and quite generally if $T(F) = F$, the occurrence of $F$-squeezing implies always a non-vanishing anti-linear part $T_a \neq 0$ of the real-linear symplectic $T$.

With the polar decomposition (1.12), we split in Subsection 3.1 the symplectic $T$ into a direct sum $T = U(e^S \oplus e^{-S})$ acting on two orthogonal *real* subspaces. Here the selfadjoint positive operator $S$ may unbounded with respect to the norm. This decomposition formula enables a general and detailed investigation of the squeezing properties for the state $\omega$ in terms of (real) Hilbert space methods, whenever the above positive symmetric real-bilinear form $v_\omega$ for the considered state $\omega$ is known. Especially, in this way we obtain results concerning the squeezing of states with bounded fluctuations (i.e., the form $v_\omega$ is bounded) when we perform an unbounded squeezing transformation $T$, i.e., with unbounded $S$.

Classical states are the most easily prepared ones in experiments. For example, the vacuum and the macroscopic coherent photon states of a maser or a laser are classical [15],

[16]. Also the thermal equilibrium states, the limiting Gibbs resp. KMS–states (with and without Bose–Einstein condensation), are classical and in addition are quasifree ([17], [8], and references therein). A classical state here is characterized by the positive definiteness of its normally ordered characteristic function, which is equivalent to a positive $P$–representation (see the Appendix A.2 and Subsection 3.4.2, also [18], [19], [20]).

The squeezing properties of the vacuum state are given in Subsection 3.2. In Subsection 3.4 we show that the fluctuations of the classical states are always larger than the vacuum fluctuations. But those of the extremal (that are the pure) classical states agree with the vacuum variances. Then some estimates concerning the fluctuations for squeezed classical states are deduced. The squeezing of quasifree states is treated in Subsection 3.5. Subsection 3.6 is devoted to the squeezing of optical coherent states, which in addition have bounded fluctuations. Bounded fluctuations for coherent states are equivalent to the square–integrability of Glauber's factorizing coherence function [15], [16], [21], [22]. It is found that the squeezing properties of the coherent states are directly connected with the spectrum of the selfadjoint $S$ occurring in the decomposition (1.12) of $T$. Optimal squeezing (here over the window $F = E$) would produce as minimal fluctuations those of the squeezed vacuum (subjected to the same squeezing procedure). Whether this optimum can be realized, depends on the relation between the squeezing operator $T$ (with rotation $U$ and strength–spectrum $\sigma(S)$) and the bounded factorizing coherent field function. In Subsection 3.6.2 it is completely analyzed for which conditions optimal squeezing of a coherent state is achieved. It is illustrated that these conditions are *almost sharp* since certain violations of them prevent optimal squeezing (for certain coherent states).

Even more detailed relations are worked out for a one–parameter family of quasifree coherent states. Here the original, classical fluctuations determine precisely the squeezing strength, which is necessary to render the squeezed state non–classical. These results lead to a refined analysis of squeezed white noise [23].

In the Conclusions (Section 4) some popular criteria for identifying a state as non–classical are discussed and compared with those derived in the present work.

Let us make some notational remarks. Throughout the paper the one–photon testfunction space $E$ is a complex separable pre–Hilbert space with norm–completion $\mathcal{H}$ and the (right linear) scalar product $\langle . \mid . \rangle$. $F^{\perp}$ means the orthogonal complement of the subset $F$ in $\mathcal{H}$ with respect to $\langle . \mid . \rangle$. With the real scalar product $( . \mid . ) := Re\langle . \mid . \rangle$, the complex Hilbert space $\mathcal{H}$ becomes a real Hilbert space, which in the sequel is denoted by $\mathcal{H}_r$. If $F \subseteq E$ is a complex subspace, then $dim_{\mathbb{C}}(F)$ denotes its complex dimension, if $F$ is real we write $dim_{\mathbb{R}}(F)$ for its real dimension, especially, $dim_{\mathbb{C}}(F) = 2\, dim_{\mathbb{R}}(F)$ for complex $F$.

On the one–particle level (on $\mathcal{H}$ resp. on $\mathcal{H}_r$), there occur complex–linear, complex–anti–linear, and real–linear operators. "linear" resp. "anti–linear" always mean complex–linear resp. complex–anti–linear, and into the notations "operator" and "unitary" we include complex–linearity on $\mathcal{H}$. Real–linear mappings are always denoted "real–linear operators" resp. "real–linear unitaries" with respect to $( . \mid . )$ on $\mathcal{H}_r$. $B|_K$ is our notation for the restriction of the real– or complex–(anti–)linear operator $B$ to the subset $K$ of its domain

$D(B)$, $ker(B)$ means the kernel of $B$. On the level of the second quantization (C*–algebraic level), however, we are concerned with complex–linear operators, only.

# 2 Squeezing Transformations

## 2.1 Preliminaries concerning the Weyl Algebra

For more details to the present Subsection we refer to [8, Section 5.2]. The C*–algebra of the Boson system is the Weyl algebra $\mathcal{W}(E)$, also called the CCR–algebra over $E$. $\mathcal{W}(E)$ is generated by the unitary Weyl operators $W(f)$, $f \in E$, satisfying the Weyl relations

$$W(f)\,W(g) \;=\; exp\big\{-\tfrac{i}{2}\; Im\,\langle f \mid g\rangle\big\}\,W(f+g)\,, \qquad W(f)^* \;=\; W(-f)\,, \quad \forall f,g \in E\,.$$

$\mathcal{S}$ denotes the convex, weak*–compact state space of $\mathcal{W}(E)$. Each element of its extreme boundary $\partial_e\mathcal{S}$ is denoted a pure states. Each $\omega \in \mathcal{S}$ is uniquely determined by its expectations of the Weyl operators, that is by its *characteristic function* [15]

$$C_\omega : E \longrightarrow \mathbb{C}\,, \quad f \longmapsto C_\omega(f) \;:=\; \langle \omega\,;\, W(f)\rangle\,. \tag{2.1}$$

A state $\omega \in \mathcal{S}$ is called *regular*, if for each $f \in E$ the map $\mathbb{R} \ni t \mapsto C_\omega(tf)$ is continuous. In the GNS–representation $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ [8, Theorem 2.3.16] of the regular $\omega \in \mathcal{S}$ the selfadjoint field operators $\Phi_\omega(f) := -i\frac{\mathrm{d}}{\mathrm{d}t}\Pi_\omega(W(tf))|_{t=0}$, $f \in E$, fulfill the CCR

$$[\Phi_\omega(f), \Phi_\omega(g)] \;\subseteq\; Im\,\langle f \mid g\rangle\,\mathbb{1} \qquad \forall\, f,g \,\in\, E\,. \tag{2.2}$$

The map $E \ni f \mapsto \Phi_\omega(f)$ is real–linear. The annihilation and creation operators, $a_\omega(f) := \sqrt{2}^{-1}(\Phi_\omega(f) + i\Phi_\omega(if))$ and $a_\omega^*(f) := \sqrt{2}^{-1}(\Phi_\omega(f) - i\Phi_\omega(if))$, associated with $\omega$ are densely defined, closed, it is $a_\omega(f)^* = a_\omega^*(f)$, $f \mapsto a_\omega(f)$ is anti–linear and $f \mapsto a_\omega^*(f)$ is linear. The CCR for the annihilation and creation operators $[a_\omega(f), a_\omega^*(g)] \subseteq \langle f \mid g\rangle\,\mathbb{1}$ follow from (2.2).

$\omega \in \mathcal{S}$ is said to be of class $\mathcal{C}^m$, or a $\mathcal{C}^m$–state, if $\mathbb{R} \ni t \mapsto C_\omega(tf)$ is $m$–times differentiable for every $f \in E$, where $m \in \mathbb{N}\bigcup\{\infty\}$. If $\omega$ is of class $\mathcal{C}^{2m}$, then the associated cyclic vector $\Omega_\omega$ is contained in the domain of each polynomial of field operators with degree $\leq m$, in which case one commonly defines

$$\langle \omega\,;\, \Phi_\omega(f_1)\cdots\Phi_\omega(f_{2m})\rangle \;:=\; \langle \Phi_\omega(f_m)\cdots\Phi_\omega(f_1)\Omega_\omega \mid \Phi_\omega(f_{m+1})\cdots\Phi_\omega(f_{2m})\Omega_\omega\rangle\,.$$

$\omega$ is called *analytic*, if for each $f \in E$ the function $\mathbb{R} \ni t \mapsto C_\omega(tf)$ is analytic in a neigborhood of the origin. If for every $f \in E$ this neighborhood is $\mathbb{R}$, then $\omega$ is *entire-analytic*.

## 2.2 Symplectic Transformations

A *symplectic transformation* on $E$ is a real-linear, bijective mapping $T : E \to E$ satisfying $Im\langle Tf \mid Tg\rangle = Im\langle f \mid g\rangle\ \forall f, g \in E$ (cf. the Introduction). The group of all symplectic transformations on $E$ is denoted by $\mathcal{T}(E)$.

Since $E$ is a complex vector space the real-linear $T \in \mathcal{T}(E)$ uniquely decomposes into its linear part $T_l$ and its anti-linear part $T_a$,

$$T = T_l + T_a, \qquad T_l = \tfrac{1}{2}(T - iTi), \qquad T_a = \tfrac{1}{2}(T + iTi).$$

Observe that the multiplication with the complex "$i$" in general does not commute with the real-linear $T$. In [7] we derived the polar decomposition (1.12) for $T \in \mathcal{T}(E)$:

**Theorem 2.1** *Let $T \in \mathcal{T}(E)$. Then on $\mathcal{H}$ there exist a unique positive selfadjoint operator $S$, a unique unitary $U$, and an anti-linear involution $J$ (that is, $J = J^* = J^{-1}$) unique on $ker(S)^\perp$, so that $J$ commutes with $S$ (especially, $J(ker(S)) = ker(S)$) and*

$$T_l = U\, cosh(S)|_E, \qquad T_a = U\, J\, sinh(S)|_E.$$

*Moreover, $E$ is a core for $\exp\{S\}$, and the following assertions are equivalent:*

*(i) $T$ commutes with the complex "$i$", i.e., $T$ is complex-linear,*

*(ii) the anti-linear part vanishes, $T_a = 0$,*

*(iii) $T$ is unitary, $T = U$,*

*(iv) the selfadjoint $S$ vanishes, $S = 0$.*

## 2.3 Bogoliubov Transformations

For each $T \in \mathcal{T}(E)$ there exists a (unique) $^*$-automorphism $\alpha_T$ on $\mathcal{W}(E)$ with

$$\alpha_T(W(f)) = W(Tf) \qquad \forall f, g \in E,$$

which is called the associated Bogoliubov or canonical transformation [8, Theorem 5.2.8]. Its dual mapping $\nu_T := \alpha_T^*$ is an affine bijection on the state space $\mathcal{S}$,

$$\langle \nu_T(\omega)\,;\, A\rangle = \langle \omega\,;\, \alpha_T(A)\rangle \qquad \forall \omega \in \mathcal{S} \quad \forall A \in \mathcal{W}(E).$$

Obviously, $(\nu_T)^{-1} = \nu_{T^{-1}}$, and $C_{\nu_T(\omega)} = C_\omega \circ T$ for all $\omega \in \mathcal{S}$. Since a symplectic $T \in \mathcal{T}(E)$ is real-linear and bijective, the state $\omega \in \mathcal{S}$ is regular, of class $\mathcal{C}^m$, analytic, or entire-analytic, if and only if $\nu_T(\omega)$ is so, respectively.

## 2.4 Fluctuations of the Field Expectation Values

For every $\mathcal{C}^2$–state $\omega \in \mathcal{S}$ the variance (fluctuation) of the expectation value for the field operator $\Phi_\omega(f)$, $f \in E$, is calculated in terms of its characteristic function $C_\omega$

$$\mathsf{Var}(\omega; f) := \left\langle \omega\,;\, \Phi_\omega(f)^2 \right\rangle - \langle \omega\,;\, \Phi_\omega(f)\rangle^2 = \left( \frac{\mathrm{d}C_\omega(tf)}{\mathrm{d}t}\Big|_{t=0} \right)^2 - \frac{\mathrm{d}^2 C_\omega(tf)}{\mathrm{d}t^2}\Big|_{t=0} . \tag{2.3}$$

A $\mathcal{C}^2$–state $\omega$ is called to have *bounded fluctuations*, if the associated quadratic form $E \ni f \mapsto \mathsf{Var}(\omega; f)$ is bounded [24],

$$\mathsf{Var}(\omega; f) \leq c\,\|f\|^2 \quad \forall f \in E\,, \quad \text{for some } c \geq 0\,. \tag{2.4}$$

For Bogoliubov transformed states one obviously obtains the relation

$$\mathsf{Var}(\nu_T(\omega); f) = \mathsf{Var}(\omega; Tf) \qquad \forall f \in E \quad \forall T \in \mathcal{T}(E)\,, \tag{2.5}$$

which allows the calculation of the variances of the transformed state $\nu_T(\omega)$ from those of the original $\mathcal{C}^2$–state $\omega \in \mathcal{S}$.

From the CCR (2.2) follows the Heisenberg uncertainty principle

$$\mathsf{Var}(\omega; f)\,\mathsf{Var}(\omega; g) \geq \tfrac{1}{4}\,|Im\,\langle f \mid g\rangle|^2 \qquad \forall\, f, g \in E\,. \tag{2.6}$$

Here we only have to demand $\omega \in \mathcal{S}$ to be of class $\mathcal{C}^2$, since the relations (2.2) are also valid in the weak sense [25]. For each real or complex subspace $F \subseteq E$ let us define the infimum of the variances with respect to $F$,

$$\mathsf{InfVar}(\omega; F) := inf\{\mathsf{Var}(\omega; f) \mid f \in F,\ \|f\| = 1\}\,,$$

and similarly the supremum $\mathsf{SupVar}(\omega; F) := sup\,\{\mathsf{Var}(\omega; f) \mid f \in F,\ \|f\| = 1\}$. Then the Heisenberg uncertainty relations (2.6) imply the

**Observation 2.2** $\mathsf{InfVar}(\omega; F) = 0$ *implies* $\mathsf{SupVar}(\omega; F) = \infty$. *If* $\mathsf{InfVar}(\omega; F) \neq 0$, *then we have* $\mathsf{InfVar}(\omega; F)\,\mathsf{SupVar}(\omega; F) \geq \frac{1}{4}$ *for each* complex *subspace* $F \subseteq E$.

