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A Classification of Classical Representations for Quantun like Systems

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Abstract. For general non-classical systems, we study the different classical representations that fulfill the specific context dependence imposed by the hidden measurement system formalism introduced in [6]. We show that the collection of non-equivalent representations has a poset structure. We also show that in general, there exists no 'smallest' representation, since this poset is not a semi-lattice. Then we study the possible representations of quantum-like measurement systems. For example, we show that there exists a classical representation of finite dimensional quantum mechanics with $\mathbb{N}$ as a set of states for the measurement context, and we build an explicit example of such a representation.

1 Introduction

For a detailed introduction on the general subject we refer to [6], where we have proved that every 'measurement system' (=m.s.) has a representation as a 'hidden measurement system' (=h.m.s.), i.e., every physical entity that is defined by a collection of states and a collection of measurements, such that for a defined initial state we have a probability measure that defines the relative occurrence of the outcomes, there exist a classical representation with a very specific kind of context dependence. Thus, also for quantum entities there exist such a classical representation. An obvious question is: which different classical representations exist for a given quantum entity. In this paper we build a classification of all possible h.m.s.-representations, for every given quantum m.s. A lot of preparing work has already be done in [6], in section 4 and in the appendix. Therefore, we will regularly refer to the theorems and definitions of that paper. For the identification of the h.m.s.-representations for quantum m.s. we proceed in three steps. First we show that the collection of possible h.m.s.-representations has a poset structure, and this will allow us to characterize the possible h.m.s.-representations for a given m.s. by its smallest h.m.s.-representations. Then we show that in general, there exists no 'preferred' smallest h.m.s.-representation, and this forces us to consider the quantum m.s. case by case. After these three steps we present two examples of explicit h.m.s.-representations which illustrate the results that we have obtained in this paper. As in [6], for a general definition of the basic mathematical objects we refer to [3] and [9]. As in [6], we remark that the results presented in this paper where made known in [5].

2 An additional assumption

In this paper we make a from a physical point of view very natural assumption, namely we only consider h.m.s. for which there exists a probability measure $\mu \in \mathbf{MX}$ such that this h.m.s. belongs to $\mathbf{HMS}(\mu)$, i.e., for every measurement of this h.m.s. we consider the same set of states of the measurement context and the same relative frequency of occurrence of these states. The motive of this additional assumption is essentially 'simplicity', since a general treatment (which is indeed possible) would not lead to any new conceptual insights or mathematical insights, but would only make things a lot more complicated. Nonetheless, in the light of section 3.3 of [6] this assumption is also very natural since most m.s. that we consider in practice have a sufficient additional structure to impose a representation for all measurements, starting with a representation for one measurement, and all of the with the same probability measure that determines the relative frequency of occurrence of the states of the measurement context. We also remark that this assumption does not influence the results of [6], and in particular does the existence theorem in section 4.3 remain valid. For this subset of $\mathbf{HMS}$ that we consider in this paper we introduce the following notation:

$$\mathbf{HMS}(\mathbf{MX}) = \{\Sigma, \mathcal{E} | \Sigma, \mathcal{E} \in \mathbf{HMS}(\mu), \mu \in \mathbf{MX}\} \tag{2.1}$$

3 A poset characteristic for possible representations

In section 4.1 in [6] we have defined a set $\mathbf{M}$, consisting of classes of isomorphic measure spaces. We also proved that for every separable measure space, there exists one and only one class in $\mathbf{M}$ to which this measure space belongs (see Lemma 1 of [6]). In Definition 8 of [6] we introduced a binary relation $\leq$ on $\mathbf{M}$. We have the following lemma:

Lemma 1 $\mathbf{M}, \leq$ *is a poset.*

Proof: The proof that $\leq$ is reflexive and transitive is straightforward, so we only have to verify whether it's anti-symmetrical. Let $\mathcal{M} \leq \mathcal{M}'$ and $\mathcal{M}' \leq \mathcal{M}$, and $\mathcal{B}, \mu \in \mathcal{M}$,

$\mathcal{B}', \mu' \in \mathcal{M}$, and $F : \mathcal{B} \to \mathcal{B}'$ and $F' : \mathcal{B}' \to \mathcal{B}$ fulfilling Definition 8 in [6]. First we prove that, given the above stated assumptions (the existence of F and F'), there are two possibilities: $\mathcal{M}, \mathcal{M}' \in \mathbf{M}_{\mathbb{X}} \cup \mathbf{M}_{\mathbb{N}}$ and $\mathcal{M}, \mathcal{M}' \in \mathbf{M}_{\mathbb{R},a}$. Suppose that $\mathcal{M} \in \mathbf{M}_{\mathbb{R},a}$ and $\mathcal{M}' \in \mathbf{M}_{\mathbb{R},a'}$ with $a \neq a'$. Consider the sets $B_0 = (\emptyset, I) \in \mathcal{B}$, and $B_0' = (\emptyset, I') \in \mathcal{B}'$ (I and I' are the greatest elements of different Borel algebras). We have $F(B_0) \neq B_0'$ since $\mu'(F(B_0)) = \mu(B_0) = a$ and $\mu'(B_0') = a' \neq a$. Thus, $F(B_0^c) \cap B_0' \neq \emptyset$ or $F(B_0^c) \subset B_0'^c$. Suppose $F(B_0^c) \cap B_0' \neq \emptyset$. According to Lemma 3 in [6], $\mathcal{B}_l, \mu_l$ is a measure space (we also use the notations of Lemma 3 in [6]), and, following Lemma 2 in [6] we find $\mathcal{B}_l, \mu_l \cong \mathcal{B}_{\mathbb{R}}, \mu$. As a consequence, $\mathcal{B}_{\mathbb{R}} \cong \mathcal{B}_l \cong \{F(B) | B \in \mathcal{B}_l\}$ (due to Proposition 3 in [6], $\{F(B) | B \in \mathcal{B}_l\}$ is a Borel subalgebra of $\mathcal{B}'$, isomorphic with $\mathcal{B}_l$). But, since $F(B_0^c) \cap B_0' \neq \emptyset$, this situation contradicts with the fact that $\mathcal{B}_{\mathbb{R}}$ cannot be embedded in a one-to-one way in $\mathcal{B}_{\mathbb{N}}$. If $F(B_0^c) \subset B_0'^c$ we can make the same reasoning by exchanging the roles of $\mathcal{B}, \mu$ and $\mathcal{B}', \mu'$. In an analogous way one proves that the following two situations: $\mathcal{M} \in \mathbf{M}_{\mathbb{R}}$, $\mathcal{M}' \in \mathbf{M}_{\mathbb{X}} \cup \mathbf{M}_{\mathbb{N}}$ and $\mathcal{M} \in \mathbf{M} \setminus \{\mathcal{M}_{\mathbb{R}}\}$, $\mathcal{M}' = \mathcal{M}_{\mathbb{R}}$ are not possible. Suppose that $\mathcal{M}, \mathcal{M}' \in \mathbf{M}_{\mathbb{N}}$. Let $\mathcal{B} = \mathcal{B}' = \mathcal{B}_{\mathbb{N}}$, $\mu = \mu_m$ and $\mu' = \mu_{m'}$. If $i, j \in \mathbb{N}$ and $i \neq j$ then $F(\{i\}) \cap F(\{j\}) = F(\{i\} \cap \{j\}) = F(\emptyset) = \emptyset$. Let n_- be the smallest integer such that $F(\{n_-\})$ is not a singleton, and let n_+ be the largest integer such that $m(n_+) = m(n_-)$. Since $(m(i))_i$ is a decreasing sequence, $(\mu_{m'}(F(\{i\})))_i = (\mu_m(\{i\}))_i = (m(i))_i$ is also a decreasing sequence. Since we have that:

