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Classical Representations for Quantum-like Systems through an Axiomatics for Context Dependence

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Abstract. We introduce a definition for a 'hidden measurement system', i.e., a physical entity for which there exist: (i) 'a set of non-contextual states of the entity under study' and (ii) 'a set of states of the measurement context', and which are such that all uncertainties are due to a lack of knowledge on the actual state of the measurement context. First we identify an explicit criterion that enables us to verify whether a given hidden measurement system is a representation of a given couple $\Sigma, \mathcal{E}$ consisting of a set of states Σ and a set of measurements $\mathcal{E}$ (= measurement system). Then we prove for every measurement system that there exists at least one representation as a hidden measurement system with $[0,1]$ as set of states of the measurement context. Thus, we can apply this definition of a hidden measurement system to impose an axiomatics for context dependence. We show that in this way we always find classical representations (hidden measurement representations) for general non-classical entities (e.g. quantum entities).

1 Introduction

In [1], Aerts introduced the 'hidden measurement approach' to quantum mechanics. He considered the quantum state as a complete representation of the entity under study, but he allowed a lack of knowledge on the interaction of the entity with its measurement context during the measurement. This idea can also be put forward as follows: *with every quantum measurement corresponds a collection of classical measurements (called hidden measurements), and there exists a lack of knowledge concerning which measurement is actually* *performed*[1]. Explicit 'hidden measurement models' have been introduced for some 'typical' quantum systems (see [1], [2], [4], [5], [6], [8], [10] and [12]).

In this paper, we apply these idea's within a much more general framework. In stead of only supposing the existence of a set of states for the physical entity (denoted by Σ), we also suppose the existence of a set of states of the measurement context (denoted by Λ) which corresponds with the collection of hidden measurements. For an as general as possible class of systems defined by a set Σ of states and a set $\mathcal{E}$ of measurements (called 'measurement systems' and abbreviated as m.s.) we will prove that there exists an equivalent representation as a 'hidden measurement system' (abbreviated as h.m.s.) such that the probabilities that occur are due to a lack of knowledge on the actual state of the measurement context. In this way we find for every m.s., and thus also for quantum mechanics, a classical representation as a h.m.s.

In section 3.3 we illustrate how an additional structure on the m.s. (for example, the geometric structure of quantum mechanics) can be induced on the h.m.s. in a natural way. Thus, the classical representations that we consider respect the symmetries of the given entity. We also identify the criterion that enables us to verify whether a given h.m.s. is a representation of a given m.s. (see section 4.2). Such a criterion is an essential tool for any further study that uses this 'hidden measurement axiomatics' for context dependence. In [10] and [11] we have build a complete classification of all possible h.m.s.-representations for a given quantum m.s., starting from this criterion.

For a general definition of the basic mathematical objects that are used in this paper (σ-fields, σ-morphisms, probability measures, measurable functions etc...) we refer to [7] and [24]. We mention that from a mathematical point of view, the representation that we introduce in this paper coincides sometimes with Gudder's proof on the existence for contextual hidden variable representations[2] of systems described by orthomodular lattices (see [17]). A first theorem on the existence of a hidden measurement representation for finite dimensional quantum mechanics was contained in [1]. A generalization of this theorem to more general finite dimensional entities can be found in [3]. The specific case of mixed states was considered in [9], and the general proof for the existence of a hidden measurement representation for infinite dimensional entities can be found in [13]. Finally, we remark that the results presented in this paper (except for section 3.3) where made known in [10].

2 Assumptions of the approach

In this section we consider a situation when there is a lack of knowledge concerning the interaction of the entity under study with its measurement context, i.e., when the state[3]

[1]For a general physical and philosophical background of the idea of hidden measurements we refer to [1], [3], [5] and [10].

[2]For the debate on this kind of representations we refer to [16], [19] and [26].