*For shortness we adopt in the following the convention, that every state $\omega \in \mathcal{S}$ is automatically of class $\mathcal{C}^2$, if we investigate its fluctuations.*

## 2.5 General Definition of Squeezing

For the qualification of noise reduction the variances of the field values in the transformed state $\nu_T(\omega)$ are compared with those of a reference state $\varphi$. The reference state $\varphi$ usually is chosen as $\omega$ itself, or as the (Fock) vacuum state $\omega_{\mathsf{vac}}$ (cf. Subsection 3.2 below). One also is interested how the set $\{\mathsf{Var}(\omega; zf) \mid z \in \mathbb{C},\ |z| = 1\}$ becomes deformed by transforming

$\omega$ with $\nu_T$. Especially the variances associated with conjugate pairs, $\Phi_\omega(f)$ and $\Phi_\omega(if)$ (quadrature components), $f \in E$, of the state $\omega$ and those of the transformed state $\nu_T(\omega)$ are considered.

We assume that before and after the squeezing procedure the field is observed by a detector sensitive in the same testmodes. Or, spoken as in the Introduction, the detection of $\omega$ and $\nu_T(\omega)$ is realized by the same $F$-window. These *observable* modes are summarized into the set $\{f \in F \mid \|f\| = 1\}$ with $F$ a real or complex subspace of $E$. The normalization of the testfunctions is necessary for comparison reasons (which is in analogy to the mathematical definition of the norm of a bounded operator on a Banach space).

We propose the following definition of squeezing using the original state for itself as reference state, respectively fixing an $F$-window.

**Definition 2.3 (Squeezing)** *Let $T \in \mathcal{T}(E)$. For a ($C^2$-) state $\omega \in \mathcal{S}$ we say:*

*(a) The state $\omega$ is squeezed by $\nu_T$ in the testmode $f \in E$, if*

$$\mathsf{Var}(\nu_T(\omega); f) \ < \ \mathsf{Var}(\omega; f)\,.$$

*(b) $\omega$ is effectively squeezed by $\nu_T$ in the subspace $F \subseteq E$, or simply $F$-squeezed, if*

$$\mathsf{InfVar}(\nu_T(\omega); F) \ < \ \mathsf{InfVar}(\omega; F)\,.$$

If the variance $\mathsf{Var}(\nu_T(\omega); f)$ is smaller than $\mathsf{Var}(\omega; f)$, in many cases the conjugate variance $\mathsf{Var}(\nu_T(\omega); if)$ becomes larger than $\mathsf{Var}(\omega; if)$ in virtue of the uncertainty relation (2.6).

Let us first demonstrate that the squeezing effect essentially arises from the anti-linear part $T_a$ of the associated symplectic transformation $T$.

Assume $T \in \mathcal{T}(E)$ with $T_a = 0$. Then by Theorem 2.1 $T$ acts unitarily on $E$. Consequently, $\{\mathsf{Var}(\nu_T(\omega); f) \mid f \in E,\ \|f\| = 1\} = \{\mathsf{Var}(\omega; f) \mid f \in E,\ \|f\| = 1\}$ for every state $\omega \in \mathcal{S}$, and one has no effect, if one is interested in all testmodes simultaneously (i.e., $E$-squeezing). However, for $T_a = 0$ one may obtain $F$-squeezing, when in the state $\omega$ some variances of the $T$-transformed testmodes $T(F)$ are smaller than the variances of the non-transformed modes $F$ (see Example 3.12 below). For a real or complex subspace $F \subseteq E$ with $T(F) = F$, however, $F$-squeezing by $\nu_T$ is impossible for $T_a = 0$.

**Proposition 2.4** *Let $F$ be a real or complex subspace of $E$ and $T \in \mathcal{T}(E)$ with $T(F) = F$. If $\omega \in \mathcal{S}$ is $F$-squeezed by $\nu_T$, then $T_a|_F \neq 0$, or equivalently $F \cap \mathsf{ker}(S)^\perp \neq \{0\}$ for the positive selfadjoint $S$ occuring in the decomposition of $T$ by Theorem 2.1.*

PROOF: Assume $T_a|_F = 0$. Then by Theorem 2.1 $F \subseteq \mathsf{ker}(S)$, and thus $\|Tf\| = \|f\|\ \forall f \in F$. Equation (2.5) now implies $\{\mathsf{Var}(\nu_T(\omega); f) \mid f \in F, \|f\| = 1\} = \{\mathsf{Var}(\omega; f) \mid f \in F, \|f\| = 1\}$, which is a contradiction to the supposed $F$-squeezing of $\omega$. ∎

From the above Proposition it especially follows, that $E$-squeezing by $\nu_T$ is always a consequence of a non-vanishing anti-linear part of $T$. But, on the other hand, also for $T_a \neq 0$ there exist some states on $\mathcal{W}(E)$ which are not $E$-squeezed by $\nu_T$ (see Example 3.13).

# 3 Field Variances in Squeezed States

Throughout the present Section (up to Subsection 3.6.3) we suppose a fixed (but arbitrary) symplectic $T \in \mathcal{T}(E)$ with the polar decomposition $T = U(cosh(S) + J\, sinh(S))$ from Theorem 2.1. The spectrum of $S \neq 0$ is denoted by $\sigma(S)$, and if $S$ is unbounded, we put $\|S\| := \infty$, writing then $exp\{-2\|S\|\} = 0$ and $exp\{2\|S\|\} = \infty$.

## 3.1 Decomposition of Testfunctions

The involution $J$ is a real-linear selfadjoint unitary on the real Hilbert space $\mathcal{H}_r$ (recall, $\mathcal{H}_r$ is the completion $\mathcal{H}$ of $E$, equipped with the real scalar product $(.\mid .) := Re\langle . \mid . \rangle$) with the eigenvalues $\pm 1$. The associated (real) eigenspaces $H_\pm$ are given by the

**Lemma 3.1** *The orthogonal eigenspaces $H_\pm$ for $J$ (with respect to $(.\mid .)$) are given by*

$$H_\pm := \{f \in \mathcal{H} \mid Jf = \pm f\} = \{h \pm Jh \mid h \in \mathcal{H}\},$$

*especially, $\mathcal{H}_r = H_+ \oplus H_-$. If $P_\pm$ are the orthogonal (with respect to $(.\mid .)$) real-linear projections from $\mathcal{H}_r$ onto $H_\pm$, the spectral projections for $J$, then $P_+ f = \frac{1}{2}(f + Jf)$ and $P_- f = \frac{1}{2}(f - Jf)$ for all $f \in \mathcal{H}_r$. Moreover, it is $H_- = iH_+$ and $P_- i = iP_+$.*

Since $S$ and $J$ commute, it follows that the real-linear $P_\pm$ commute with the (complex-linear) spectral projection $E_S(\mathcal{B})$ of $S$ for every Borel subset $\mathcal{B}$ of $\mathbb{R}$. Especially $exp\{\pm S\}$ leave $H_+$ and $H_-$ invariant.

The symplectic $T \in \mathcal{T}(E)$ is a real-linear closable operator on the real Hilbert space $\mathcal{H}_r$, [7]. Obviously, its closure $\overline{T} = U(cosh(S) + Jsinh(S))$ decomposes according to the direct sum $\mathcal{H}_r = H_+ \oplus H_-$ as

$$\overline{T} = U\left(exp\{S\}\,|_{H_+} \oplus exp\{-S\}\,|_{H_-}\right) = U\left(exp\{S\}\,P_+ + exp\{-S\}\,P_-\right), \tag{3.1}$$

which implies $\|Tf\|^2 = \|exp\{S\}\,P_+ f\|^2 + \|exp\{-S\}\,P_- f\|^2$ for all $f \in E$.

From equation (3.1) it immediately follows that

$$T^{-1} = \left(exp\{-S\}\,|_{H_+} \oplus exp\{S\}\,|_{H_-}\right) U^*|_E = \left(exp\{-S\}\,P_+ + exp\{S\}\,P_-\right) U^*|_E, \tag{3.2}$$

which is in accordance with the relations $(T^{-1})_l = T_l^*|_E$ and $(T^{-1})_a = -T_a^*|_E$ for the linear resp. anti-linear part of $T^{-1} \in \mathcal{T}(E)$ known from [7].

**Proposition 3.2** *Let $\omega \in \mathcal{S}$ have bounded fluctuations (cf. equation (2.4)). Then $\|S\| = \infty$ implies $\mathsf{InfVar}(\nu_T(\omega); E) = 0$.*

PROOF: With the equation (2.5) we obtain $0 \leq \mathsf{Var}(\nu_T(\omega); f) = \mathsf{Var}(\omega; Tf) \leq c\,\|Tf\|^2$. Since $E$ is a core for $exp\{S\}$ (by Theorem 2.1) it follows from equation (3.1) that $inf\{\|Tf\| \mid f \in E,\ \|f\| = 1\} = inf\{\|e^{-S}P_-f\| \mid f \in \mathcal{H},\ \|f\| = 1\}$, which gives the result. ■

**Corollary 3.3** *Let $\|S\| = \infty$. Suppose $E$ to be a core for $exp\{(1+\tau)S\}$ for some $0 \leq \tau < 1$ and $U^*(E) \subseteq \mathcal{D}(exp\{\tau S\})$. Then for the $C^2$-state $\omega \in \mathcal{S}$ with $\mathsf{Var}(\omega; f) \leq c\,\|exp\{\tau S\}\,U^*f\|^2$ $\forall f \in E$ for some $c > 0$ we have $\mathsf{InfVar}(\nu_T(\omega); E) = 0$.*

PROOF: Is analoguously to the proof of the foregoing Proposition. ■

## 3.2 Squeezing of the Vacuum

The characteristic function $C_{\text{vac}}$ of the vacuum state $\omega_{\text{vac}} \in \mathcal{S}$ is given by [8, Subsection 5.2.3]

$$C_{\text{vac}}(f) \;=\; \langle \omega_{\text{vac}}\,;\, W(f)\rangle \;=\; exp\left\{-\tfrac{1}{4}\,\|f\|^2\right\} \qquad \forall f \in E. \tag{3.3}$$

The vacuum fluctuations are the variances of the field values for the vacuum state $\omega_{\text{vac}}$. With formula (2.3) they are easily determined to be

$$\mathsf{Var}(\omega_{\text{vac}}; f) \;=\; \tfrac{1}{2}\,\|f\|^2\,, \qquad f \in E. \tag{3.4}$$

Equation (2.5) gives the variances for the Bogoliubov transformed vacuum state $\nu_T(\omega_{\text{vac}})$

$$\mathsf{Var}(\nu_T(\omega_{\text{vac}}); f) \;=\; \tfrac{1}{2}\,\|Tf\|^2\,, \qquad f \in E.$$

For the $E$–squeezing properties of the vacuum we have the

**Proposition 3.4** *The minimal squeezing fluctuation is*

$$\mathsf{InfVar}(\nu_T(\omega_{\text{vac}}); E) \;=\; \tfrac{1}{2}\,exp\{-2\,\|S\|\}\,,$$

*which is strictly smaller than the vacuum fluctuations $\mathsf{InfVar}(\omega_{\text{vac}}; E) = \frac{1}{2}$, if and only if $S \neq 0$, or equivalently, if and only if $T_a \neq 0$.*

*Furtheron, $\mathsf{SupVar}(\nu_T(\omega_{\text{vac}}); E) = \frac{1}{2}\,exp\{2\,\|S\|\}$, which agrees with Observation 2.2.*

PROOF: The spectral calculus for the positive selfadjoint operator $S$ gives $\|e^S f\| \geq \|f\| \geq \|e^{-S} f\|$. But by equation (3.1) we have $inf\{\|Tf\| \mid f \in E,\ \|f\| = 1\} = inf\{\|e^{-S}P_-f\| \mid f \in \mathcal{H},\ \|f\| = 1\}$, and $sup\{\|Tf\| \mid f \in E,\ \|f\| = 1\} = sup\{\|e^S P_+ f\| \mid f \in \mathcal{D}(e^S),\ \|f\| = 1\}$, which gives the result. ■

## 3.3 Fluctuations and Normally Ordered Characteristic Function

The normally ordered characteristic function $P_\omega : E \to \mathbb{C}$ of a state $\omega \in \mathcal{S}$ is defined by $P_\omega := C_\omega / C_{\text{vac}}$, i.e., $P_\omega(f) = exp\left\{\frac{1}{4}\|f\|^2\right\} C_\omega(f)\ \forall f \in E$. For an entire-analytic state $\omega$, the function $P_\omega$ decomposes in terms of the normally ordered expectations $\left\langle \omega\,;\, a^*_\omega(f)^k a_\omega(f)^l \right\rangle$,

$$P_\omega(f) \;=\; \sum_{k,l=0}^{\infty} \left(\frac{i}{\sqrt{2}}\right)^{k+l} \frac{1}{k!}\,\frac{1}{l!}\,\left\langle \omega\,;\, a^*_\omega(f)^k a_\omega(f)^l \right\rangle\,, \tag{3.5}$$

which converges absolutely for every testfunction $f \in E$ [15]. For finite dimensional $E$, the function $P_\omega$ is directly connected with the $P$–representation (see Appendix A.2, [12], [14]).