1) $(\mu_{m'}(F(\{i\})))_i$ is a decreasing sequence
2) $(\mu_{m'}(i))_i = (m'(i))_i$ is a decreasing sequence
3) $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}$ such that $i \in F(\{j\})$
4) $\forall i \in \mathbb{X}_{n_- -1} : F(\{i\})$ is a singleton

we have $\cup_{j=1}^{j=n_- -1} F(\{j\}) = \mathbb{X}_{n_- -1}$. Moreover, since for all $i \in F(\{n_-\}) : \mu_{m'}(F(\{n_-\}) \setminus \{i\}) > 0$, we can conclude with the following image:

$$\begin{aligned}
\forall i \in \cup_{j=1}^{j=n_- -1} F(\{j\}) &: m'(i) = \mu_{m'}(\{i\}) = \mu_{m'}(F(\{i\})) = m(i) \geq m(n_-) \\
\forall i \in F(\{n_-\}) &: m'(i) = \mu_{m'}(\{i\}) < \mu_{m'}(F(\{n_-\})) = m(n_-) \\
\forall i \in \cup_{j=n_- +1}^{j=n_+} F(\{j\}) &: m'(i) \leq m(n_-) \\
\forall i \notin \cup_{j=1}^{j=n_+} F(\{j\}) &: m'(i) < m(n_-)
\end{aligned}$$

In the case that $n_- + 1 \leq j \leq n_+$, $i \in F(\{j\})$ and $m'(i) = m(n_-)$, we have $\{i\} = F(\{j\})$. As a consequence, there are at most $n_+ - 1$ integers i such that $m'(i) \geq m(n_+) = m(n_-)$, and thus we have $m(n_+) > m'(n_+)$. All this results in the following equations:

$$\forall i \in \mathbb{X}_{n_+ -1} : m(i) \geq m'(i), m(n_+) > m'(n_+) \tag{3.1}$$

Analogously, with the same reasoning on F', we find that there exists an integer n'_+ such that:

$$\forall i \in \mathbb{X}_{n'_+ -1} : m(i) \leq m'(i), m(n'_+) < m'(n'_+) \tag{3.2}$$

The contradiction between eq.3.1 and eq.3.2 indicates that there exist no such integers n_+ and n'_+. As a consequence there exist no integers n_- and n'_-, and thus, F and F' are onto, i.e., they are isomorphisms of measure spaces (see Proposition 3 in [6]). As a consequence, $\mathcal{B}, \mu \cong \mathcal{B}', \mu'$, and thus we have $\mathcal{M} = \mathcal{M}'$. The case $\mathcal{M}, \mathcal{M}' \in \mathbf{M}_{\mathbb{X}}$ proceeds along the same lines. It is also clear that the case $\mathcal{M} \in \mathbf{M}_{\mathbb{X}}$ and $\mathcal{M}' \in \mathbf{M}_{\mathbf{N}}$ is not possible. For the case $\mathcal{M}, \mathcal{M}' \in \mathbf{M}_{\mathbb{R},a}$, we already know from the foregoing part of this proof that $F(B_0) = B_0'$. Along the same lines as for the case $\mathcal{M}, \mathcal{M}' \in \mathbf{M}_{\mathbb{X}} \cup \mathbf{M}_{\mathbf{N}}$, and by applying Lemma 3 in [6], we find that $\mathcal{M} = \mathcal{M}'$. •

Moreover, $\mathbf{M}, \leq$ has a greatest element, namely $\mathcal{M}_{\mathbb{R}}$ (see Lemma 6 in [6]). Thus, $\mathbf{M}, \leq$ is a poset with a greatest element. As we will show now, this poset characterizes the possible different candidates for h.m.s.-representations. First we introduce 'inclusion up to mathematical equivalence' based on the definition of 'belonging up to mathematical equivalence' (see [6]). If $\Sigma, \mathcal{E} \in \mathbf{N}$ implies that $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{N}$ we write:

$$\mathbf{N} \underset{\sim}{\subset} \mathbf{N}' \tag{3.3}$$

If both $\mathbf{N} \underset{\sim}{\subset} \mathbf{N}'$ and $\mathbf{N}' \underset{\sim}{\subset} \mathbf{N}$ are valid, we write:

$$\mathbf{N} \approx \mathbf{N}' \tag{3.4}$$

One easily sees that the condition '$\Sigma, \mathcal{E} \in \mathbf{N}$ implies that $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{N}'$' is equivalent with '$\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{N}$ implies that $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{N}'$'. Thus, if $\mathbf{N} \approx \mathbf{N}'$, then $\mathbf{N}$ and $\mathbf{N}'$ are equivalent sets of representations, i.e., if a measure space has a representation $\Sigma, \mathcal{E}$ in $\mathbf{N}$, then it has a representation $\Sigma', \mathcal{E}'$ in $\mathbf{N}'$ such that $\Sigma, \mathcal{E} \sim \Sigma', \mathcal{E}'$, and vice versa. As we saw in section 4.2 in [6], with every $\mu \in \mathbf{MX}$ we can relate $\mathcal{M}_\mu \in \mathbf{M}$. For every $\mathcal{M} \in \mathbf{M}$ we can introduce the collection of all h.m.s. in $\mathbf{HMS}(\mathbf{MX})$ which are such that $\mathcal{M}_\mu = \mathcal{M}$:

- $\mathbf{HMS}(\mathcal{M}) = \cup_{\mathcal{M}_\mu = \mathcal{M}} \mathbf{HMS}(\mu)$

This allows us to introduce the following collection of classes of h.m.s.:

- $\mathbf{M}_{\mathbf{HMS}} = \{\mathbf{HMS}(\mathcal{M}) | \mathcal{M} \in \mathbf{M}\}$

The importance if this collection of classes of h.m.s. follows from the following two theorems:

Theorem 1 $\forall \mu, \mu' \in \mathbf{MX} : \mathcal{M}_\mu = \mathcal{M}_{\mu'} \Rightarrow \mathbf{HMS}(\mu) \approx \mathbf{HMS}(\mu')$.

Proof: If $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mu)$, then $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\mu)$. As a consequence of Theorem 1 in [6] we have $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}_\mu = \mathcal{M}_{\mu'}$, and thus, again due to Theorem 1 in [6], $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\mu')$. This results in $\mathbf{HMS}(\mu) \underset{\sim}{\subset} \mathbf{HMS}(\mu')$. Analogously, we find $\mathbf{HMS}(\mu') \underset{\sim}{\subset} \mathbf{HMS}(\mu)$, and thus $\mathbf{HMS}(\mu) \approx \mathbf{HMS}(\mu')$. •

Thus, for a given m.s., if μ and μ' are such that $\mathcal{M}_\mu = \mathcal{M}_{\mu'}$, then $\mathbf{HMS}(\mu)$ and $\mathbf{HMS}(\mu')$ are equivalent collections of h.m.s.-representations. As a consequence, for all $\mathcal{M} \in \mathbf{M}$, $\mathbf{HMS}(\mathcal{M})$ can be considered as a collection of equivalent h.m.s.-representations. The different collections of equivalent h.m.s.-representations are ordered in the following way.

Theorem 2 $\mathbf{M}, \leq$ *and* $\mathbf{M_{HMS}}, \subsetneq$ *are isomorphic posets.*

Proof: We have to prove that $\mathbf{HMS}(\mathcal{M}) \subsetneq \mathbf{HMS}(\mathcal{M}') \Leftrightarrow \mathcal{M} \leq \mathcal{M}'$.

i) $\mathcal{M} \leq \mathcal{M}' \Rightarrow \mathbf{HMS}(\mathcal{M}) \subsetneq \mathbf{HMS}(\mathcal{M}')$: Since $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mathcal{M})$ implies $\Sigma, \mathcal{E} \lneqq \mathbf{HMS}(\mathcal{M})$, there exists $\mu \in \mathbf{MX}$ such that $\Sigma, \mathcal{E} \lneqq \mathbf{HMS}(\mu)$. As a consequence of Theorem 1 in [6] we have $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}_\mu = \mathcal{M}$. If $\mathcal{M} \leq \mathcal{M}'$ then $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}' = \mathcal{M}_{\mu'}$ for every μ' such that $\mathcal{M}_{\mu'} = \mathcal{M}'$. Thus, again due to Theorem 1 in [6] we have $\Sigma, \mathcal{E} \lneqq \mathbf{HMS}(\mu') \subsetneq \mathbf{HMS}(\mathcal{M}')$, and thus $\mathbf{HMS}(\mathcal{M}) \subsetneq \mathbf{HMS}(\mathcal{M}')$.