[3]We exclude the situation of a lack of knowledge concerning the state, i.e., if we write 'state', we mean 'pure state'. For a well-founded definition of state we refer to [21].

of the entity does not determine the outcome anymore. In such a case, when we perform a measurement e on an entity in a state p, we might even be lucky if we manage to find a formalizable statistical regime in the occurring outcomes. As a consequence, a general theoretical treatment of these measurements is a priori not possible. Nevertheless, after stating a few reasonable assumptions, it is possible to construct a framework to study these situations:

Assumption 1 *There exists*[4] *a set of possible descriptions of the measurement context on the precise time that we decide to perform the measurement, i.e., there exists a set of 'relevant' parameters for the measurement context. We call this set of relevant parameters the 'states of the measurement context'.*

Assumption 2 *The result of a measurement, which is the result of the interaction between the entity and the measurement context, is completely determined by the state of the entity and the state of the measurement context, i.e., there is a 'deterministic dependence' on the initial conditions.*

Assumption 3 *There exists a statistical description for the relative frequency of occurrence of the states of the measurement context during the measurement.*

We suppose that all these assumptions are fulfilled. In the next sections, we will denote the set of states of the measurement context as Λ. For a fixed state of the measurement context $\lambda \in \Lambda$, the measurement process is strictly classical[5] (because of the deterministic dependence), and thus, for every such strictly classical *hidden measurement* there exists a strictly classical observable:

$$\varphi_\lambda : \Sigma \rightarrow O_e \tag{2.1}$$

Where Σ is the set of states of the physical entity and O_e is the set of possible outcomes of measurement e. Thus, we have the following set of strictly classical observables that correspond with the different possible states of the measurement context:

$$\Phi_\Lambda = \{\varphi_\lambda | \lambda \in \Lambda\} \tag{2.2}$$

Since there exists a relative frequency of occurrence for states of the measurement context, there exists a probability measure:

$$\mu_\Lambda : \mathcal{B}_\Lambda \rightarrow [0,1] \tag{2.3}$$

[4]We remark that 'existence' is not equivalent with 'knowledge'. Thus, we don't have to know the set of possible descriptions of the measurement context.

[5]We use 'strictly classical' in stead of 'classical' since we exclude the situations of unstable equilibrium that occur in most classical theories.

Where $\mathcal{B}_\Lambda$ is a σ-field of subsets of Λ. Thus, we are able to compute a probability defined on subsets of the set of outcomes, for every given initial state, i.e., we obtain an 'outcome probability' for every measurement e on the entity in a state p:

$$P_{p,e} : \mathcal{B}_e \to [0,1] \tag{2.4}$$

Where $\mathcal{B}_e$ is a σ-field of subsets of O_e. In fact, we have summarized, and represented, the 'unknown but relevant information' of the measurement process (i.e., all possible interactions during the measurement, for all possible initial states), in a couple consisting in: a set of strictly classical observables Φ_Λ and, a probability measure μ_Λ defined on these observables. In the last section of [11] we illustrate how these mathematical objects are encountered in Aerts' model system for a spin-$\frac{1}{2}$ quantum entity.

3 An axiomatics for context dependence

In this section we translate the assumptions of the previous section in an axiomatic way.

3.1 Measurement systems (m.s.)

We characterize the physical entities that we consider by the following objects:

- a set of states Σ and a set of measurements $\mathcal{E}$.
- $\forall e \in \mathcal{E}$, a set of outcomes O_e represented as a measurable subset of the real line.
- $\forall p \in \Sigma, \forall e \in \mathcal{E}$: a probability measure $P_{p,e} : \mathcal{B}_e \to [0,1]$, where $\mathcal{B}_e$ are the measurable subsets of O_e.