If $\omega \in \mathcal{S}$ is of class $\mathcal{C}^2$, then $\mathbb{R} \ni t \mapsto P_\omega(tf)$ is two times continuously differentiable, and similarly to equation (2.3) one obtains

$$\langle \omega\,;\, \Phi_\omega(f)\rangle \;=\; -i\,\frac{\mathrm{d}C_\omega(tf)}{\mathrm{d}t}\bigg|_{t=0} \;=\; -i\,\frac{\mathrm{d}P_\omega(tf)}{\mathrm{d}t}\bigg|_{t=0},$$

and the variances may be expressed in terms of $P_\omega$ and the vacuum fluctuations (3.4)

$$\mathsf{Var}(\omega; f) = \mathsf{Var}(\omega_{\text{vac}}; f) + \Delta(\omega; f), \qquad \Delta(\omega; f) := \left(\frac{\mathrm{d}P_\omega(tf)}{\mathrm{d}t}\bigg|_{t=0}\right)^2 - \frac{\mathrm{d}^2 P_\omega(tf)}{\mathrm{d}t^2}\bigg|_{t=0}. \tag{3.6}$$

## 3.4 Squeezing of Classical States

### 3.4.1 The Generalized Glauber States

Let us denote by $a_F(f)$ and $a^*_F(f)$, where $f \in \mathcal{H}$, the usual annihilation and creation operators acting on the (Bose–) Fock space $F_+(\mathcal{H})$ over the completion $\mathcal{H}$ of $E$. With the Fock field operators, $\Phi_F(f) = 2^{-1/2}(a_F(f) + a^*_F(f))$, the Fock–Weyl operators $W_F(f) = exp\{i\Phi_F(f)\}$ are constructed for each $f \in \mathcal{H}$. The (abstract) Weyl operators $W(f) \in \mathcal{W}(E)$, however, are defined for the testfunctions from $E$, only. Thus the Fock representation $\Pi_F$ of $\mathcal{W}(E)$ on $F_+(\mathcal{H})$ fulfills $\Pi_F(W(f)) = W_F(f)$ only for $f \in E$. The (normalized) vacuum vector $\Omega_{\text{vac}} \in F_+(\mathcal{H})$ satisfies $a_F(f)\Omega_{\text{vac}} = 0\ \forall f \in \mathcal{H}$ (e.g. [8, Subsection 5.2.1], [26, Section X.7], [6], and also [27, Section 8.1]).

The Glauber vector $G(h) \in F_+(\mathcal{H})$ (in quantum optics called a coherent state vector), $h \in \mathcal{H}$, is given by the displacement of the vacuum vector, $G(h) = W_F(-i\sqrt{2}h)\,\Omega_{\text{vac}}$, [18], [14]. Especially, $G(0) = \Omega_{\text{vac}}$ for $h = 0$. The associated Glauber state $\omega^G_h$ on the Weyl algebra, $\langle \omega^G_h\,;\, A\rangle = \langle G(h) \mid \Pi_F(A)\, G(h)\rangle$, $A \in \mathcal{W}(E)$, has the characteristic function [28]

$$C^G_h(f) \;=\; C_{\text{vac}}(f)\, exp\left\{i\,\sqrt{2}\; Re\langle h \mid f\rangle\right\} \qquad \forall f \in E\,. \tag{3.7}$$

For $h = 0$ the vacuum state $\omega_{\text{vac}} = \omega^G_0$ from Subsection 3.2 is obtained.

The mapping $E \ni f \mapsto exp\{i\sqrt{2}\, Re\langle h \mid f\rangle\}$ appearing in the Glauber characteristic function $C_h^G$ is a character[3] on $E$. This observation gives rise to the following generalization of the Glauber states: If $\chi$ is an element of the character group $\widehat{E}$ of the additive group $E$, then the characteristic function

$$C_\chi(f) \quad = \quad C_{\text{vac}}(f)\, \chi(f) \qquad \forall f \in E \tag{3.8}$$

generalizes (3.7) and determines the unique states $\varphi_\chi$ on $\mathcal{W}(E)$. Indeed, it is $\varphi_\chi = \omega_{\text{vac}} \circ \gamma_\chi$ with the *-automorphism $\gamma_\chi$ on $\mathcal{W}(E)$ satisfying $\gamma_\chi(W(f)) = \chi(f)W(f)\ \forall f \in E$ (gauge transformation of the second kind). Since the vacuum $\omega_{\text{vac}}$ is a pure state, this relation reveals each generalized Glauber state $\varphi_\chi$ to be pure, too, that is, $\varphi_\chi \in \partial_e\mathcal{S}$.

### 3.4.2 Classical States: the Mixtures of the Generalized Glauber States

For $dim_{\mathbb{C}}(E) < \infty$ the classical states from the usual quantum optics literature are described in the Appendix A.2. Here we deal with the infinite dimensional generalization.

Let the additive group $E$ be topologyzed with the discrete topology, then its character group $\widehat{E}$ is compact in the $\Delta$-topology (the topology of pointwise convergence: $lim_i\, \chi_i = \chi$ in $\widehat{E}$, if and only if $lim_i\, \chi_i(f) = \chi(f)\ \forall f \in E$, [29]). For each (positive) probability measures $\mu$ on $\widehat{E}$ the mixture (weak*-topology) of the generalized Glauber states $\varphi_\chi$,

$$\omega \quad := \quad \int_{\widehat{E}} \varphi_\chi \, \mathrm{d}\mu(\chi)\,, \tag{3.9}$$

defines a state $\omega \in \mathcal{S}$. Its characteristic function $C_\omega = C_{\text{vac}}\, \widehat{\mu}$ incorporates the "Fourier" transform $\widehat{\mu}(f) = \int_{\widehat{E}} \chi(f)\, \mathrm{d}\mu(\chi)$, $f \in E$, which by Subsection 3.3 agrees with the normally ordered characteristic function, $P_\omega = \widehat{\mu}$. Because of its positivity, $\mu$ resp. $P_\omega = \widehat{\mu}$ may be regarded as a statistical state of a classical field with phase space $E$. By Bochner's theorem [29] the positive-definite functions on the additive group $E$ agree with the Fourier transformed positive measures on $\widehat{E}$.

**Definition 3.5 (Classical States)** *A state $\omega \in \mathcal{S}$ is called classical, if its normally ordered characteristic function $P_\omega = C_\omega / C_{\text{vac}}$ is a positive-definite function on the additive group $E$. The set of all classical states on $\mathcal{W}(E)$ is denoted by $\mathcal{S}_{cl}$.*

*Moreover, if $P : E \to \mathbb{C}$ is a positive-definite function with the normalization $P(0) = 1$, then there exists a unique $\omega \in \mathcal{S}_{cl}$ with $C_\omega = C_{\text{vac}} P$, i.e., with $P = P_\omega$.*

An immediate consequence of the integral representation (3.9) is the unique decomposition of each $\omega \in \mathcal{S}_{cl}$ into the extreme ones $\varphi_\chi \in \partial_e\mathcal{S}_{cl}$, which is a typical property of the state spaces in classical statistical mechanics: $\mathcal{S}_{cl}$ is a Bauer simplex (in the weak*-topology). The extreme boundary $\partial_e\mathcal{S}_{cl}$ consists just of the pure, classical states $\varphi_\chi$, $\chi \in \widehat{E}$, especially $\varphi_1 = \omega_{\text{vac}} \in \partial_e\mathcal{S}_{cl}$ for $\chi \equiv 1$, [20], [10].

[3] A character is a function $\chi : E \to \{z \in \mathbb{C} \mid |z| = 1\}$ satisfying $\chi(f+g) = \chi(f)\, \chi(g)\ \forall f, g \in E$.

### 3.4.3 The Classical Field Fluctuations

The support of an arbitrary probability measure $\mu$ on $\widehat{E}$ in general contains also non-continuous characters $\chi$. Thus it is not possible to specify the smoothness properties of the resulting $\omega \in \mathcal{S}_{cl}$, as regularity, class $\mathcal{C}^m$, or analyticity, if only the measure $\mu$ is known. However, if the normally ordered characteristic function $P_\omega$ is continuous with respect to a nuclear topology $\tau$ on $E$, which is stronger than the norm topology, then $P_\omega$ decomposes according to the Bochner–Minlos theorem in terms of $\tau$–continuous characters, which arise from $\tau$–continuous linear forms $L : E \to \mathbb{C}$,

$$P_\omega(f) \;=\; \int_{E'} exp\big\{i\,\sqrt{2}\; Re(L(f))\big\}\,\mathrm{d}\mu_\omega^{\mathrm{BM}}(L)\,, \tag{3.10}$$

where $E'$ is the complex–linear dual space of $E$ with respect to $\tau$. To avoid the delicate mathematics of the Bochner–Minlos probability measure $\mu_\omega^{\mathrm{BM}}$ we restrict the classical regular state $\omega \in \mathcal{S}_{cl}$ to a finite dimensional complex subspace $D$ of $E$, that is, to the CCR–subalgebra $\mathcal{W}(D) \subseteq \mathcal{W}(E)$. The regularity of the restricted state $\omega|_{\mathcal{W}(D)}$ on $\mathcal{W}(D)$ implies its Fock normality by the Stone–von Neumann uniqueness theorem [8]. Consequently, the restrictions of the characteristic functions $C_\omega$ and $P_\omega$ to $D$ are norm–continuous. The Bochner (–Minlos) decomposition $P_\omega(f) = \int_D exp\big\{i\sqrt{2}\; Re\langle f \mid h\rangle\big\}\,\mathrm{d}\mu_\omega^D(h)$ for all $f \in D$ (cf. [26, Theorem IX.9] and the Appendix A.2), of the continuous restricted positive–definite $P_\omega|_D$ may be viewed as the marginal measure $\mu_\omega^D$ obtained by restricting $\mu_\omega^{\mathrm{BM}}$ to $D \subset E'$.

**Proposition 3.6** *Let $\omega \in \mathcal{S}_{cl}$ (of class $\mathcal{C}^2$). Then for each non–vanishing testmode $f \in E$ there is a unique probability measure $\rho_f$ on $\mathbb{R}$ with $P_\omega(tf) = \int_{\mathbb{R}} exp\{itx\}\,\mathrm{d}\rho_f(x)$, and thus*

$$\langle \omega\,;\, \Phi_\omega(f)\rangle = \int_{\mathbb{R}} x\,\mathrm{d}\rho_f(x) \;=:\; \langle x\rangle_f\,, \quad \Delta(\omega; f) \;=\; \int_{\mathbb{R}} (x - \langle x\rangle_f)^2\,\mathrm{d}\rho_f(x) \;\geq\; 0\,.$$

*Consequently, the variances for each $\omega \in \mathcal{S}_{cl}$ are larger than the vacuum fluctuations,*

$$\mathsf{Var}(\omega; f) \;=\; \tfrac{1}{2}\,\|f\|^2 \,+\, \Delta(\omega; f) \;\geq\; \tfrac{1}{2}\,\|f\|^2 \;=\; \mathsf{Var}(\omega_{\mathrm{vac}}; f) \qquad \forall f \in E\,.$$

*Furthermore, for the classical state $\omega \in \mathcal{S}_{cl}$ we have the following equivalences:*

*(i)* $\mathsf{Var}(\omega; f) \;=\; \tfrac{1}{2}\,\|f\|^2 \;=\; \mathsf{Var}(\omega_{\mathrm{vac}}; f) \;\; \forall f \in E\,,$

*(ii) $\omega$ is a pure state, that is, $\omega = \varphi_\chi \in \partial_e\mathcal{S}_{cl}$ for some character $\chi \in \widehat{E}$.*

PROOF: Let $0 \neq g \in E$. Because of $P_\omega(0) = 1$ and by Bochner's theorem there exists a probability measure $\rho$ on $\mathbb{R}$ with $P_\omega(tg) = \int_{\mathbb{R}} exp\{itx\}\,\mathrm{d}\rho(x)\;\forall t \in \mathbb{R}$. On the Hilbert space $\mathsf{L}^2(\mathbb{R}, \rho)$ of $\rho$–square integrable functions we now define the selfadjoint multiplication operator $(B\eta)(x) := x\,\eta(x)$ for $\rho$–almost all $x \in \mathbb{R}$, where $\eta$ is an element of its domain $\mathcal{D}(B) = \{\xi \in \mathsf{L}^2(\mathbb{R}, \rho) \mid \int_{\mathbb{R}} x^2\,|\xi(x)|^2\,\mathrm{d}\rho(x) < \infty\}$. Obviously, $1(x) \equiv 1 \in \mathsf{L}^2(\mathbb{R}, \rho)$. Since $\omega \in \mathcal{S}_{cl}$ is of class $\mathcal{C}^2$ the mapping $\mathbb{R} \ni t \mapsto P_\omega(tg) = \langle 1 \mid exp\{itB\}\,1\rangle$ is two–times continuously differentable, which implies $1 \in \mathcal{D}(B)$. Differentiating as in Subsection 3.3 gives $\Delta(\omega; g) = \|B1 - \langle 1 \mid B1\rangle\,1\|^2 \geq 0$.

Now let $\omega \in \partial_e \mathcal{S}_{cl}$, i.e., $P_\omega$ is a character on $E$, respectively $\mathbb{R} \ni t \mapsto P_\omega(tg)$ is a continuous character on $\mathbb{R}$, which implies $\rho$ to be the point measure at some $x_0 \in \mathbb{R}$. Thus 1 is an eigenvector for $B$ with eigenvalue $x_0$, which gives $\|B1 - \langle 1 \mid B1 \rangle 1\| = 0$.