ii) $\mathbf{HMS}(\mathcal{M}) \subsetneq \mathbf{HMS}(\mathcal{M}') \Rightarrow \mathcal{M} \leq \mathcal{M}'$: If $\mathbf{HMS}(\mathcal{M}) \subsetneq \mathbf{HMS}(\mathcal{M}')$, then $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mathcal{M})$ implies $\Sigma, \mathcal{E} \lneqq \mathbf{HMS}(\mathcal{M}')$, and thus, following Theorem 1 in [6], $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mathcal{M})$ implies $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}'$. Now we'll prove that there exists $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mathcal{M})$ such that $\mathcal{M} = \Delta\mathbf{M}(\Sigma, \mathcal{E})$. Let μ_Λ be such that $\mathcal{M}_\mu = \mathcal{M}$. We'll define a measure system $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mu_\Lambda)$. For all $e \in \mathcal{E}$, let $O_e = \Lambda$, $\mathcal{B}_e = \mathcal{B}_\Lambda$, and define the set of strictly classical observables $\{\varphi_{\lambda,e} | e \in \mathcal{E}, \lambda \in \Lambda\}$ such that $\forall p \in \Sigma, e \in \mathcal{E}, \forall \lambda \in \Lambda : \varphi_{\lambda,e}(p) = \lambda$. Thus, $\forall p \in \Sigma, e \in \mathcal{E}, \forall B \in \mathcal{B}_\Lambda : \Delta\Lambda^B_{p,e} = B$, and thus, as a consequence of Proposition 2 in [6], $\forall p \in \Sigma, e \in \mathcal{E}, \forall B \in \mathcal{B}_e : P_{p,e}(B) = \mu_\Lambda(\Delta\Lambda^B_{p,e}) = \mu_\Lambda(B)$. Since $\forall p \in \Sigma, e \in \mathcal{E} : P_{p,e} = \mu_\Lambda$, we have $\forall p \in \Sigma, e \in \mathcal{E} : \mathcal{M}_{p,e} = \mathcal{M}_\mu$, and thus $\Delta\mathbf{M}(\Sigma, \mathcal{E}) = \{\mathcal{M}\}$. All this leads us to $\mathcal{M} \leq \mathcal{M}'$. •

4 H.m.s.-representations for a given m.s.

In the previous subsection, we have 'projected' the partial order structure of $\mathbf{M}$ on $\mathbf{M_{HMS}}$. This allows us to translate the explicit structure of $\mathbf{M}$ in terms of 'representation of a m.s. as h.m.s.' In fact, the proof of the theorem on the existence of h.m.s.-representations (see [6]) was a first example of this approach. In this subsection, we'll translate some more structural properties of $\mathbf{M}$ to $\mathbf{M_{HMS}}$. Let us introduce the following notations referring to the collection of all h.m.s.-representations in $\mathbf{HMS}(\mathbf{MX})$, for a given $\Sigma, \mathcal{E} \in \mathbf{MS}$:

- $\mathbf{HMS}(\Sigma, \mathcal{E}) = \{\Sigma', \mathcal{E}' \in \mathbf{HMS}(\mathbf{MX}) | \Sigma, \mathcal{E} \sim \Sigma', \mathcal{E}'\}$
- $\mathbf{M_{HMS}}(\Sigma, \mathcal{E}) = \{\mathbf{HMS}(\mathcal{M}) \in \mathbf{M_{HMS}} | \Sigma, \mathcal{E} \lneqq \mathbf{HMS}(\mathcal{M})\}$

One can always enlarge the set of possible h.m.s.-representations if one knows the 'small' $\mathbf{HMS}(\mathcal{M}) \in \mathbf{M_{HMS}}(\Sigma, \mathcal{E})$, where we consider a h.m.s.-representation smaller than an other one if their respective measure spaces related to $(\mathcal{B}_\Lambda, \mu_\Lambda)$ have this same ordering (in the sense of Definition 8 in [6]). This fact is a straightforward consequence of Theorem 1 in [6], and in particular of the presence of an order condition in the righthandside of eq.15.

4.1 General considerations

There exists a smallest collection of mathematical equivalent representations for $\Sigma, \mathcal{E}$ (i.e., a smallest element for the relation $\lesssim$ in the collection $\mathbf{M_{HMS}}(\Sigma, \mathcal{E})$) if there exists $\mathcal{M} \in \mathbf{M}$ such that:

- $\Sigma, \mathcal{E} \lnsim \mathbf{HMS}(\mathcal{M})$
- $\forall \mathcal{M}' \in \mathbf{M}$ with $\Sigma, \mathcal{E} \lnsim \mathbf{HMS}(\mathcal{M}')$: $\mathbf{HMS}(\mathcal{M}) \lesssim \mathbf{HMS}(\mathcal{M}')$

In the following theorem we prove that in general, there exists no smallest h.m.s.-representation. First we prove a lemma.

Lemma 2 *For all* $\mathbf{N} \subseteq \mathbf{M}$, *there exists* $\Sigma, \mathcal{E} \in \mathbf{MS}$ *such that:*

$$\Delta\mathbf{M}(\Sigma, \mathcal{E}) = \mathbf{N} \tag{4.1}$$

Proof: Clearly, for all $\mathcal{M} \in \mathbf{N}$, there exists $\mu_{\mathcal{M}} \in \mathbf{MX}$ such that the related measure space is in $\mathcal{M}$. Take $\mathbf{N}$ as set of states and $\mathcal{E}$ as set of measurements, with for all $e \in \mathcal{E}$, $\mathcal{B}_e = \mathcal{B}_{\mathbb{R}}$. For all $\mathcal{M} \in \mathbf{N}$ and all $e \in \mathcal{E}$, define the outcome probability $P_{\mathcal{M},e} : \mathcal{B}_{\mathbb{R}} \to [0,1]$ such that $P_{\mathcal{M},e} = \mu_{\mathcal{M}}$. Thus, as the class in $\mathbf{M}$ related to $P_{\mathcal{M},e}$ we find $\mathcal{M}_{\mathcal{M},e} = \mathcal{M}$, and thus $\Delta\mathbf{M}(\mathbf{N}, \mathcal{E}) = \mathbf{N}$. •

We remark that this proof can also be applied to show that for all $\mathbf{N} \subseteq \mathbf{M}$, there exists $\Sigma, e \in \mathbf{MS}$ such that $\Delta\mathbf{M}(\Sigma, e) = \mathbf{N}$, i.e., it suffices to consider one measurement systems to obtain all $\mathbf{N} \subseteq \mathbf{M}$.

Lemma 3 $\mathbf{M}, \leq$ *is not a semi-lattice.*

Proof: We have to give a counterexample, i.e., a set $\mathbf{N} \subset \mathbf{M}$ which has no smallest upper bound. Consider the set $\mathbf{N} = \{\mathcal{M}_2^{m_1}, \mathcal{M}_2^{m_2}\}$ where $m_1(1) = \frac{2}{3}$, $m_1(2) = \frac{1}{3}$, $m_2(1) = \frac{3}{4}$ and $m_2(2) = \frac{1}{4}$. We have that both $\mathcal{M}_3^{m_3}$ and $\mathcal{M}_3^{m_4}$, defined by $m_3(1) = \frac{2}{3}$, $m_3(2) = \frac{1}{4}$, $m_3(3) = \frac{1}{12}$, $m_4(1) = \frac{1}{3}$, $m_4(2) = \frac{5}{12}$ and $m_4(3) = \frac{1}{4}$, are upper bounds of $\mathbf{N}$, but one easily verifies that $\mathcal{M}_3^{m_3} \not\leq \mathcal{M}_3^{m_4}$ and $\mathcal{M}_3^{m_4} \not\leq \mathcal{M}_3^{m_3}$, and thus they cannot be smallest upper bounds. If $\mathcal{M}$ would be a smallest upper bound then we should have $\mathcal{M}_2^{m_1} \leq \mathcal{M}$, $\mathcal{M}_2^{m_2} \leq \mathcal{M}$, $\mathcal{M} \leq \mathcal{M}_3^{m_3}$ and $\mathcal{M} \leq \mathcal{M}_3^{m_4}$. But, we also have $\mathcal{M}_2^{m_1} \neq \mathcal{M}$, $\mathcal{M}_2^{m_2} \neq \mathcal{M}$, $\mathcal{M} \neq \mathcal{M}_3^{m_3}$ and $\mathcal{M} \neq \mathcal{M}_3^{m_4}$. Due to Lemma 1 in [6] this is not possible since there does not exist an element of $\mathbf{M}$ that fulfills these conditions. •

Theorem 3 *There exist* $\Sigma, \mathcal{E} \in \mathbf{MS}$ *such that* $\mathbf{M_{HMS}}(\Sigma, \mathcal{E})$ *contains no smallest element for the relation* $\lesssim$, *i.e., the collection* $\mathbf{HMS}(\Sigma, \mathcal{E})$ *contains no 'smallest' h.m.s.-representation.*

Proof: Following Lemma 3, $\mathbf{M}$ is not a semi-lattice, and as a consequence, there exists $\mathbf{N} \subset \mathbf{M}$ which contains no smallest element. By Lemma 2 we know that $\forall \mathbf{N} \subseteq \mathbf{M}$ there exists $\Sigma, \mathcal{E} \in \mathbf{MS}$ such that $\Delta\mathbf{M}(\Sigma, \mathcal{E}) = \mathbf{N}$. Thus, there exists no smallest $\mathcal{M} \in \mathbf{M}$ such that $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}$, i.e., there exists no smallest $\mathcal{M} \in \mathbf{M}$ such that $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\mathcal{M})$ (see Theorem 1 in [6]). Thus, as a consequence of Theorem 2, there exists no smallest $\mathbf{HMS}(\mathcal{M}) \in \mathbf{M_{HMS}}$ such that $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\mathcal{M})$. If there exist a smallest $\Sigma', \mathcal{E}' \in \mathbf{HMS}(\Sigma, \mathcal{E})$, then there exists $\mathbf{HMS}(\mathcal{M}') \in \mathbf{M_{HMS}}(\Sigma, \mathcal{E})$ with $\Sigma', \mathcal{E}' \in \mathbf{HMS}(\mathcal{M}')$. By Theorem 2 we know that $\mathbf{HMS}(\mathcal{M}') \underset{\sim}{\subset} \mathbf{HMS}(\mathcal{M}'')$ for all $\mathbf{HMS}(\mathcal{M}'') \in \mathbf{M_{HMS}}(\Sigma, \mathcal{E})$. This contradicts with the first part of this proof. •

As a consequence of this theorem, we cannot make general statements concerning a smallest h.m.s.-representation for a general m.s. Thus, we have to look for specific m.s. that are such that we can make some explicit statements concerning the existence (or nonexistence) of a smallest h.m.s.-representation. In the next section we will identify the collection of all h.m.s.-representations of the m.s. that are encountered in quantum mechanics.

4.2 H.m.s-representations for quantum-like m.s.

In this subsection, we will study the possible h.m.s.-representations for some specific classes of m.s. that allow some explicit statements concerning the existence (or nonexistence) of a smallest h.m.s.-representation. We will show that these classes of m.s. about which we can make some statements are of major importance for the case of quantum mechanics, since they contain all quantum m.s. In fact, the statements that we are going to prove in this section suffices to identify all possible h.m.s.-representations for all quantum m.s. We have the following expressions that characterize certain quantum-like m.s.:

- If we consider an entity (e.g. a quantum entity) with only measurements with n outcomes, we clearly have $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \subseteq \mathbf{M}_n$.
- If we consider a quantum entity with measurements with a finite number of outcomes we have $\mathbf{M}_{\mathbb{X}} = \Delta\mathbf{M}(\Sigma, \mathcal{E})$, as a consequence of the definition of a quantum state.
- If we consider a quantum entity with measurements with an at most countable number of outcomes we have $\mathbf{M}_{\mathbb{N}} = \Delta\mathbf{M}(\Sigma, \mathcal{E})$.

As we will show at the end of this section, these expressions characterize the quantum m.s. in a sufficient way to identify all their h.m.s.-representations. We start with a theorem which states that if $\mathbf{M}_{\mathbb{N}} \subseteq \Delta\mathbf{M}(\Sigma, \mathcal{E})$, then this m.s. has no smaller h.m.s.-representation than the ones contained in $\mathbf{HMS}([0, 1])$, i.e., there exists a smallest h.m.s.-representation, but it is one that needs the 'maximal' number of strictly classical observables, namely a continuous set.

Lemma 4 $\mathbf{M}_{\mathbb{X}}$ *has a smallest upper bound in* $\mathbf{M}$, *namely* $\mathcal{M}_{\mathbb{R}}$.

Proof: We only have to prove that there exists no smaller upper bound for $\mathbf{M}_{\mathbb{X}}$ in $\mathbf{M}$ than $\mathcal{M}_{\mathbb{R}}$, the greatest element of $\mathbf{M}$ itself. Suppose that $\mathcal{M}$ is such a smaller upper bound. If $\mathcal{M} \neq \mathcal{M}_{\mathbb{R}}$ and $\mathcal{B}, \mu \in \mathcal{M}$, there exists a set $B \in \mathcal{B}$ such that for all $B' \in \mathcal{B}$, $B' \subseteq B$ implies $B' = B$ or $B' = \emptyset$ (see Lemma 2 in [6]). Let $\mu(B) = a$. Define N as the smallest integer such that $N > 1/a$. Consider $\mathcal{M}_N^m$ where $\forall i \in \mathbb{X}_N : m(i) = 1/N$. Since $\mathbf{M}_{\mathbb{X}} \leq \mathcal{M}$ we have $\mathcal{M}_N^m \leq \mathcal{M}$. Thus, there exists a σ-morphism $F : \mathcal{B}_{\mathbb{N}} \to \mathcal{B}$ fulfilling Definition 8 in [6]. Clearly, for all $i \in \mathbb{X}_N : B \cap F(\{i\}) = \emptyset$ or $B \cap F(\{i\}) = B$. Since for all $i \in \mathbb{X}_N$ we have $F(\{i\}) \in \mathcal{B}$ and $\cup_{i \in \mathbb{N}}(B \cap F(\{i\})) = B \cap (\cup_{i\in\mathbb{N}} F(\{i\})) = B \cap I = B$, there exists $i \in \mathbb{X}_N$ such that $B \cap F(\{i\}) = B$, i.e., $B \subseteq F(\{i\})$, and thus, $\mu(F(\{i\})) \geq a$. This contradicts with $\mu(F(\{i\})) = m(i) = 1/N < a$. •

Lemma 5 *$\mathbf{M}_{\mathbb{N}}$ has a smallest upper bound in $\mathbf{M}$, namely $\mathcal{M}_{\mathbb{R}}$.*

Proof: If we consider $\mathcal{M}_{\mathbb{N}}^m$, where for all i in $\mathbb{X}_{N-1} : m(i) = 1/N$, and for all integers $i \geq N : m(i) = N^{-1}.2^{N-i-1}$ (m defines a probability measure because $\sum_{i\in\mathbb{N}} m(i) = (N-1)(1/N) + \sum_{i\in\mathbb{N}}(2^{-i}/N) = (N - 1 + \sum_{i\in\mathbb{N}} 2^{-i})/N = 1$), we can prove this proposition along the same lines as the proof of Lemma 4. •

Theorem 4 *Let $\Sigma, \mathcal{E} \in \mathbf{MS}$. We have:*

$$\mathbf{M}_{\mathbb{N}} \subseteq \Delta\mathbf{M}(\Sigma, \mathcal{E}) \Longrightarrow \mathbf{HMS}(\Sigma, \mathcal{E}) \subsetneq \mathbf{HMS}([0,1]) \tag{4.2}$$

Proof: Following Theorem 3 in [6] we know that $\mathbf{HMS}([0,1]) = \mathbf{HMS}(\mathcal{M}_{\mathbb{R}}) \in \mathbf{M}_{\mathbf{HMS}}(\Sigma, \mathcal{E})$. If there exists $\mathcal{M} \in \mathbf{M}$ such that $\mathcal{M} < \mathcal{M}_{\mathbb{R}}$ and $\mathbf{HMS}(\mathcal{M}) \in \mathbf{M}_{\mathbf{HMS}}(\Sigma, \mathcal{E})$, then, as a consequence of Theorem 1 in [6], $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}$. Thus, $\mathbf{M}_{\mathbb{N}} \leq \mathcal{M}$. This contradicts with Lemma 5 which states that $\mathcal{M}_{\mathbb{R}}$ is the supremum of $\mathbf{M}_{\mathbb{N}}$, and thus, there exist no smaller upper bounds. As a consequence, $\mathbf{M}_{\mathbf{HMS}}(\Sigma, \mathcal{E}) = \{\mathbf{HMS}(\mathcal{M}_{\mathbb{R}})\}$, and $\mathbf{HMS}(\Sigma, \mathcal{E}) \subsetneq \mathbf{HMS}([0,1])$. •

We proceed with a theorem which states that the smallest h.m.s.-representations in the case that $\mathbf{M}_{\mathbb{X}} \subseteq \Delta\mathbf{M}(\Sigma, \mathcal{E})$ is also contained in $\mathbf{HMS}([0,1])$.