Conversely, let $\omega \notin \partial_e \mathcal{S}_{cl}$, then $P_\omega = \widehat{\mu_\omega}$ for some probability measure $\mu_\omega$ on $\widehat{E}$, which is not a point measure. Then there exist some $g \in E$ with $|P_\omega(g)| < 1$. Consequently, the measure $\rho$ from above cannot be a point measure, too, or equivalently the unit function 1 is not an eigenvector for $B$. Thus $\Delta(\omega; g) = \|B1 - \langle 1 \mid B1 \rangle 1\|^2 > 0$. ■

As mentioned above, for the classical state $\omega \in \mathcal{S}_{cl}$ the normally ordered characteristic function $P_\omega = \widehat{\mu_\omega}$ — with probability measure $\mu_\omega$ according to (3.9) — represents the statistical distribution of a classical field over the phase space $E$. Thus the above result exhibits that indeed the field variances of $\omega \in \mathcal{S}_{cl}$ decompose additively into the vacuum fluctuations (3.4) plus the classical fluctuations $\Delta(\omega; f) \geq 0$ form (3.6),

$$\mathsf{Var}(\omega; f) \;=\; \mathsf{Var}(\omega_{\text{vac}}; f) + \Delta(\omega; f), \qquad f \in E. \tag{3.11}$$

### 3.4.4 Estimates of the Squeezed Field Fluctuations

The transformation of the classial state $\omega \in \mathcal{S}_{cl}$ from equation (3.9) with $\nu_T$ gives

$$\nu_T(\omega) \;=\; \int_{\widehat{E}} \nu_T(\varphi_\chi)\, \mathrm{d}\mu(\chi),$$

that is the decomposition of the squeezed classical state $\nu_T(\omega)$ into the pure states $\nu_T(\varphi_\chi)$, $\chi \in \widehat{E}$. Using equation (2.5) one obtains from Proposition 3.6 some estimates for the fluctuations of squeezed classical states.

**Proposition 3.7** *For $\omega \in \mathcal{S}_{cl}$ it holds*

$$\mathsf{Var}(\nu_T(\omega); f) \;\geq\; \tfrac{1}{2}\|Tf\|^2 \;=\; \mathsf{Var}(\nu_T(\omega_{\text{vac}}); f) \qquad \forall f \in E.$$

*For $\omega \in \partial_e \mathcal{S}_{cl}$ we have the same squeezing properties as for the vacuum $\omega_{\text{vac}}$,*

$$\mathsf{Var}(\nu_T(\omega); f) \;=\; \tfrac{1}{2}\|Tf\|^2 \;=\; \mathsf{Var}(\nu_T(\omega_{\text{vac}}); f) \qquad \forall f \in E.$$

Combining the Propositions 3.2 and 3.6 one easily gets the following result.

**Proposition 3.8** *Suppose $\omega \in \mathcal{S}_{cl}$ to have bounded fluctuations. Then $\|S\| = \infty$ implies*

$$0 \;=\; \mathsf{InfVar}(\nu_T(\omega_{\text{vac}}); E) \;=\; \mathsf{InfVar}(\nu_T(\omega); E) \;<\; \tfrac{1}{2} \;=\; \mathsf{InfVar}(\omega_{\text{vac}}; E) \;\leq\; \mathsf{InfVar}(\omega; E).$$

### 3.4.5 Optimal and Non–Optimal Squeezing of Classical Sates

By Proposition 3.7 it is for each real or complex subspace $F \subseteq E$,

$$\mathsf{InfVar}(\nu_T(\omega); F) \geq \mathsf{InfVar}(\nu_T(\omega_{\text{vac}}); F), \qquad \forall \omega \in \mathcal{S}_{cl}.$$

Thus a squeezed classical state may reach at the best the squeezed vacuum fluctuations, provided the same squeezing Bogoliubov transformation $\nu_T$ is applied. It is clear that the smallest fluctuations are obtained when taking $F = E$, which with Proposition 3.4 leads to the following definition of a qualitative degree of squeezing by $\nu_T$: A ($\mathcal{C}^2$–) state $\omega \in \mathcal{S}_{cl}$ is called *optimally squeezed* by $\nu_T$, if

$$\mathsf{InfVar}(\nu_T(\omega); E) = \mathsf{InfVar}(\nu_T(\omega_{\text{vac}}); E) = \tfrac{1}{2} \exp\{-2 \, \|S\|\} ,$$

whereas $\omega \in \mathcal{S}_{cl}$ is called *non–optimally squeezed* by $\nu_T$, if

$$\mathsf{InfVar}(\nu_T(\omega); E) > \mathsf{InfVar}(\nu_T(\omega_{\text{vac}}); E) = \tfrac{1}{2} \exp\{-2 \, \|S\|\} .$$

Obviously, by Proposition 3.7 the pure classical states $\omega \in \partial_e \mathcal{S}_{cl}$ are optimally squeezed. Optimal squeezing also occurs in the situaton of Proposition 3.8. The white noise states $\omega_b$ from Subsection 3.5.3 below (they are classical and quasifree) are non–optimally squeezed by $\nu_T$ for each $b > 0$ (cf. Proposition 3.14). The description of optimal and non–optimal squeezing for classical coherent states is the content of Theorem 3.16. Especially one–mode squeezing of coherent states is often non–optimal as is illustrated in Subsection 3.6.3 below.

## 3.5 Squeezing of Quasifree States

### 3.5.1 Quasifree States and their Field Fluctuations

Quasifree states — also called Gaussian states — play an important role in statistical physics, since, e.g., the thermodynamic equilibrium states (limiting Gibbs and KMS states) for photons resp. the free Boson gas (with and without Bose–Einstein condensation) are quasifree and classical states [17], [8].

The characteristic function of a *quasifree* state $\omega \in \mathcal{S}$ is given by

$$C_\omega(f) = exp\Big\{i \, \ell_\omega(f) - \tfrac{1}{4} \, s_\omega(f,f)\Big\} \qquad \forall f \in E, \tag{3.12}$$

where $\ell_\omega : E \to \mathbb{R}$ is a real–linear form and $s_\omega : E \times E \to \mathbb{R}$ is a positive symmetric real–bilinear form satisfying

$$|Im \langle f \mid g \rangle|^2 \leq s_\omega(f,f) \, s_\omega(g,g) \qquad \forall f,g \in E \tag{3.13}$$

(cf. [8], [30], and [10]). Conversely, for each real–linear form $\ell$ and positive symmetric form $s$ fulfilling (3.13) there exists a unique (quasifree) state $\omega \in \mathcal{S}$ with $\ell_\omega = \ell$ and $s_\omega = s$. The set of all quasifree states on $\mathcal{W}(E)$ is denoted by $\mathcal{S}_{qf}$. Each $\omega \in \mathcal{S}_{qf}$ is entire–analytic, and has the field fluctuations (recall, $\Delta(\omega; f) = \mathsf{Var}(\omega; f) - \mathsf{Var}(\omega_{\text{vac}}; f)$ by (3.6))

$$\mathsf{Var}(\omega; f) = \tfrac{1}{2}\, s_\omega(f,f)\,, \qquad \Delta(\omega; f) = \tfrac{1}{2}\left(s_\omega(f,f) - \|f\|^2\right), \qquad \forall f \in E\,, \tag{3.14}$$

which follow from the differentiation of the mapping $\mathbb{R} \ni t \mapsto C_\omega(tf)$ as in (2.3).

By Proposition 3.6 the classicality of an arbitrary $\omega \in \mathcal{S}$ implies for the field fluctuations $\mathsf{Var}(\omega; f) \geq \mathsf{Var}(\omega_{\text{vac}}; f)$. Here for the quasifree states this implication has a converse.

**Proposition 3.9** *Let $\omega \in \mathcal{S}_{qf}$ with associated bilinear form $s_\omega$ according to (3.12). The normally ordered characteristic function $P_\omega$ defined in Subsection 3.3 is given by $P_\omega(f) = \exp\left\{i\ell_\omega(f) - \frac{1}{4}\left(s_\omega(f,f) - \|f\|^2\right)\right\}$ $\forall f \in E$. The subsequent conditions are equivalent:*

*(i)* $\omega$ *is classical, or by definition,* $P_\omega$ *is positive-definite,*

*(ii)* $s_\omega(f,f) \geq \|f\|^2$ *for all* $f \in E$,

*(iii)* $\Delta(\omega; f) \geq 0$ *for all* $f \in E$,

*(iv)* $\mathsf{InfVar}(\omega; E) \geq \frac{1}{2} = \mathsf{InfVar}(\omega_{\text{vac}}; E)$.

PROOF: Consequence of Proposition 3.6 and equation (3.14), cf. also [10]. ■

### 3.5.2 Bogoliubov Transformations of Quasifree States

**Proposition 3.10** *It holds:* $\nu_T(\mathcal{S}_{qf}) = \mathcal{S}_{qf}$ , *for all* $T \in \mathcal{T}(E)$.

PROOF: From (3.12) it follows $C_{\nu_T(\omega)}(f) = \exp\left\{i\ell_\omega(Tf) - \frac{1}{4}\, s_\omega(Tf,Tf)\right\}$. But $\ell_{\nu_T(\omega)} = \ell_\omega \circ T$ is a real-linear form on $E$. (1.9) and (3.13) valid for $\omega$ imply for $s_{\nu_T(\omega)}(f,g) = s_\omega(Tf,Tg)$,

$$|Im\,\langle f \mid g\rangle|^2 = |Im\,\langle Tf \mid Tg\rangle|^2 \leq s_\omega(Tf,Tf)\, s_\omega(Tg,Tg) = s_{\nu_T(\omega)}(f,f)\, s_{\nu_T(\omega)}(f,f) \quad \forall f,g \in E.$$

Consequently, $\nu_T(\omega) \in \mathcal{S}_{qf}$, and thus $\nu_T(\mathcal{S}_{qf}) \subseteq \mathcal{S}_{qf}$. The same argumentation for $T^{-1} \in \mathcal{T}(E)$ and $(\nu_T)^{-1} = \nu_{T^{-1}}$ yields the result. ■

Since $\nu_T(\omega) \in \mathcal{S}_{qf}$ for each $\omega \in \mathcal{S}_{qf}$, Proposition 3.9 also applies to the transformed $\nu_T(\omega)$.

**Corollary 3.11** *Let $\omega \in \mathcal{S}_{qf}$ with bilinear form $s_\omega$ according to (3.12). Then $\nu_T(\omega)$ is classical, if and only if* $\frac{1}{2}s_\omega(Tf,Tf) = \mathsf{Var}(\nu_T(\omega); f) \geq \mathsf{Var}(\omega_{\text{vac}}; f) = \frac{1}{2}\,\|f\|^2$ $\forall f \in E$.

Let us now present the examples mentioned in Subsection 2.5. They are of structural interest for more insight into the Definition 2.3 of squeezing.

**Example 3.12** *Let $T_a = 0$, or equivalently, $T = U$ is unitary by Theorem 2.1.*

*(a) Assume $Th \neq \pm h$ for an $h \in E$. Define the real-linear form $\rho$ on $E$ by $\rho(f) := Re\langle h \mid f\rangle = (h \mid f)$ and the state $\omega \in \mathcal{S}_{qf} \bigcap \mathcal{S}_{cl}$ with the characteristic function $C_\omega(f) = C_{\text{vac}}(f)\exp\{-\frac{1}{2}\rho(f)^2\}$, $f \in E$. Then we have $\mathsf{Var}(\nu_T(\omega); h) < \mathsf{Var}(\omega; h)$, that is, $\omega$ is squeezed by $\nu_T$ in the testmode $h \in E$.*

*(b) Assume $F$ to be a complex subspace of $E$ with $U(\overline{F}) \bigcap F^\perp \neq \emptyset$. Let $Q_F$ be the orthogonal projection onto the closure $\overline{F}$ of $F$. Then the state $\omega \in \mathcal{S}$ with $C_\omega(f) := C_{\text{vac}}(f)\exp\{-\frac{1}{2}\|Q_F f\|^2\}$, $f \in E$, is quasifree, classical, and $F$-squeezed by $\nu_T$.*

PROOF: (a): From (3.14) it follows $\mathsf{Var}(\omega_\rho; f) = \frac{1}{2}\|f\|^2 + \rho(f)^2$. Since $Th \neq \pm h$ we have $|Re\langle h \mid Th\rangle| < \|h\|^2$ (e.g. [24] Theorem 1.4). Consequently, using (2.5) we obtain $\mathsf{Var}(\nu_T(\omega); h) = \mathsf{Var}(\omega; Th) < \mathsf{Var}(\omega; h)$. (b): From (3.14) it follows $\mathsf{Var}(\omega; f) = \frac{1}{2}\|f\|^2 + \|Q_F f\|^2$. There exists a normalized $g \in \overline{F}$ with $Ug \in F^\perp$. It follows

$$0 = \|Q_F U g\| = \inf\{\|Q_F T f\| \mid f \in F, \|f\| = 1\} < \inf\{\|Q_F f\| \mid f \in F, \|f\| = 1\} = 1,$$

which by the use of (2.5) proves the stated squeezing property. ∎

**Example 3.13** *Let $\omega \in \mathcal{S}_{qf} \bigcap \mathcal{S}_{cl}$ with $C_\omega(f) = C_{\text{vac}}(f)\exp\{-\frac{1}{4}\|T^{-1}f\|^2\}$, $f \in E$. Then it holds $\mathsf{InfVar}(\nu_T(\omega); E) = \mathsf{InfVar}(\omega; E)$.*

PROOF: (3.1) and (3.2) imply $\inf\{\|Tf\| \mid f \in E, \|f\| = 1\} = \inf\{\|T^{-1}f\| \mid f \in E, \|f\| = 1\}$. Now the assertion follows from (3.14) and (2.5). ∎

### 3.5.3 Squeezing of the White Noise States

In the quantum stochastic calculus the white noise or temperature states are the classical, quasifree states $\omega_b \in \mathcal{S}$ given for each real parameter $b \geq 0$ by the characteristic function

$$C_{\omega_b}(f) = C_{\text{vac}}(f)\exp\{-\tfrac{b}{4}\|f\|^2\}, \qquad f \in E$$

($b = e^\beta - 1$ for the inverse temperature $\beta$; for $\dim_{\mathbb{C}}(E) = 1$ see [27], [11]). With the canonical transformation $\nu_T$ they turn into the squeezed white noise states [23]. For $b = 0$ we obtain the vacuum state $\omega_0 = \omega_{\text{vac}}$, and for different parameters $b$ the states $\omega_b$ are not quasi-equivalent ($E$ infinite dimensional, [31]), especially for $b > 0$ the white noise state $\omega_b$ is not given by a density operator on Fock space. It is $\omega_b \in \partial_e\mathcal{S}$ (a pure state), if and only if $b = 0$. With (3.14) we have the bounded fluctuations

$$\mathsf{Var}(\omega_b; f) = \tfrac{1+b}{2}\|f\|^2 = \mathsf{Var}(\omega_{\text{vac}}; f) + \tfrac{b}{2}\|f\|^2, \qquad f \in E,$$

and thus $\Delta(\omega_b; f) = \frac{b}{2} \|f\|^2$ for the classical field fluctuations (3.11). With equation (2.5) it follows for the Bogoliubov transformed states $\nu_T(\omega_b)$, $b \geq 0$,

$$\mathsf{Var}(\nu_T(\omega_b); f) = \tfrac{1+b}{2} \|Tf\|^2 = \mathsf{Var}(\nu_T(\omega_{\text{vac}}); f) + \tfrac{b}{2} \|Tf\|^2 , \qquad f \in E . \tag{3.15}$$

It is $\nu_T(\omega_b) = \omega_b$, if and only if $T_a = 0$, or equivalently, if and only if $S = 0$. Since $\nu_T(\mathcal{S}_{qf}) = \mathcal{S}_{qf}$, the transformed states $\nu_T(\omega_b)$ are quasifree, too. Let us use Corollary 3.11 to determine for which parameters $b \geq 0$ the squeezed white noise states $\nu_T(\omega_b)$ are classical.

**Proposition 3.14** *It holds* $\mathsf{InfVar}(\nu_T(\omega_b); E) = \frac{1+b}{2} \exp\{-2 \|S\|\}$ $\forall b \geq 0$, *i.e., if* $S \neq 0$, *then* $\omega_b$ *is* $E$*–squeezed by* $\nu_T$. *Furthermore, we have the following equivalences:*

*(i)* $\nu_T(\omega_b) \in \mathcal{S}_{cl}$ ,

*(ii)* $\mathsf{InfVar}(\nu_T(\omega_b); E) \geq \frac{1}{2} = \mathsf{InfVar}(\omega_{\text{vac}}; E)$ ,

*(iii)* $b \geq \exp\{2 \|S\|\} - 1$ .

*Especially, for* $\|S\| = \infty$ *we have* $\nu_T(\omega_b) \notin \mathcal{S}_{cl}$ *for all* $b \geq 0$.

PROOF: $\mathsf{InfVar}(\nu_T(\omega_b); E)$ is obtained with (3.15) analoguously to the proof of Proposition 3.4. Now apply Corollary 3.11 resp. Proposition 3.9. ∎

Summary: Here the squeezing strength of our squeezing transformation $\nu_T$ is given by $\|S\|$. Only if the classical fluctuations $\Delta(\omega_b; f) = \frac{b}{2} \|f\|^2$ for the white noise state $\omega_b$ are large enough compared with the squeezing strength, $b \geq \exp\{2 \|S\|\} - 1$, then the tendency of $\nu_T$ to diminish some fluctuations is counterbalanced by their wide range and $\nu_T(\omega_b)$ remains classical. For white noise states $\omega_b$ below the critical value, $b < \exp\{2 \|S\|\} - 1$, the squeezing operation $\nu_T$ is strong enough to render them non–classical, $\nu_T(\omega_b) \notin \mathcal{S}_{cl}$.

## 3.6 Squeezing of Classical Coherent States

A smearing procedure of Glauber's original factorization condition [18] leads to the algebraic formulation of quantum optical coherence [15], [28], where Glauber's complex factorizing coherence function is replaced by a (complex–) linear form $G : E \to \mathbb{C}$. An analytic state $\omega \in \mathcal{S}$ is called $G$–coherent in $n$–th order, if the normally ordered expectations factorize up to degree $n$, where $n \in \mathbb{N} \cup \{\infty\}$,

$$\langle \omega \, ; \, a^*_\omega(f_1) \cdots a^*_\omega(f_j) \, a_\omega(g_1) \cdots a_\omega(g_j) \rangle = G(f_1) \cdots G(f_j) \, \overline{G(g_1)} \cdots \overline{G(g_j)} \tag{3.16}$$

for all $f_k, g_l \in E$ and each $1 \leq j \leq n$. Let us denote by $\mathcal{S}^{(n)}_{coh}(G)$ the set of all $n$–th order coherent states on $\mathcal{W}(E)$ factorizing with $G$, and by $\mathcal{S}^{(n)}_{coh,cl}(G)$ those $n$–th order coherent states which in addition are classical, $\mathcal{S}^{(n)}_{coh,cl}(G) = \mathcal{S}^{(n)}_{coh}(G) \bigcap \mathcal{S}_{cl}$.

Starting from the factorization condition the characteristic functions of all (classical and non–classical) quantum optical coherent states on $\mathcal{W}(E)$ have been determined in [15], [16], [21], [22]. Especially, it is $\mathcal{S}^{(n)}_{coh,cl}(G) = \mathcal{S}^{(n)}_{coh}(G)$, if and only if the linear form $G$ is unbounded, and $\omega \in \mathcal{S}^{(n)}_{coh}(G)$ is Fock normal, if and only if $G$ is bounded.

### 3.6.1 Classical Coherent States as Mixtures of Generalized Glauber States

For every linear form $L : E \to \mathbb{C}$ let us denote by $\mathcal{S}^L_{cl}$ the set of those classical states $\omega \in \mathcal{S}_{cl}$, which are the mixtures of the generalized Glauber states $\varphi_{zL} := \varphi_{\chi_{zL}}$ corresponding to the characters $\chi_{zL}(f) = exp\{i\sqrt{2}\ Re(zL(f))\}$, $f \in E$, where $z$ ranges over the complex plane $\mathbb{C}$. That is, the states $\omega \in \mathcal{S}^L_{cl}$ are in one–to–one correspondence with the probability measures $\mu^L_\omega$ on $\mathbb{C}$, so that according to (3.9),

$$\omega \;=\; \int_{\mathbb{C}} \varphi_{zL}\, \mathrm{d}\mu^L_\omega(z)\,, \qquad \omega \in \mathcal{S}^L_{cl}\,. \tag{3.17}$$

The states $\omega \in \mathcal{S}^L_{cl}$ are regular with the characteristic functions

$$C_\omega(f) \;=\; C_{\text{vac}}(f) \int_{\mathbb{C}} exp\Big\{i\sqrt{2}\ Re(zL(f))\Big\}\ \mathrm{d}\mu^L_\omega(z) \quad \forall f \in E\,. \tag{3.18}$$

The trivial case $L = 0$ implies $\mathcal{S}^0_{cl} = \{\omega_{\text{vac}}\}$; thus let us suppose $L \neq 0$. The associated moments $c^L_\omega(k,l)$ have the form (if they exist)

$$c^L_\omega(k,l) \;:=\; \int_{\mathbb{C}} z^k\, \overline{z}^l\ \mathrm{d}\mu^L_\omega(z)\,, \quad 0 \le k,l < \infty\,, \tag{3.19}$$

and determine for arbitrary testfunctions $f_i, g_j \in E$ the normally ordered expectations

$$\langle \omega\,;\, a^*_\omega(f_1)\cdots a^*_\omega(f_k)\, a_\omega(g_1)\cdots a_\omega(g_l)\rangle \;=\; c^L_\omega(k,l)\, L(f_1)\cdots L(f_k)\, \overline{L(g_1)}\cdots\overline{L(g_l)}\,. \tag{3.20}$$

**Lemma 3.15** *Let $\omega \in \mathcal{S}^L_{cl}$. Then for the classical fluctuations it holds*

$$\Delta(\omega; f) \;=\; Re\Big(a^L_\omega\, L(f)^2\Big) \;+\; b^L_\omega\, |L(f)|^2\,,$$

*where the coefficients are given in terms of the centered moments*

$$a^L_\omega \;:=\; c^L_\omega(2,0) - c^L_\omega(1,0)^2\,, \qquad b^L_\omega \;:=\; c^L_\omega(1,1) - |c^L_\omega(1,0)|^2\,.$$

*Moreover, $b^L_\omega \ge |a^L_\omega|$. It holds $b^L_\omega = a^L_\omega = 0$, if and only if $\omega$ is pure, or equivalently, if and only if $\mu^L_\omega$ is a point measure.*

PROOF: The variances are obtained by differentiating (3.18) as in equation (2.3) resp. (3.6). The rest easily follows from $\Delta(\omega; f) \ge 0$, and the fact that $\Delta(\omega; f) = 0$, if and only if $\omega$ is pure, by Proposition 3.6. ∎

The classical coherent states of $n$-th order turn out to constitute certain subsets of $\mathcal{S}_{cl}^L$. Comparing (3.16) with equation (3.20) it follows that for $\omega \in \mathcal{S}_{cl}^L$ the factorization takes place, if and only if the moments (3.19) satisfy $c_\omega^L(j,j) = |\lambda|^{2j}$ for every $1 \leq j \leq n$ for some non-zero $\lambda \in \mathbb{C}$, which yields the factorization (3.16) with respect to the linear form $G(f) = \lambda L(f)$, $f \in E$. Indeed, by [16] the set $\mathcal{S}_{coh,cl}^{(n)}(\lambda L)$ consists of those states $\omega \in \mathcal{S}_{cl}^L$, for which the associated measure $\mu_\omega^L$ is analytic and satisfies $c_\omega^L(j,j) = |\lambda|^{2j}$ for all $1 \leq j \leq n$.

There is some redundancy in the condition for the moments: If $c_\omega^L(j,j) = |\lambda|^{2j}$ for $j = 1, 2$, then $c_\omega^L(j,j) = |\lambda|^{2j}$ for every $j \in \mathbb{N}$, or equivalently, the probability measure $\mu_\omega^L$ is concentrated on $\{z \in \mathbb{C} \mid |z| = |\lambda|\}$. This implies that a classical coherent $\omega$ of order two, automatically is all order (fully) coherent. Thus we have the proper inclusions

$$\mathcal{S}_{coh,cl}^{(\infty)}(\lambda L) = \mathcal{S}_{coh,cl}^{(n)}(\lambda L) = \mathcal{S}_{coh,cl}^{(2)}(\lambda L) \subset \mathcal{S}_{coh,cl}^{(1)}(\lambda L) \subset \mathcal{S}_{cl}^L, \qquad \forall n \geq 2, \quad (3.21)$$

for each $0 \neq \lambda \in \mathbb{C}$. Observe, that for all $n \in \mathbb{N} \cup \{\infty\}$ it is $\mathcal{S}_{coh,cl}^{(n)}(\lambda_1 L) = \mathcal{S}_{coh,cl}^{(n)}(\lambda_2 L)$ for $|\lambda_1| = |\lambda_2|$, whereas it holds $\mathcal{S}_{coh,cl}^{(n)}(\lambda_1 L) \cap \mathcal{S}_{coh,cl}^{(n)}(\lambda_2 L) = \emptyset$ for $|\lambda_1| \neq |\lambda_2|$, but in both cases one has $\mathcal{S}_{cl}^L = \mathcal{S}_{cl}^{\lambda_1 L} = \mathcal{S}_{cl}^{\lambda_2 L}$.

### 3.6.2 Estimates of the Squeezed Field Fluctuations for Bounded Linear Form

From Lemma 3.15 it is immediately seen that the non-pure $\omega \in \mathcal{S}_{cl}^L$ has bounded fluctuations (2.4), if and only if the linear form $L$ is bounded. Throughout the present Subsection let us suppose a bounded non-zero $L$, which then is given with a unique $0 \neq h \in \mathcal{H}$ by $L(f) = \langle h \mid f \rangle \ \forall f \in E$ according to the Riesz Lemma [26, Theorem II.4], i.e., here the mode $\overline{\lambda} h$ agrees with Glauber's factorizing function for the coherent states $\mathcal{S}_{coh}^{(n)}(\lambda L)$.

For bounded $L$ not all coherent states are classical (cf. Subsection 3.6.3 for an example of a non-classical coherent state). However, we are interested in the squeezing properties of the *classical* coherent states $\mathcal{S}_{coh,cl}^{(n)}(\lambda L)$ resp. of $\mathcal{S}_{cl}^L$, only. Since $L(f) = \langle h \mid f \rangle \ \forall f \in E$ each state $\omega \in \mathcal{S}_{cl}^L$ decomposes into the Glauber states $\varphi_{zL} = \varphi_{\chi_{zL}} = \omega_{\overline{z}h}^G$ according to (3.17),

$$\omega = \int_{\mathbb{C}} \omega_{\overline{z}h}^G \, d\mu_\omega^L(z), \qquad \omega \in \mathcal{S}_{cl}^L. \quad (3.22)$$

The boundedness of $L$ implies optimal squeezing for $\|S\| = \infty$, i.e., $\mathsf{InfVar}(\nu_T(\omega); E) = 0$ for all $\omega \in \mathcal{S}_{cl}^L$, by Proposition 3.2. Thus we suppose $S$ to be bounded, too, for a refined discussion.