Theorem 5 *Let $\Sigma, \mathcal{E} \in \mathbf{MS}$. We have:*

$$\mathbf{M}_{\mathbb{X}} \subseteq \Delta\mathbf{M}(\Sigma, \mathcal{E}) \Longrightarrow \mathbf{HMS}(\Sigma, \mathcal{E}) \subsetneq \mathbf{HMS}([0,1]) \tag{4.3}$$

Proof: The proof of this theorem is the same as the proof of Theorem 4, if we replace 'Lemma 5' by 'Lemma 4'. •

In the last two theorems of this section we identify measurement systems that do have smaller h.m.s.-representations than the ones contained in $\mathbf{HMS}([0,1])$, but for which there exists no smallest ones. More precisely, if we consider $\mathbf{M}_n$ for some fixed $n \in \mathbb{N}$ in stead of $\mathbf{M}_{\mathbb{X}}$, then, measurement systems $\Sigma, \mathcal{E} \in \mathbf{MS}$ fulfilling $\mathbf{M}_n = \Delta\mathbf{M}(\Sigma, \mathcal{E})$ do have smaller h.m.s.-representations than the ones fulfilling $\mathbf{M}_{\mathbb{X}} = \Delta\mathbf{M}(\Sigma, \mathcal{E})$, but there doesn't exists a smallest one.

Lemma 6 *Let $a \in [0,1]$. There exist one and at most two sequences $(a_i)_i$, with $\forall i \in \mathbb{N}$: $a_i \in \{0,1\}$ and such that:*

$$a = \sum_{i \in \mathbb{N}} a_i/2^i \tag{4.4}$$

i.e., there are at most two non-equal subsets $B_1, B_2 \in \mathbb{N}$ such that:

$$\sum_{i \in B_1} 1/2^i = \sum_{i \in B_2} 1/2^i \tag{4.5}$$

Proof: This lemma expresses the well known representation of reals as decimal numbers, but now in the scale 2 in stead of the scale 10. For a detailed proof we refer to [8] p.107-112. Here, we'll only sketch how these two possible decompositions can be constructed. A first decomposition is defined by: $\forall i \in \mathbb{N}$ such that $a - \sum_{j \in \mathbb{X}_{i-1}} a_j/2^j > 1/2^i$: $a_i = 1$, otherwise: $a_i = 0$. A second decomposition is defined by: $\forall i \in \mathbb{N}$ such that $a - \sum_{j \in \mathbb{X}_{i-1}} a_j/2^j \geq 1/2^i$: $a_i = 1$, otherwise: $a_i = 0$. One can prove that for both decompositions, we find the same sum $a = \sum_{i \in \mathbb{N}} a_i/2^i$, and that for almost[1] all $a \in [0,1]$, both decompositions give the same sequence $(a_i)_i$. If we define $B_1 = \{j | a_j = 1\}$ for the first decomposition, and $B_2 = \{j | a_j = 1\}$ for the second one, we find eq.(4.5). •

Lemma 7 *If $\forall i \in \mathbb{X}_n$, $Q_i \in \mathbb{R}$ and $m_i = Q_i.(1 - \sum_{j=1}^{i-1} m_j)$ then:*

$$\forall i \in \mathbb{X}_n : m_i = Q_i. \prod_{j=1}^{i-1}(1 - Q_j) \tag{4.6}$$

$$1 - \sum_{j=1}^{n} m_i = \prod_{i=1}^{n}(1 - Q_i) \tag{4.7}$$

Proof: We prove this lemma by induction. For $n = 1$ we have $m_1 = Q_1$ and $1 - m_1 = 1 - Q_1$. If the lemma is true $\forall n' \in \mathbb{X}_{n-1}$, then eq.4.6 is true $\forall i \in \mathbb{X}_{n-1}$.

$$\begin{aligned} m_n &= Q_n(1 - \sum_{i=1}^{n-1} m_i) = Q_n(1 - \sum_{i=1}^{n-1} Q_i \prod_{j=1}^{i-1}(1 - Q_j)) \\ &= Q_n((1 - Q_1) - \sum_{i=2}^{n-1} Q_i \prod_{j=1}^{i-1}(1 - Q_j)) \end{aligned}$$

[1]We mean for all $a \in [0,1]$ except for a set of measure zero.

$$
\begin{aligned}
&= Q_n(1-Q_1)(1-\sum_{i=2}^{n-1} Q_i \prod_{j=2}^{i-1}(1-Q_j)) \\
&= Q_n \prod_{i=1}^{n-1}(1-Q_i)
\end{aligned}
$$

$$
\begin{aligned}
1-\sum_{j=1}^{n} m_j &= (1-\sum_{i=1}^{n} Q_i \prod_{j=1}^{i-1}(1-Q_j)) \\
&= (1-Q_1)(1-\sum_{i=2}^{n} Q_i \prod_{j=2}^{i-1}(1-Q_j)) \\
&= \prod_{i=1}^{n}(1-Q_i)
\end{aligned}
$$

what completes the proof. •

Lemma 8 *For all $n \in \mathbb{N}$, $\mathbf{M}_n$ has an upper bound in $\mathbf{M}_{\mathbb{N}}$.*

Proof: For all $m \in M_n$, let $Q_1 = m(1)$, and $\forall i \in \mathbb{X}_{n-1} \setminus \{1\}$: let $Q_i = 0$ if $1 - \sum_{j=1}^{i-1} m(j) = 0$ and let $Q_i = m(i)(1-\sum_{j=1}^{i-1} m(j))^{-1}$ if $1-\sum_{j=1}^{i-1} m(j)) \neq 0$. Since for all $i \in \mathbb{X}_{n-1}$, $Q_i \in [0,1]$, we can define a sequence $(q_{i,j})_j$, with $\forall i,j \in \mathbb{N} : q_{i,j} \in \{0,1\}$, such that $(q_{i,j})_j$ represents a binary decomposition[2] of Q_i fulfilling $\forall i \in \mathbb{X}_{n-1} : Q_i = \sum_{j\in\mathbb{N}}(q_{i,j}/2^j)$. Since $\mathbb{N}^{n-1}$ is countable, there exists a one-to-one map $\xi : \mathbb{N} \to \mathbb{N}^{n-1}$. We choose one fixed ξ, and $\forall i \in \mathbb{N}$ we denote $\xi(i) \in \mathbb{N}^{n-1}$ as $i_1, i_2, \ldots, i_{n-1}$. Define $m' \in M_{\mathbb{N}}$ such that $\forall i \in \mathbb{N} : m'(\{i\}) = \prod_{j=1}^{m-1}(1/2^{i_j})$. m' defines a probability measure (all summations are taken over $\mathbb{N}$):

$$
\begin{aligned}
m'(\mathbb{N}) &= \sum_i \prod_{j=1}^{m-1}(1/2^{i_j}) = \sum_{i_1}\sum_{i_2}\ldots\sum_{i_{n-1}}\prod_{j=1}^{n-1}(1/2^{i_j}) \\
&= \sum_{i_1}(1/2^{i_1})\sum_{i_2}(1/2^{i_2})\ldots\sum_{i_{n-1}}(1/2^{i_{n-1}}) = (\sum_j 1/2^j)^{n-1} = 1
\end{aligned}
$$

We define $F : \mathcal{B}_n \to \mathcal{B}_{\mathbb{N}}$ such that for all $j \in \mathbb{X}_{n-1}$:

$$
\begin{aligned}
F(\{n\}) &= \{i | i \in \mathbb{N}, \forall j \in \mathbb{X}_{n-1} : q_{j,i_j} = 0\} \\
F(\{j\}) &= \{i | i \in \mathbb{N}, \forall k \in \mathbb{X}_{j-1} : q_{k,i_k} = 0, q_{j,i_j} = 1\}
\end{aligned}
$$

and for all $B \in \mathcal{B}_n : F(B) = \cup_{i\in B} F(\{i\})$. For all $i,j \in \mathbb{X}_n$ such that $i \neq j$ (i.e., $\{i\}\cap\{j\} = \emptyset$) we have $F(\{i\}) \cap F(\{j\}) = \emptyset$. Since:

$$
\begin{aligned}
F(\mathbb{X}_n) &= \cup_{i\in\mathbb{N}} F(\{i\}) \\
&= \{i | j \in \mathbb{X}_{n-1}, q_{j,i_j} = 1, \forall k \in \mathbb{X}_{j-1} : q_{k,i_k} = 0\} \\
&\quad \cup \{i | \forall j \in \mathbb{X}_{n-1} : q_{j,i_j} = 0\} \\
&= \{i | j \in \mathbb{X}_{n-1} : q_{j,i_j} = 1 \text{ or } \forall j \in \mathbb{X}_{n-1} : q_{j,i_j} = 0\} = \mathbb{N}
\end{aligned}
$$

[2] See Lemma 6.

F is a σ-morphism. We still have to verify if $\mu_m = \mu_{m'}$ (see Definition 8 in [6]). For all $j \in \mathbb{X}_{n-1}$:

$$\begin{aligned}
m'(j) &= \sum_{i\in F(j)} \prod_{k=1}^{m-1} \frac{1}{2^{i_k}} = \sum_{i\in F(j)} \Big(\prod_{k=1}^{j-1}\big(1-\frac{q_{k,i_k}}{2^{i_k}}\big)\Big)\frac{q_{j,i_j}}{2^{i_j}} \prod_{k=j+1}^{m-1} \frac{1}{2^{i_k}} \\
&= \sum_{i\in\mathbb{N}}\Big(\prod_{k=1}^{j-1}\big(1-\frac{q_{k,i_k}}{2^{i_k}}\big)\Big)\frac{q_{j,i_j}}{2^{i_j}} \prod_{k=j+1}^{m-1} \frac{1}{2^{i_k}} \\
&= \sum_{i_1\in\mathbb{N}}\sum_{i_2\in\mathbb{N}}\cdots\sum_{i_{m-1}\in\mathbb{N}} \Big(\prod_{k=1}^{j-1}\big(1-\frac{q_{k,i_k}}{2^{i_k}}\big)\Big)\frac{q_{j,i_j}}{2^{i_j}} \prod_{k=j+1}^{m-1} \frac{1}{2^{i_k}} \\
&= \prod_{k=1}^{j-1}\Big(\sum_{i_k\in\mathbb{N}}\big(1-\frac{q_{k,i_k}}{2^{i_k}}\big)\Big) \sum_{i_j\in\mathbb{N}} \frac{q_{j,i_j}}{2^{i_j}} \prod_{k=j+1}^{m-1} \sum_{i_k\in\mathbb{N}} \frac{1}{2^{i_k}} \\
&= \prod_{k=1}^{j-1}\Big(\sum_{k\in\mathbb{N}}\frac{1}{2^k} - \sum_{k\in\mathbb{N}}\frac{q_{k,k}}{2^k}\Big) \sum_{k\in\mathbb{N}} \frac{q_{j,k}}{2^k} \prod_{k=j+1}^{m-1} \sum_{k\in\mathbb{N}} \frac{1}{2^k} \\
&= \prod_{k=1}^{j-1}\Big(1-\sum_{k\in\mathbb{N}}\frac{q_{k,k}}{2^k} \sum_{j\in\mathbb{N}}\frac{q_{j,j}}{2^j}\Big) \\
&= \prod_{k=1}^{j-1}(1-Q_k)Q_j = m(j) \\
m'(n) &= \sum_{i\in F(n)} \prod_{j=1}^{m-1}\big(1-\frac{q_{j,i_j}}{2^{i_j}}\big) = \prod_{j=1}^{m-1}(1-Q_j) = 1-\sum_{j=1}^{m-1} m(j) = m(n)
\end{aligned}$$

as a consequence of Lemma 7. •

Theorem 6 *Let $\Sigma, \mathcal{E} \in$ **MS** and $n \in \mathbb{N}$. We have:*

$$\Delta\mathbf{M}(\Sigma,\mathcal{E}) \subseteq \mathbf{M}_n \Longrightarrow \Sigma,\mathcal{E} \lesssim \mathbf{HMS}(\Sigma, O_{\mathcal{E}}, \mathbb{N}) \tag{4.8}$$

Proof: For all $n \in \mathbb{N}$ we know that $\mathbf{M}_n$ has an upper bound $\mathcal{M} \in \mathbf{M}_\mathbb{N}$ (see Lemma 8). Since $\mathcal{M} \in \mathbf{M}_\mathbb{N}$ we have $\mathbf{HMS}(\mathcal{M}) \lesssim \mathbf{HMS}(\mathbb{N})$. Thus, if $\mathbf{M}_n = \Delta\mathbf{M}(\Sigma,\mathcal{E})$, then $\Delta\mathbf{M}(\Sigma,\mathcal{E}) \leq \mathcal{M}$ and thus, according to Theorem 1 in [6], we have $\Sigma,\mathcal{E} \lesssim \mathbf{HMS}(\mathcal{M}) \lesssim \mathbf{HMS}(\mathbb{N})$. Thus, $\Sigma,\mathcal{E} \lesssim \mathbf{HMS}(\Sigma, O_{\mathcal{E}}, \mathbb{N})$. •

Lemma 9 *For all $n \in \mathbb{N}$, there exists no smallest upper bound for $\mathbf{M}_n$ in $\mathbf{M}$.*