With (2.5) and Lemma 3.15 it is for $\omega \in \mathcal{S}_{cl}^L$,

$$\mathsf{Var}(\nu_T(\omega); f) = \tfrac{1}{2} \|Tf\|^2 + Re\left(a_\omega^L \langle h \mid Tf \rangle^2\right) + b_\omega^L \, |\langle h \mid Tf \rangle|^2, \quad (3.23)$$

Since our symplectic transformation $T = U(exp\{S\} \, P_+ + exp\{-S\} \, P_-)$ in general is only real-linear, we have to be very carefully, when shifting $T$ from the right to the left side in the scalar product $\langle h \mid Tf \rangle = \langle U^* h \mid (exp\{S\} \, P_+ + exp\{-S\} \, P_-) f \rangle$, so that $exp\{\pm S\}$ act on $U^*h$. Such a detailed analysis leads to the subsequent Theorem about the squeezing properties of $\omega \in \mathcal{S}_{cl}^L$ by $\nu_T$, which depend on the relation between the spectral properties of $S$ and the vector $U^*h$.

Before formulating the results let us collect some spectral notions: Since $S$ is a positive selfadjoint operator on $\mathcal{H}$ it follows that $\|S\| \in \sigma(S) \subseteq [0, \|S\|]$. Recall, an isolated point of the spectrum $\sigma(S)$ always is an eigenvalue for $S$ (e.g. [24]), but there may exist non-isolated eigenvalues in $\sigma(S)$. By Subsection 3.1 the real-linear spectral projections $P_\pm$ of $J$ commute with the complex-linear spectral projections $E_S(\mathcal{B})$ of $S$. Let $P_S := E_S(\{\|S\|\})$ be the spectral projection onto the spectral value $\|S\|$; it is $P_S \neq 0$, if and only if $\|S\|$ is an eigenvalue of $S$, in which case the associated eigenspace $P_S\mathcal{H} = P_+P_S\mathcal{H} + P_-P_S\mathcal{H}$ decomposes, where $dim_{\mathbb{C}}(P_S\mathcal{H}) = dim_{\mathbb{R}}(P_\pm P_S\mathcal{H})$ (since $P_\pm$ commute with $P_S$).

**Theorem 3.16** *With the notions of optimal and non-optimal degrees of squeezing from Subsection 3.4.5 the following assertions hold:*

*(I) If one of the following spectral conditions for $S$ is fulfilled,*

- *(a) $P_SU^*h = 0$,*
- *(b) $\|S\|$ is not an isolated point of the spectrum of $S$,*
- *(c) the dimension of the eigenspace $P_S\mathcal{H}$ corresponding to the eigenvalue $\|S\|$ is larger than or equal to three, $\dim_{\mathbb{C}}(P_S\mathcal{H}) \geq 3$,*

*then all states $\omega \in \mathcal{S}_{cl}^L$ are optimally squeezed by $\nu_T$.*

*(II) Consider the remaining case, where $P_SU^*h \neq 0$, and $\|S\|$ is an isolated point of $\sigma(S)$, and $\dim_{\mathbb{C}}(P_S\mathcal{H}) = 1$ or $= 2$. Let $\omega \in \mathcal{S}_{cl}^L$ and put $\alpha_\omega := \exp\{-\frac{i}{2}\arg(a_\omega^L)\}$.*

- *(a) Let $b_\omega^L \neq a_\omega^L$ (actually, $b_\omega^L > |a_\omega^L|$ by Lemma 3.15) and suppose:*

  *$dim_{\mathbb{C}}(P_S\mathcal{H}) = 2$, then $\omega$ is optimally squeezed by $\nu_T$, if and only if $P_-\alpha_\omega P_SU^*h$ and $P_-i\alpha_\omega P_SU^*h$ are real-linearly dependent;*

  *$dim_{\mathbb{C}}(P_S\mathcal{H}) = 1$, then $\omega$ is non-optimally squeezed by $\nu_T$.*
- *(b) Let $b_\omega^L = |a_\omega^L| \neq 0$ and suppose:*

  *$dim_{\mathbb{C}}(P_S\mathcal{H}) = 2$, then $\omega$ is optimally squeezed by $\nu_T$;*

  *$dim_{\mathbb{C}}(P_S\mathcal{H}) = 1$, then $\omega$ is optimally squeezed by $\nu_T$, if and only if $P_-\alpha_\omega P_SU^*h = 0$.*
- *(c) Let $b_\omega^L = a_\omega^L = 0$. Then $\omega \in \partial_e\mathcal{S}_{cl} \cap \mathcal{S}_{cl}^L$ is pure and optimally squeezed by $\nu_T$.*

Non-optimal squeezing especially occurs for one-mode squeezing transformations, that is, $S = s\,|e_0\rangle\langle e_0|$ for the normalized mode $e_0 \in E$ and $s > 0$. This case is treated for physically relevant Gaussian coherent states with factorizing coherence function $h = e_0$ in the next Subsection.

The remainder of the present Subsection is devoted to the proof of the foregoing Theorem. First let us give a Lemma, which allows an approximation of $\mathsf{InfVar}(\nu_T(\omega); E)$.

**Lemma 3.17** *Let $\omega \in \mathcal{S}^L_{cl}$ and again $\alpha_\omega = exp\left\{-\frac{i}{2}\, arg(a^L_\omega)\right\}$.*

*(a) For each Borel set $\mathcal{B} \subseteq [0, \|S\|]$ with $\mathcal{B} \cap \sigma(S) \neq \emptyset$ it holds*

$$\begin{aligned} \mathsf{InfVar}(\nu_T(\omega_{\text{vac}}); E) \;&=\; \tfrac{1}{2}\, exp\{-2\,\|S\|\} \;\leq \\ \leq\; \mathsf{InfVar}(\nu_T(\omega); E) \;&\leq\; \left(\tfrac{1}{2} + 2\, b^L_\omega\, \|E_S(\mathcal{B})U^*h\|^2\right) exp\{-2\; inf(\mathcal{B})\}\,. \end{aligned}$$

*(b) Suppose $\|S\|$ to be an eigenvalue of $S$, then we have for every $f \in P_- P_S \mathcal{H}$,*

$$\begin{aligned} \mathsf{Var}(\nu_T(\omega); f) \;&=\; exp\{-2\,\|S\|\}\, \Big[\tfrac{1}{2}\, \|f\|^2 \;+ \\ &+\; \left(b^L_\omega + |a^L_\omega|\right)(P_-\alpha_\omega P_S U^* h \mid f)^2 \;+\; \left(b^L_\omega - |a^L_\omega|\right)(P_- i\alpha_\omega P_S U^* h \mid f)^2\Big]\,. \end{aligned}$$

PROOF: $\omega$ is a classical state, and the first inequality sign in (a) follows from the Propositions 3.7 and 3.4. Observe $Im\,\langle \alpha_\omega h \mid g\rangle = Re\,\langle i\alpha_\omega h \mid g\rangle = (i\alpha_\omega h \mid g)$ for all $g \in E$. Then (3.23) rewrites as

$$\begin{aligned} \mathsf{Var}(\nu_T(\omega); f) \;&=\; \tfrac{1}{2}\,\|Tf\|^2 \;+\; |a^L_\omega|\; Re(\langle \alpha_\omega h \mid Tf\rangle^2) \;+\; b^L_\omega\, |\langle \alpha_\omega h \mid Tf\rangle|^2 \\ &=\; \tfrac{1}{2}\,\|Tf\|^2 \;+\; \left(b^L_\omega + |a^L_\omega|\right)[Re\,\langle \alpha_\omega h \mid Tf\rangle]^2 + \left(b^L_\omega - |a^L_\omega|\right)[Im\,\langle \alpha_\omega h \mid Tf\rangle]^2\,, \\ &=\; \tfrac{1}{2}\left[\|e^S P_+ f\|^2 + \|e^{-S} P_- f\|^2\right] \\ &\quad+\; \left(b^L_\omega + |a^L_\omega|\right)\left[(P_+\alpha_\omega U^* h \mid e^S f) + (P_-\alpha_\omega U^* h \mid e^{-S} f)\right]^2 \\ &\quad+\; \left(b^L_\omega - |a^L_\omega|\right)\left[(P_+ i\alpha_\omega U^* h \mid e^S f) + (P_- i\alpha_\omega U^* h \mid e^{-S} f)\right]^2\,, \end{aligned}$$

which leads to (b). Now let $E_S(\mathcal{B}) \neq 0$. With $P_- i = iP_+$ by Lemma 3.1 we obtain for each $f \in P_- E_S(\mathcal{B})\mathcal{H}$

$$\begin{aligned} &\mathsf{Var}(\nu_T(\omega); f) \;\leq \\ &\leq\; \left[\tfrac{1}{2} + \left(b^L_\omega + |a^L_\omega|\right)\|P_-\alpha E_S(\mathcal{B})U^* h\|^2 + \left(b^L_\omega - |a^L_\omega|\right)\|P_+\alpha E_S(\mathcal{B})U^* h\|^2\right]\|e^{-S} f\|^2 \\ &\leq\; \left[\tfrac{1}{2} + 2\, b^L_\omega\, \|E_S(\mathcal{B})U^* h\|^2\right]\|e^{-S} f\|^2. \end{aligned}$$

Now observe $\mathsf{InfVar}(\nu_T(\omega); E) \leq \mathsf{InfVar}(\nu_T(\omega); P_- E_S(\mathcal{B})\mathcal{H})$. If $E_S(\mathcal{B}) = 0$, then we choose a sequence of Borel sets $\mathcal{B}_n$, $n \in \mathbb{N}$, with $lim_n\, E_S(\mathcal{B}_n) = E_S(\mathcal{B})$ in the strong operator topology and $lim_n\, inf(\mathcal{B}_n) = inf(\mathcal{B})$. ∎

PROOF OF THEOREM 3.16: (I)(a) is an immediate consequence of Lemma 3.17 (a).
(I)(b): Since $\|S\| \in \sigma(S)$ is not isolated, there exists a sequence $\{s_n \mid n \in \mathbb{N}\} \subset \sigma(S)$ with $s_m \neq s_n \neq \|S\|$ for all $m \neq n$, which converges to $\|S\|$. Then $\sum_n E_S(\{s_n\})U^* h$ converges with respect to the norm of $\mathcal{H}$, which implies $lim_n\, \|E_S(\{s_n\})U^* h\| = 0$. By Lemma 3.17 (a) one has for all $n \in \mathbb{N}$

$$\tfrac{1}{2}\, exp\{-2\,\|S\|\} \;\leq\; \mathsf{InfVar}(\nu_T(\omega); E) \;\leq\; \left(\tfrac{1}{2} + 2\, b^L_\omega\, \|E_S(\{s_n\})U^* h\|^2\right) exp\{-2\, s_n\}\,.$$

The limit $n \to \infty$ gives the result.
(I)(c): Because of $dim_{\mathbb{R}}(P_- P_S \mathcal{H}) \geq 3$ we may choose a normalized $f \in P_- P_S \mathcal{H}$, which is orthogonal

to $P\_\alpha_\omega P_S U^* h$ and $P\_i\alpha_\omega P_S U^* h$ with respect to $(. \mid .)$. Then Lemma 3.17 (b) gives $\mathsf{Var}(\nu_T(\omega); f) = \frac{1}{2}\, exp\{-2\, \|S\|\}$, optimal squeezing.

(II): $\|S\|$ is an isolated spectral value, thus there exists and $0 \leq s < \|S\|$ with $\sigma(S)\backslash\|S\| \subseteq [0, s]$. Then the proof of Lemma 3.17 implies $\mathsf{Var}(\nu_T(\omega); f) \geq \frac{1}{2}\, exp\{-2s\}$ for all normalized $f \in (\mathbb{1} - P_S)\mathcal{H}$. Thus optimal squeezing is only obtainable with $f \in P_S\mathcal{H}$. Using Lemma 3.17 (b) the remaining results follow by an analysis similarly to (I)(c). Part (II)(c) also follows from Lemma 3.15 and the Propositions 3.6 and 3.7. ∎

### 3.6.3 One–Mode Squeezing of Quasifree Classical Coherent States

As in equation (1.13) of the Introduction or in [10, Section 6] we specify here the symplectic $T \in \mathcal{T}(E)$ to be a one–mode transformation, where $E = \mathcal{H}$ for convenience.

Let $e_0 \in E$ be the single, normalized photon mode under consideration. Then $T$ is given with $S = s\, |e_0\rangle\langle e_0|$ for the isolated eigenvalue $s = \|S\| > 0$, the anti–linear involution $J$ satisfying $Je_0 = e_0$, and arbitrary unitary $U$. If $P_{e_0}^{\perp}$ denotes the projection onto the orthogonal complement of $e_0$ with respect to $\langle . \mid . \rangle$, then the one–mode symplectic $T = T_l + T_a$ decomposes according to Theorem 2.1 as

$$T_l \;=\; cosh(s)\, \langle e_0 \mid . \rangle\, Ue_0 \,+\, UP_{e_0}^{\perp}\,, \qquad T_a \;=\; sinh(s)\, \langle . \mid e_0 \rangle\, Ue_0\,.$$

Thus, by Theorem 3.16 (II) in this situation one may obtain non–optimal squeezing by $\nu_T$ for some states $\omega \in \mathcal{S}_{cl}^L$, when the linear form is chosen as $L(f) := \langle Ue_0 \mid f \rangle\; \forall f \in E$.

For each $\lambda \geq 0$ there exists by [16] a (unique) classical, first order coherent, quasifree state $\omega_\lambda \in \mathcal{S}_{coh,cl}^{(1)}(\lambda L) \bigcap \mathcal{S}_{qf}$ with the (positive–definite) normally ordered characteristic function

$$P_{\omega_\lambda}(f) \;=\; exp\Big\{-\tfrac{\lambda^2}{2}\, |L(f)|^2\Big\} \qquad \forall f \in E\,. \tag{3.24}$$

Its moments are given by $c_{\omega_\lambda}^L(k, l) = \delta_{k,l}\, l!\, \lambda^{k+l}$, and the factorization for first order coherence (cf. the equations (3.19), (3.20), and (3.16)) has the form

$$\Big\langle \omega_\lambda\,;\, a_{\omega_\lambda}^*(f)\, a_{\omega_\lambda}(g) \Big\rangle \;=\; \lambda L(f)\, \overline{\lambda L(g)} \qquad \forall f, g \in E\,.$$

With the Fourier transformation formula for $a > 0$,

$$exp\Big\{-a^2 k^2\Big\} \;=\; \frac{1}{\sqrt{2\pi}\, a} \int_{\mathbb{R}} exp\Big\{\pm i\sqrt{2}\, x\, k\Big\}\, exp\Big\{-\frac{x^2}{2a^2}\Big\}\, \mathrm{d}x \qquad \forall k \in \mathbb{R}\,, \tag{3.25}$$

one immediately calculates the decomposition (3.22) of the state $\omega_\lambda$ into the Glauber states,

$$\omega_\lambda \;=\; \int_{\mathbb{C}} \omega_{\bar{z}Ue_0}^G\, \mathrm{d}\mu_{\omega_\lambda}^L(z)\,, \qquad \mathrm{d}\mu_{\omega_\lambda}^L(z) \;=\; \frac{exp\Big\{-|z|^2/\lambda^2\Big\}}{\pi\, \lambda^2}\, \mathrm{d}^2 z\,,$$

where $\mathrm{d}^2 z = \mathrm{d}Re(z)\, \mathrm{d}Im(z)$. Since this incoherent superposition of the pure coherent states $\omega_{\bar{z}Ue_0}^G$, $z \in \mathbb{C}$, is performed in terms of a (positive) Gaussian $P$–representation $\mathrm{d}\mu_{\omega_\lambda}^L(z)$ depending on $|z|$ only, one has an equipartition of the phases $arg(z)$.

Equation (3.14) yields $\mathsf{Var}(\omega_\lambda; f) = \frac{1}{2}\|f\|^2 + \lambda^2 |\langle Ue_0 \mid f\rangle|^2$, which gives for the complex subspace $F \subseteq E$

$$\mathsf{InfVar}(\omega_\lambda; F) = \begin{cases} \frac{1}{2} = \mathsf{InfVar}(\omega_{\text{vac}}, F), & \text{for } dim_{\mathbb{C}}(F) \geq 2\,, \\ \frac{1}{2} + \lambda^2 \quad\quad , & \text{for } F = \mathbb{C}Ue_0\,, \\ \frac{1}{2} + \lambda^2 |\langle Ue_0 \mid e_0\rangle|^2, & \text{for } F = \mathbb{C}e_0\,. \end{cases} \tag{3.26}$$

Let us turn to the transformed states $\nu_T(\omega_\lambda)$, $\lambda \geq 0$, which by Proposition 3.10 are quasifree, too. With (2.5) one easily finds $\mathsf{Var}(\nu_T(\omega_\lambda); f) = \mathsf{Var}(\omega_{\text{vac}}; f) + \Delta(\nu_T(\omega_\lambda); f)$ for all $f \in E$ (cf. the equations (3.6) and (3.14)), where

$$\begin{aligned} \Delta(\nu_T(\omega_\lambda); f) &= \left(\left[\tfrac{1}{2} + \lambda^2\right] exp\{2s\} - 1\right) (Re\langle e_0 \mid f\rangle)^2 \\ &\quad + \left(\left[\tfrac{1}{2} + \lambda^2\right] exp\{-2s\} - 1\right) (Im\langle e_0 \mid f\rangle)^2\,. \end{aligned} \tag{3.27}$$

**Proposition 3.18** *For each $\lambda \geq 0$ the normally ordered characteristic function of the transformed $\nu_T(\omega_\lambda)$ is given by*

$$P_{\nu_T(\omega_\lambda)}(f) = exp\left\{-\tfrac{1}{2}\,\Delta(\nu_T(\omega_\lambda); f)\right\} \qquad \forall f \in E\,. \tag{3.28}$$

*$\nu_T(\omega_\lambda)$ is a first order coherent state, exactly, $\nu_T(\omega_\lambda) \in \mathcal{S}^{(1)}_{coh}(\kappa(\lambda)Q) \bigcap \mathcal{S}_{qf}$, where $Q(f) = \langle e_0 \mid f\rangle$ $\forall f \in E$ and $\kappa(\lambda) := \sqrt{sinh(s)^2 + sinh(2s)\lambda^2}$, implying the factorization*

$$\left\langle \nu_T(\omega_\lambda)\,;\, a^*_{\nu_T(\omega_\lambda)}(f)\, a_{\nu_T(\omega_\lambda)}(g)\right\rangle = \kappa(\lambda)Q(f)\,\overline{\kappa(\lambda)Q(g)} \qquad \forall f, g \in E\,. \tag{3.29}$$

*Moreover, for $\nu_T(\omega_\lambda)$ being classical we have the following equivalent conditions:*

*(i)* $\nu_T(\omega_\lambda) \in \mathcal{S}_{cl}$,

*(ii)* $\lambda \geq \lambda_c(s)$ *with the critical value* $\lambda_c(s)^2 := \frac{1}{2}(exp\{2s\} - 1)$,

*(iii)* $\Delta(\nu_T(\omega_\lambda); f) \geq 0$ $\forall f \in E$.

*For $\lambda > \lambda_c(s)$ we have the decomposition $\nu_T(\omega_\lambda) = \int_{\mathbb{C}} \omega^G_{\overline{z}e_0}\, \mathrm{d}\mu^Q_{\nu_T(\omega_\lambda)}(z)$ into Glauber states with the probability measure*

$$\mathrm{d}\mu^Q_{\nu_T(\omega_\lambda)}(z) = N(\lambda, s)\, exp\left\{-\frac{2\,(Re(z))^2}{[1 + 2\lambda^2]\,exp\{2s\} - 1} - \frac{2\,(Im(z))^2}{[1 + 2\lambda^2]\,exp\{-2s\} - 1}\right\} \mathrm{d}^2 z\,,$$

*with the normalization $N(\lambda, s) := \frac{2}{\pi}\left\{([1 + 2\lambda^2]\,exp\{2s\} - 1)([1 + 2\lambda^2]\,exp\{-2s\} - 1)\right\}^{-1/2}$.*

PROOF: (3.28) is a consequence of the results obtained for quasifree states in Section 3.5. Define for each $f \in E$ the entire-analytic mapping $\mathbb{C}^2 \ni (u, v) \mapsto N(u, v; f)$,

$$N(u, v; f) := exp\left\{\alpha(\lambda, s)\,|\langle e_0 \mid f\rangle|^2\, u\, v + \beta(\lambda, s)\left(\langle e_0 \mid f\rangle^2\, u^2 + \langle f \mid e_0\rangle^2\, v^2\right)\right\},$$

$\alpha(\lambda, s) := -\frac{1}{4}\left([1+2\lambda^2]\,[cosh(s)^2 + sinh(s)^2] - 1\right)$ and $\beta(\lambda, s) := -\frac{1}{4}\,[1+2\lambda^2]\,cosh(s)\,sinh(s)$, then it follows $N(z,\bar{z};f) = P_{\nu_T(\omega_\lambda)}(zf)\ \forall z \in \mathbb{C}$. Now [15] (cf. also equation (3.5)) leads to

$$\left\langle \nu_T(\omega_\lambda)\,;\, a^*(f)^k\, a(f)^l \right\rangle = \left(\tfrac{\sqrt{2}}{i}\right)^{k+l} k!\; l!\; \left.\frac{\partial^{k+l} N(.,.;f)}{\partial u^k\, \partial v^l}\right|_{u=v=0}, \qquad \forall\, k,l \in \mathbb{N}_0\,,$$

which gives the factorization (3.29), when taking $k = l = 1$ and the polarization identity. The equivalences (i) to (iii) immediately follow from equation (3.27) and Proposition 3.9. For $\lambda > \lambda_c(s)$ the decomposition of $\nu_T(\omega_\lambda)$ into the Glauber states follows from (3.28) and the Fourier formula (3.25) by observing that the integral decomposition from (3.18) here writes as

$$\begin{aligned} C_{\nu_T(\omega_\lambda)}(f) &= C_{\rm vac}(f)\, P_{\nu_T(\omega_\lambda)}(f) \\ &= \int_{\mathbb{C}} C_{\rm vac}(f)\, exp\left\{i\sqrt{2}\; Re\langle \bar{z}e_0 \mid f\rangle\right\}\, d\mu^Q_{\nu_T(\omega_\lambda)}(z) \end{aligned}$$

and ranges over the characteristic functions $C^G_{\bar{z}e_0}$ of the Glauber states $\omega^G_{\bar{z}e_0}$ from eq. (3.7), $z \in \mathbb{C}$. ■

Especially below the critical value, $\lambda < \lambda_c(s)$, the transformed $\nu_T(\omega_\lambda)$ is a non-classical, first order coherent state with a non-positive $P$-representation. From the proof it is seen that for deriving the $P$-representation in this case one needs the Fourier transform of $\mathbb{R} \ni x \mapsto exp\{ax^2\}$ with $a \geq 0$, which according to [32] is a highly singular distribution in $\mathcal{Z}'$ — the space of analytical functionals — and not a signed measure or an element of $\mathcal{S}'$.

Let us turn now to the squeezing properties. If $f$ is orthogonal to $e_0$ with respect to $\langle . \mid . \rangle$, then $\Delta(\nu_T(\omega_\lambda); f) = 0$ and the variances (3.27) reduce to the vacuum fluctuations, $\mathsf{Var}(\nu_T(\omega_\lambda); f) = \mathsf{Var}(\omega_{\rm vac}); f)$. If $0 \neq z \in \mathbb{C}$, then

$$\mathsf{Var}(\nu_T(\omega_\lambda); ze_0) = \left(\tfrac{1}{2} + \lambda^2\right)|z|^2 \begin{cases} exp\{2s\}\ , & \text{for } z \in \mathbb{R}\,, \\ exp\{-2s\}, & \text{for } z \in i\mathbb{R}\,. \end{cases}$$

Thus we obtain for the complex subspace $F \subseteq E$

$$\mathsf{InfVar}(\nu_T(\omega_\lambda); F) = \begin{cases} \frac{1}{2} = \mathsf{InfVar}(\omega_{\rm vac}; F) \quad , & \text{for } F \perp e_0\,, \\ \left(\frac{1}{2}+\lambda^2\right) exp\{-2s\} < \frac{1}{2}, & \text{for } \lambda < \lambda_c(s)\,,\ e_0 \in F\,, \\ \frac{1}{2} \quad , & \text{for } \lambda = \lambda_c(s)\,,\ e_0 \in F\,, \\ \left(\frac{1}{2}+\lambda^2\right) exp\{-2s\} > \frac{1}{2}, & \text{for } \lambda > \lambda_c(s)\,,\ F = \mathbb{C}e_0\,, \\ \frac{1}{2} \quad , & \text{for } \lambda \geq \lambda_c(s)\,,\ e_0 \in F\,,\ dim_{\mathbb{C}}(F) \geq 2\,. \end{cases} \tag{3.30}$$

Observe that for $\lambda = 0$ it is $\omega_0 = \omega_{\rm vac}$. This demonstrates that $\omega_\lambda$ is non-optimally squeezed by $\nu_T$ for $0 < \lambda < \lambda_c(s)$, or for each $\lambda > 0$ whenever $dim_{\mathbb{C}}(E) = 1$, in both cases it holds: $\mathsf{InfVar}(\nu_T(\omega_\lambda); E) = \left(\frac{1}{2}+\lambda^2\right) exp\{-2s\} > \frac{1}{2}\, exp\{-2s\} = \mathsf{InfVar}(\nu_T(\omega_{\rm vac}); E)$.

Let us turn to some squeezing properties according to Definition 2.3 by comparing (3.26) with (3.30). If $dim_{\mathbb{C}}(F) \geq 2$, and $e_0 \in F$, and $\lambda < \lambda_c(s)$, then it follows that $\omega_\lambda$ is $F$-squeezed by $\nu_T$, more precisely, $\mathsf{InfVar}(\nu_T(\omega_\lambda); F) = \left(\frac{1}{2}+\lambda^2\right) exp\{-2s\} < \frac{1}{2} = \mathsf{InfVar}(\omega_\lambda; F)$. And for $F = \mathbb{C}e_0$ we have $\mathsf{InfVar}(\omega_\lambda; \mathbb{C}e_0) = \frac{1}{2} + \lambda^2\, |\langle Ue_0 \mid e_0\rangle|^2$ and $\mathsf{InfVar}(\nu_T(\omega_\lambda); \mathbb{C}e_0) = \left(\frac{1}{2}+\lambda^2\right) exp\{-2s\}$, which implies $\mathbb{C}e_0$-sqeezing also for suitable parameters $\lambda > \lambda_c(s)$.

# 4 Conclusions on the Non–Classicality of States

In the foregoing investigations a main point has been to identify the non–classical character of a state on the (photon) field algebra $\mathcal{W}(E)$ by means of its field fluctuations. In Subsection 3.4 the general definition of a classical state $\omega$ is given, which requires the normally ordered characteristic function $P_\omega(f)$ to be (normalized and) positive–definite. The field variances (3.11) of $\omega$ for the testmode $f$ contain the vacuum fluctuations plus the classical fluctuations and demonstrates a simple but important fact: if for a state $\omega \in \mathcal{S}$ one has

$$\mathsf{Var}(\omega; f) \;<\; \tfrac{1}{2}\,\|f\|^2 \;=\; \mathsf{Var}(\omega_{\mathrm{vac}}; f) \tag{4.1}$$

for a single (non–vanishing) testmode $f \in E$, then $\omega$ is necessarily non–classical by Proposition 3.6.