Proof: We prove that there exists no smallest upper bound in $\mathbf{M}_\mathbb{N}$ for the case $n = 2$. For $n > 2$, the proof goes along the same lines, but gets very complicated on a notational level. From the proof of Lemma 8 we know that $\mathcal{M}^m_\mathbb{N}$, with $\forall i \in \mathbb{N} : m(\{i\}) = 1/2^i$, is an upper bound. First we will prove that there exists no upper bound for $\mathbf{M}_2$, smaller than $\mathcal{M}^m_\mathbb{N}$. Suppose that $\mathcal{M}^{m\prime}_\mathbb{N}$ is such a smaller upper bound (it's clear that such a smaller upper bound cannot be an element of $\mathbf{M}_{n'}$ for some $n' \in \mathbb{N}$). Let $F : \mathcal{B}_{\mathbb{N}} \to \mathcal{B}_{\mathbb{N}}$ be a σ-morphism fulfilling Definition 8 in [6], and which is not onto, i.e., there exists $i \in \mathbb{N}$ with $\{i\} \not\subset \{F(B)|B \in \mathcal{B}_{\mathbb{N}}\}$ (otherwise, $\{F(B)|B \in \mathcal{B}_{\mathbb{N}}\} = \mathcal{B}_{\mathbb{N}}$ because of the σ-additivity). Since $i \in \mathbb{N} = \cup_{j\in\mathbb{N}} F(\{j\})$, there exists $j \in \mathbb{N}$ such that $i \in F(\{j\})$. Denote $B = F(\{j\})$, and let $n \in B$ be such that there exists $n' \in B : n' < n$. Let $B_1 = \{n, n+1\}$ and $B_2 = \{n\} \cup \{i|i \in \mathbb{N}, i > n+1\}$. Since $1/2^n + 1/2^{n+1} = 1/2^n + \sum_{i=n+1}^{i=\infty} 1/2^{i+1}$, we have $\sum_{i\in B_1} 1/2^i = \sum_{i\in B_2} 1/2^i$. For all $i \in B_1$ and for all $i \in B_2$ we have $i \geq n > n'$, and thus we have $B \cap B_1 \subset B$ and $B \cap B_2 \subset B$. Thus, $B_1 \notin \{F(B)|B \in \mathcal{B}_{\mathbb{N}}\}$ and $B_2 \notin \{F(B)|B \in \mathcal{B}_{\mathbb{N}}\}$. Consider $\mathcal{M}_2^{m''}$ such that $m''(1) = 1/2^n + 1/2^{n+1}$. Since $\mathcal{M}_{\mathbb{N}}^{m'}$ is an upper bound for $\mathbf{M}_2$, there exists a σ-morphism $F' : \mathcal{B}_2 \to \mathcal{B}_{\mathbb{N}}$ fulfilling Definition 8 in [6]. Let $B_3 = F \circ F'(\{1\})$. We have $\mu_m(B_3) = \mu_{m''}(\{1\}) = 1/2^n + 1/2^{n+1}$, and thus, $\sum_{i\in B_3} 1/2^i = 1/2^n + 1/2^{n+1}$. Since $B_3 \in \{F(B)|B \in \mathcal{B}_{\mathbb{N}}\}$ we have $B_1 \neq B_3$ and $B_2 \neq B_3$, and thus, there are three subsets of $\mathbb{N}$ such that $\sum_{i\in B_1} 1/2^i = \sum_{i\in B_2} 1/2^i = \sum_{i\in B_3} 1/2^i$. This contradicts with Lemma 6. Now we'll construct an upper bound $\mathcal{M}_{\mathbb{N}}^{m'}$ such that $\mathcal{M}_{\mathbb{N}}^{m'} \not\geq \mathcal{M}_{\mathbb{N}}^{m}$. Consider $\mathcal{M}_{\mathbb{N}}^{m'}$ where $m'(1) = m'(2) = 1/3$, and for all integers $i \geq 3 : m'(i) = (3.2^{i-2})^{-1}$. If $\mathcal{M}_{\mathbb{N}}^{m'} \geq \mathcal{M}_{\mathbb{N}}^{m}$, there exists a σ-morphism $F : \mathcal{B}_{\mathbb{N}} \to \mathcal{B}_{\mathbb{N}}$ fulfilling Definition 8 in [6]. Since, $\cup_{i\in\mathbb{N}} F(\{i\}) = \mathbb{N}$, there exist $i, j \in \mathbb{N}$ such that $1 \in F(\{i\})$ and $2 \in F(\{j\})$. If $i = j$ then $\{1,2\} \subseteq F(\{i\})$ and thus, $\mu_m(\{i\}) \geq m'(1) + m'(2) = 2/3$, which is not possible. Thus we have $m(i) = \mu_{m'}(F(\{i\})) \geq \mu_{m'}(\{1\}) = m'(1) = 1/3$ and $m(j) = \mu_{m'}(F(\{j\})) \geq \mu_{m'}(\{2\}) = m'(2) = 1/3$, and this contradicts with $\{i|i \in \mathbb{N}, m(i) \geq 1/3\} = \{1\}$. We still have to prove that $\mathcal{M}_{\mathbb{N}}^{m'} \geq \mathbf{M}_2$. Consider $\mathcal{M}_2^{m''}$ such that $m''(1) = a$ and $m''(2) = 1 - a$. If $a < 1/3$, let $B_1 = \emptyset$ and $a' = 3a$. If $1/3 \leq a < 2/3$, let $B_1 = \{1\}$ and $a' = 3a - 1$. If $a \geq 2/3$, let $B_1 = \{1,2\}$ and $a' = 3a - 2$. Consider a binary decomposition $a' = \sum_{i\in\mathbb{N}} a'_i/2^i$ and define $B_2 = \{i|a'_i = 1\}$. One easily verifies that the map $F : \mathcal{B}_2 \to \mathcal{B}_{\mathbb{N}}$ defined by $F(\{1\}) = B_1 \cup B_2$ and $F(\{2\}) = \mathbb{N} \setminus (B_1 \cup B_2)$ fulfills Definition 8 in [6]. •

Theorem 7 $\forall\Sigma, \mathcal{E} \in \mathbf{MS}$, $n \in \mathbb{N}$ *with* $\mathbf{M}_n = \Delta\mathbf{M}(\Sigma, \mathcal{E})$: $\mathbf{HMS}(\Sigma, \mathcal{E})$ *contains no smallest h.m.s.-representation.*

Proof: If there exists $\mathbf{HMS}(\mathcal{M}) \in \mathbf{M_{HMS}}(\Sigma, \mathcal{E})$ such that for all $\mathbf{HMS}(\mathcal{M}') \in \mathbf{M_{HMS}}(\Sigma, \mathcal{E})$ we have $\mathbf{HMS}(\mathcal{M}) \leq \mathbf{HMS}(\mathcal{M}')$. Then, as a consequence of Theorem 1 in [6], $\mathbf{HMS}(\mathcal{M})$ is the smallest element in $\mathbf{M_{HMS}}$ such that $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}$, i.e., $\mathcal{M}$ is the smallest element in $\mathbf{M}$ such that $\mathbf{M}_n \leq \mathcal{M}$ (see Theorem 2). This contradicts with Lemma 9 which states that $\mathbf{M}_n$ has no smallest upper bound in $\mathcal{M}$. •

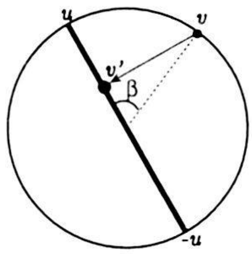
With these theorems we are now able to identify the possible h.m.s.-representations of quantum m.s. The different possible candidates for a h.m.s.-representation correspond with the different possible measure spaces in $\mathbf{M}$, and which are given by Lemma 1 in [6], and which are essentially measure spaces related to a continuous, a countable or a finite set of states of the measurement context. From Theorem 4 and Theorem 5 follows that if we consider quantum m.s. with measurements with an unbounded number of outcomes, we are not able to find smaller h.m.s.-representations than the one we have identified in Theorem 3 in [6], i.e., a h.m.s.-representation with a continuous set of states of the measurement context. If we consider a m.s. with measurements with n-outcomes (e.g. a spin-$\frac{n-1}{2}$ quantum entity) we find h.m.s.-representations with a countable set of states of the measurement context. Nonetheless, there exists no 'preferred' smallest h.m.s.-representation[3].

5 Two explicit examples

We end this paper with two examples, namely Aerts' model system for a spin-$\frac{1}{2}$ quantum entity (introduced in [1] and discussed in more detail in for example [5] and [7]), and a 'reduced' version of this model system that is implemented by the results of this paper. These two model systems will enable us to visualize the results of this paper. We start with Aerts' model system[4]. As a representation of the states of a spin-$\frac{1}{2}$ quantum entity we consider the Poincaré representation, i.e., all states are represented on a sphere in $\mathbb{R}^3$, orthogonal states correspond with antipodic point on the sphere. If a point has coordinates v, we denote the state corresponding to this point as p_v. A measurement e_u on the entity in a state p_v is defined in the following way:

- Consider a straight line segment with one of its endpoints in the point u of the sphere, and the other endpoint in the diametrically opposite point $-u$. We'll denote this segment as $[u, -u]$.
- We project v orthogonally on $[-u, u]$ and obtain the point v'. This point defines two segments $[-u, v']$ and $[v', u]$ (see Fig. 1).
- Consider a stochastic variable λ located on the segment $[-u, u]$, and suppose that relative frequency of appearance of the possible λ is uniformly distributed on $[-u, u]$. If $\lambda \in [-u, v']$, the point corresponding to the state of the entity moves to u along $[v', u]$ and we obtain a state p_u. If $\lambda \in]v', u]$, the point moves to $-u$ along $[v', -u]$ and we obtain p_{-u}.

As a consequence, there are two outcome states for this measurement e_u: p_u and p_{-u}. As been shown in earlier publications, this model system gives rise to the same probability structure as a spin-$\frac{1}{2}$ quantum entity.



[3]This obviously also implies that there exist no h.m.s.-representations with a finite set of states of the measurement context.

[4]Since Aerts' model system has already been published many times, we won't go in to much details. For these details we refer to some of these earlier publications by Aerts and his collaborators.