Let us compare this with other criteria used in the literature. We consider only the one–mode testfunction space $E = \mathbb{C}f$, $\|f\| = 1$, and set $a_\omega(f) =: a$. There are the following notions:

(1) *Two–point correlations* (with zero–time delay)

$$g_\omega^{(2)}(0) \;:=\; \frac{\langle \omega\,;\, a^* a^* a a\rangle}{\langle \omega\,;\, a^* a\rangle^2} \;=\; \frac{\langle \omega\,;\, a^* a a^* a\rangle - \langle \omega\,;\, a^* a\rangle}{\langle \omega\,;\, a^* a\rangle^2}$$

with non–classical regime $g_\omega^{(2)}(0) < 1$ (anti–bunching) [11].

(2) *Fano–factor*

$$F_\omega \;:=\; \frac{\langle \omega\,;\, \Delta^2 a^* a\rangle}{\langle \omega\,;\, a^* a\rangle} \;=\; \frac{\langle \omega\,;\, a^* a a^* a\rangle - \langle \omega\,;\, a^* a\rangle^2}{\langle \omega\,;\, a^* a\rangle}\,,$$

which expresses sub–Poissonian counting distributions for $F_\omega < 1$ [33].

(3) *Mandel's Q–factor*

$$Q_\omega \;:=\; \frac{\langle \omega\,;\, \Delta^2 a^* a\rangle - \langle \omega\,;\, a^* a\rangle}{\langle \omega\,;\, a^* a\rangle}\,,$$

which for $Q_\omega < 0$ should determine non–classicality [34], [35].

**Observation 4.1** *It holds for each state $\omega$ on $\mathcal{W}(E)$ and every testmode $f \in E$, $\|f\| = 1$,*

$$Q_\omega \;=\; F_\omega - 1 \;=\; \langle \omega\,;\, a^* a\rangle \left(g_\omega^{(2)}(0) - 1\right). \tag{4.2}$$

*Therefore*

$$Q_\omega < 0 \quad \Leftrightarrow \quad F_\omega < 1 \quad \Leftrightarrow \quad g_\omega^{(2)}(0) < 1\,. \tag{4.3}$$

*The validity of (one relation of) (4.3) is sufficient for $\omega$ to be non–classical.*

**Observation 4.2** *For every second order coherent state $\omega$ one has for all $f \in E$, $\|f\| = 1$,*

$$Q_\omega = 0\,, \quad \textit{and} \quad F_\omega = 1 = g^{(2)}_\omega(0)\,.$$

*Since there are non-classical coherent states in any order [36], [19], [21], [22], the inequalities (4.3) are not necessary for $\omega$ to be non-classical.*

The surprising fact is, that it is much harder to calculate the quadratic field variances for non-classical coherent states than the fourth order quantities in (4.2) for these states. Up to now we did not find a non-classical coherent state, which violates (4.1). In some sense the quadratic field variances seem to contain more information than the mentioned fourth order quantities $g^{(2)}_\omega(0)$, $F_\omega$, and $Q_\omega$. This point of view is supported by Proposition 3.9 stating that (4.1) is for quasifree states necessary and sufficient (i.e., equivalent) to be non-classical.

Since classical states are much easier to prepare experimentally than non-classical ones, they are the natural starting point for discussing the efficiency of a squeezing device. If the squeezing strength $\|S\|$ is finite (where as before $T = U(\mathrm{e}^S \oplus \mathrm{e}^{-S})$) then the investigations in Subsection 3.6 reveal that it is possible only under certain conditions to reach the optimal degree of squeezing (cf. Theorem 3.16)

$$\mathsf{InfVar}(\nu_T(\omega); E) \;=\; \mathsf{InfVar}(\nu_T(\omega_{\mathrm{vac}}); E)\,,$$

which is given by the minimal variances of the squeezed vacuum.

Let us use related considerations for a simple criterion for non-classicality.

**Observation 4.3** *If for a state $\omega$ on $\mathcal{W}(E)$ there is any squeezing transformation $\nu_T$, $T \in \mathcal{T}(E)$, and any testmode $f \in E$ with*

$$\mathsf{Var}(\nu_T(\omega); f) \;<\; \mathsf{Var}(\nu_T(\omega_{\mathrm{vac}}); f)\,, \tag{4.4}$$

*that is, $\omega$ is better squeezed in $f$ than the vacuum $\omega_{\mathrm{vac}}$, then $\omega$ is non-classical.*

The reason is of course, that by the inverse squeezing transformation (4.4) would lead to (4.1) for the mode $Tf$.

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# Appendix

## A.1 Degenerate and Non–Degenerate Squeezing Hamiltonians

For the theoretical descriptions of squeezing there are used mainly quadratic Hamiltonians of the photon field [1]–[7]. The Hamiltonians are meant to describe essential features of the dynamics which takes place in the non–linear optical medium. As in the Introduction let the finite dimensional testfunction space $E$ be spanned by the orthonormalized photon modes $\{e_1, \ldots, e_N\}$. Usually two types of quadratic expressions are distinguished:

- The degenerate squeezing Hamiltonian from equation (1.4)

$$H_{\mathrm{d}} = \frac{1}{2}\sum_{n=1}^{N}\left(\zeta_n\, a^*(e_n)^2 + \overline{\zeta_n}\, a(e_n)^2\right) \tag{A.1}$$

  with the squeezing parameters $\zeta_n \in \mathbb{C}$.

- The non–degenerate squeezing Hamiltonian

$$H_{\mathrm{nd}} = \frac{1}{2}\sum_{k,l=1}^{N}\left(\eta_{k,l}\, a^*(e_k)\, a^*(e_l) + \overline{\eta_{k,l}}\, a(e_k)\, a(e_l)\right) \tag{A.2}$$

  with $\eta_{k,l} \in \mathbb{C}$. Here in general the modes $\{e_1, \ldots, e_N\}$ are subdivided into the signal and the idler modes. Obviously, the terms with $k = l$ give the degenerate parts of $H_{\mathrm{nd}}$. Thus the Hamiltonian $H_{\mathrm{nd}}$ is strictly non–degenerate, only if $\eta_{k,k} = 0$.

These two cases of squeezing quadratic Hamiltonians are formally not so different as they seem to be: By superposing the modes in the smeared field formalism, we now transform the non–degenerate squeezing Hamiltonian $H_{\mathrm{nd}}$ into the form (A.1).

Since the creation operators $a^*(e_k)$ and $a^*(e_l)$ commute (and also the annihilation operators), we may assume without restriction of generality that $\eta_{k,l} = \eta_{l,k}$ for all indices $k, l \in \{1, \ldots, N\}$. From $H_{\mathrm{nd}}$ we extract the anti–linear operator $D$ on $E$,

$$Df = \sum_{k,l=1}^{N} \eta_{k,l}\, \langle f \mid e_k \rangle\, e_l \qquad \forall f \in E\,, \tag{A.3}$$

and construct a new orthonormal basis for $E$, which diagonalizes $D$. $\eta_{k,l} = \eta_{l,k}$ implies the selfadjointness of the anti–linear $D$, i.e., $\langle f \mid Dg \rangle = \langle g \mid Df \rangle$ $\forall f, g \in E$.

Let $D = V\,|D|$ be the polar decomposition of $D$ with unique anti–linear partial isometry $V$ and linear absolut value $|D| = \sqrt{D^2}$ (observe that $D^2 = D^*D \geq 0$ is linear). The selfadjointness of $D$ yields $\mathit{ker}(|D|) = \mathit{ker}(D) = \mathit{ker}(D^*)$. Hence the initial space and final space of $V$ are both $\mathit{ker}(D)^\perp$. The selfadjointness of $D$ also ensures $V = V^*$ to commute with $|D|$. Hence $V$ is an anti–linear involution on $\mathit{ker}(D)^\perp$. The diagonalization of $|D|$ gives a new orthonormal basis $\{u_1, \ldots, u_N\}$ for $E$ and eigenvalues $d_n \geq 0$ with $|D|\, u_n = d_n u_n$. $V$ commutes with $|D|$, thus the $u_n$ may be chosen such that $V u_n = u_n$, which implies

$$Df \;=\; \sum_{n=1}^{N} d_n \langle f \mid u_n \rangle\, u_n \qquad \forall f \in E\,. \tag{A.4}$$

Calculating the matrix elements $\langle u_n \mid D\, u_m \rangle$ with (A.4) and (A.3) yields $d_n\, \delta_{m,n} = \sum_{k,l=1}^{N} \eta_{k,l} \langle u_m \mid e_k \rangle \langle u_n \mid e_l \rangle$ ($\delta_{m,n}$ is the common Kronecker symbol: $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$). Insert this and the decomposition $e_k = \sum_{n=1}^{N} \langle u_n \mid e_k \rangle\, u_n$ into (A.2). Then the linearity of $E \ni f \mapsto a^*(f)$ and the anti-linearity of $E \ni f \mapsto a(f)$ implies

$$H_{\mathrm{nd}} \;=\; \frac{1}{2} \sum_{n=1}^{N} d_n \left( a^*(u_n)^2 \,+\, a(u_n)^2 \right)\,,$$

which has the form of a degenerate quadratic Hamiltonian.

## A.2 Optical States with a Positive $P$–Representation

In quantum optics the description of states often is given in terms of the phase space formalism, however, for finite dimensional one–photon testfunction spaces $E$, only. Since $dim_{\mathbb{C}}(E) = N < \infty$, by the Stone–von Neumann uniqueness theorem [8, Corollary 5.2.15] the regular photon field states $\omega$ on $\mathcal{W}(E)$ are given by the density operators $\rho$ on the Fock space $F_+(E)$ by means of $\langle \omega\,;\, A \rangle = \mathrm{tr}[\rho\, \Pi_F(A)]$ for all $A \in \mathcal{W}(E)$.

### The Phase Space Description of States in Quantum Optics

Taking an orthonormal basis $\{e_1, \ldots, e_N\}$ for $E$ each $f \in E$ decomposes according to $f = \sum_{n=1}^{N} \beta_n e_n$ with $\beta_n = \langle e_n \mid f \rangle$, defining a unitary representation of the testfunction space $E$ as the phase space $\mathbb{C}^N \cong \mathbb{R}^{2N}$ with phase space points $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{C}^N$.

Decomposing the characteristic function $C_\omega$ from equation (2.1) as

$$C_\omega(-i\sqrt{2}\textstyle\sum_{n=1}^{N} \beta_n e_n) \;=\; C_\omega^S(\beta) \;=\; exp\left\{-\frac{1}{2}\,|\beta|^2\right\} C_\omega^N(\beta) \;=\; exp\left\{\frac{1}{2}\,|\beta|^2\right\} C_\omega^A(\beta)\,,$$

where $|\beta|^2 = \sum_{n=1}^{N} |\beta_n|^2$, leads to the characteristic functions $C_\omega^S$, $C_\omega^N$, and $C_\omega^A$ in resp. symmetric (or Weyl), normal, and antinormal ordering. Especially, (3.3) implies for the vacuum $\omega_{\mathrm{vac}}$,

$$C_{\mathrm{vac}}^S(\beta) \;=\; exp\left\{-\tfrac{1}{2}\,|\beta|^2\right\}\,, \quad C_{\mathrm{vac}}^N(\beta) \;=\; 1\,, \quad C_{\mathrm{vac}}^A(\beta) \;=\; exp\left\{-\,|\beta|^2\right\}\,.$$

Fourier transformation of $C_\omega^j : \mathbb{C}^N \to \mathbb{C}$, where $j \in \{S, N, A\}$, finally gives the $W$– (or Wigner–), $P$–, and $Q$–representation of our photon state $\omega$, respectively, [12], [14]. Fourier transforms, however, may lead to completely singular distributions. This restricts the usefulness of the $W$–, $P$–, resp. $Q$–representation for a photon state $\omega \in \mathcal{S}$.

### The $P$–Representation: Decomposition into Glauber States

The $P$–representation determines the decomposition of the state $\omega \in \mathcal{S}$ into the Glauber states $\omega_h^G$, $h \in E$, from Subsection 3.4.1. Here we renounce on mathematical rigorousity. Suppose the $P$–representation of the state $\omega$ to be given by the (possibly non–positive) measure $\mu$ on $E \cong \mathbb{C}^N$. The Fourier (back–) transform $\widehat{\mu}(f) = \int_E exp\left\{i\sqrt{2}\ Re\langle h \mid f\rangle\right\} \mathrm{d}\mu(h)$ agrees (up to some factor in the argument) with the normally ordered characteristic function of $\omega$, that is, $C_\omega = C_{\text{vac}}\,\widehat{\mu}$. With formula (3.7) we arrive at the decomposition

$$\omega \;=\; \int_E \omega_h^G \,\mathrm{d}\mu(h) \tag{A.5}$$

for our state $\omega \in \mathcal{S}$. Obviously, the associated density operator $\rho_\omega$ on Fock space is given by the (possibly non–positive) "mixture" $\rho_\omega = \int_E |G(h)\rangle\langle G(h)|\,\mathrm{d}\mu(h)$ (cf. [27, Section 8.2]).

If $\mu$ is a (positive) probability measure on $E$, then (A.5) indeed defines a regular state $\omega \in \mathcal{S}$, which is a genuine (convex) mixture of the Glauber states. In this case of a positive $P$–representation, the state $\omega$ commonly is denoted to be classical. Thus classical photon field states on $\mathcal{W}(E)$ are in one–to–one correspondence with the probability measures $\mu$ on $E$ — the statistical states of the "classical meachanical system" with phase space $E$ —, or by Bochner's theorem [26, Theorem IX.9] with the continuous, normalized positive–definite functions $\widehat{\mu} : E \to \mathbb{C}$ on the additive group $E$ (Fourier transform of $\mu$).

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