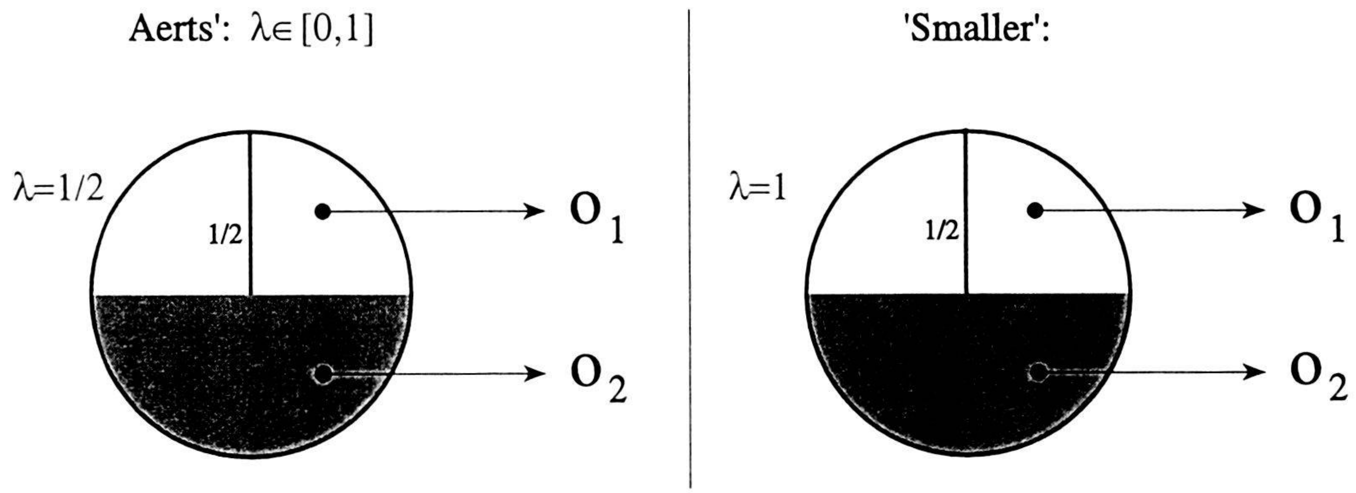
Fig.1: Illustration of a measurement e_u on an Aerts' spin-$\frac{1}{2}$ entity when the initial state is p_v.

Within the formalism of [6] and this paper, Σ corresponds in a one-to-one way with the sphere S, Λ corresponds with the line segment $[u, -u]$, μ_Λ is the uniformly distributed probability measure of the stochastic variable λ and $O_e = \{p_u, p_{-u}\}$. For a clear description of φ_λ we refer to Fig. 2. The different measurements in $\mathcal{E}$ correspond with the different possible choices of antipodic points, and their h.m.s.-representations are related in the way as described in section 3.3 in [6]. Since it is shown in Theorem 6 that for measurements with a finite and bounded number of outcomes there exists a h.m.s.-representation with $\Lambda = \mathbb{N}$, we should be able to find such a 'reduced' representation for Aerts' spin-$\frac{1}{2}$ quantum entity. The straightforward way to do this is by explicitly applying the model used in the proof of Lemma 8[5]. We find the model system as outlined in Fig. 2.

6 Conclusion

We were able to identify the possible h.m.s.-representations of quantum m.s., and we showed that it suffices to know the 'small' h.m.s.-representations in order to know the complete collection of all h.m.s.-representations. Quantum m.s. with measurements with an unbounded number of outcomes (i.e., a finite but unbounded, an infinite but countable, or a continuous number of outcomes) all have a 'preferred' smallest h.m.s.-representation with a continuous set of states of the measurement context. Quantum m.s. with a finite and bounded number of outcomes (e.g. spin-$\frac{n-1}{2}$ quantum m.s.) have h.m.s.-representation with a countable set of states of the measurement context (see for example the alternative for Aerts' spin-$\frac{1}{2}$ h.m.s.-representation in section 5), but they do not have a preferred smallest h.m.s.-representation.

[5]For the specific case of $n = 2$, the rather complicated model of Lemma 8 reduces to a rather simple model system.



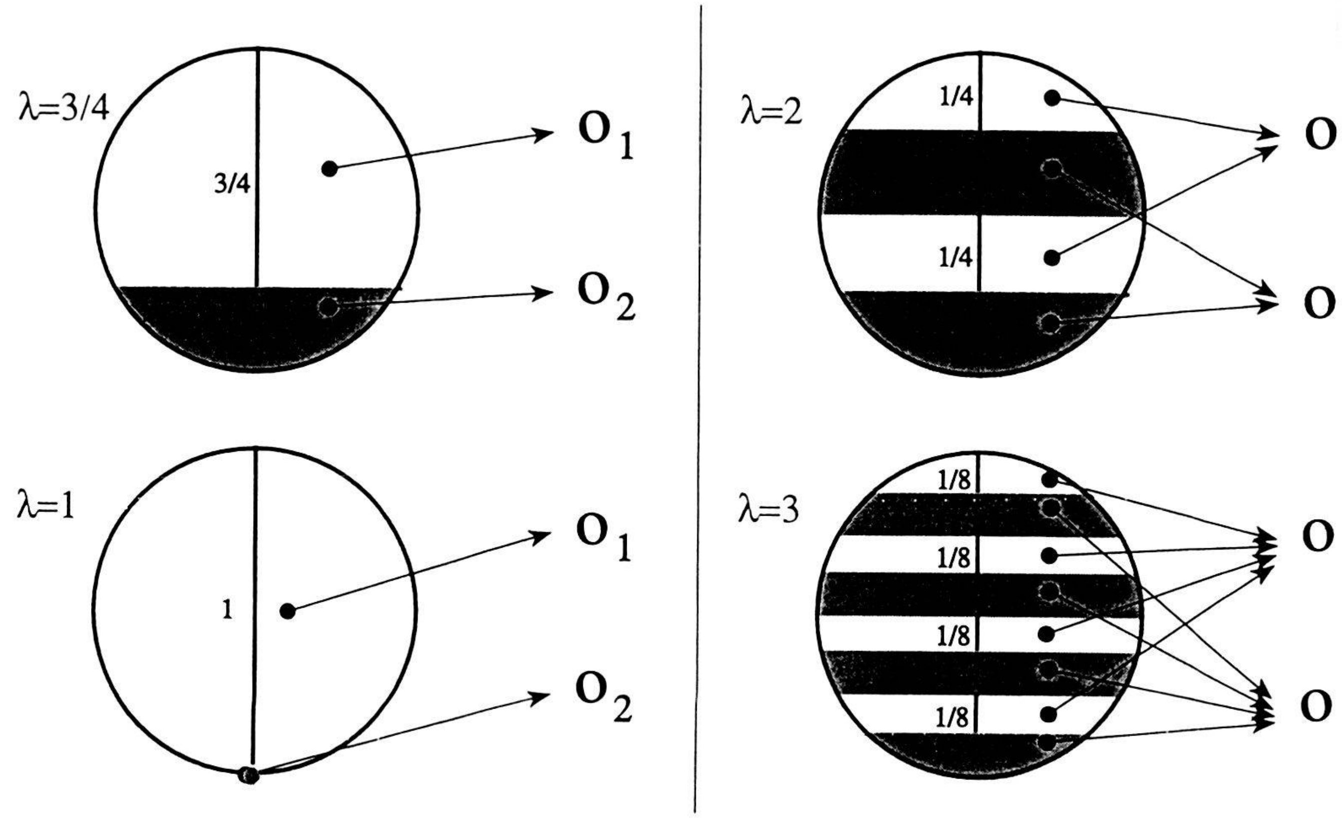


Fig. 2: A comparison of Aerts' h.m.s. with a 'smaller' h.m.s. for a spin-1/2 quantum entity. In the case of Aerts' h.m.s., for every $\lambda \in [0,1]$ the state space is divided in an upper and a lower half, depending on the value of λ. If the entity is in a state located in the upper half we obtain an outcome o_1 and if the entity is in a state located in the lower half we obtain an outcome o_2. The arrows in the drawing correspond with the map $\varphi_\lambda : \Sigma \to O_e$. In the case of the 'smaller' h.m.s., for every $\lambda \in \mathbb{N}$ the state space is divided in 2^λ spherical bands. If the entity is in a state located in the upper band we obtain an outcome o_1, if it is in a state located in the second band we obtain an outcome o_2, if it is in a state located in the third band we obtain an outcome o_1, ... This h.m.s. is completely defined by the relation $\forall\lambda \in \mathbb{N}$: $\mu_{\mathbb{N}}(\{\lambda\}) = 1/2^\lambda$. One easily verifies that this 'smaller' h.m.s. is mathematically equivalent with Aerts' h.m.s.

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