We call $\Sigma, \mathcal{E}$ a m.s. and denote the collection of all m.s. as **MS**. Let $O_\mathcal{E} = \cup_{e\in\mathcal{E}} O_e$, $\mathcal{B}_\mathcal{E} = \{B \in \mathcal{B}_e | e \in \mathcal{E}\}$ and $\mathcal{P}_\mathcal{E} = \{B \subseteq O_e | e \in \mathcal{E}\}$. For a fixed set of outcomes O and a fixed set of states Σ, the set of all $\Sigma, \mathcal{E} \in \mathbf{MS}$ with $O_\mathcal{E} \subseteq O$ is denoted as $\mathbf{MS}(\Sigma, O)$. If $\mathcal{E}$ contains only one measurement e we call it a *one measurement system* (abbreviated as 1m.s.), and we denote it as Σ, e. The collection of all 1m.s. is denoted as $\mathbf{MS}_0$. To summarize all probability measures that characterize a m.s. within one mathematical object we introduce a map $P_{\Sigma,\mathcal{E}} : \Sigma \times \mathcal{E} \times \mathcal{B}_\mathcal{E} \to [0,1]$, which is such that $\forall p \in \Sigma, \forall e \in \mathcal{E}$: $P_{p,e}$ is the trace of $P_{\Sigma,\mathcal{E}}$ for a restricted domain $\{p\} \times \{e\} \times \mathcal{B}_e$, and for all $B \in \mathcal{B}_\mathcal{E}$:

$$P_{p,e}(B) = P_{p,e}(B \cap O_e) \tag{3.1}$$

For this collection of m.s. we express in the following definition the relation '... is representable as ...' in a mathematical way.

Definition 1 *Two m.s. $\Sigma, \mathcal{E}$ and $\Sigma', \mathcal{E}'$ are called mathematically equivalent (denoted by $\Sigma, \mathcal{E} \sim \Sigma', \mathcal{E}'$) if there exist two maps $\zeta : \Sigma \rightarrow \Sigma'$ and $\eta : \mathcal{E} \rightarrow \mathcal{E}'$, both one to one and onto, and if $\forall e \in \mathcal{E}$, there exists a σ-isomorphism $\nu : \mathcal{B}_e \rightarrow \mathcal{B}_{\eta(e)}$ such that:*

$$\forall p \in \Sigma, \forall B \in \mathcal{B}_e : P_{p,e}(B) = P_{\zeta(p),\eta(e)}(\nu(B)) \tag{3.2}$$

Clearly, theorems on the existence of certain representations of a m.s. can be expressed in terms of mathematical equivalence. We end this section the notion of 'belonging up to mathematical equivalence'. Let $\Sigma, \mathcal{E} \in \mathbf{MS}$ and $\mathbf{N}, \mathbf{N}' \subseteq \mathbf{MS}$. If there exists $\Sigma', \mathcal{E}' \in \mathbf{N}$ such that $\Sigma', \mathcal{E}' \sim \Sigma, \mathcal{E}$ we write:

$$\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{N} \tag{3.3}$$

3.2 Hidden measurement systems (h.m.s.)

In the following definition we introduce these m.s. that are related to parameterized sets of 'compatible' strictly classical observables, i.e., strictly classical observables with a common set of states and a common set of outcomes.

Definition 2 *$\Sigma, \mathcal{E} \in \mathbf{MS}$ is called 'strictly classical' if $\forall e \in \mathcal{E}$, e is a 'strictly classical measurement', i.e., $\forall p \in \Sigma, \forall B \in \mathcal{B}_e : P_{p,e}(B) \in \{0, 1\}$.*

If $\Sigma, \mathcal{E}$ is a strictly classical m.s. then, $\forall e \in \mathcal{E}$ there always exists a strictly classical observable $\varphi_e : \Sigma \rightarrow O_e$ such that $\forall p \in \Sigma$ and $\forall B \in \mathcal{B}_e$ we have $P_{p,e}(B) = \mathbf{1}_B[\varphi_e(p)]$ ($\mathbf{1}_B$ is the indicator[6] of B). We use this property in the following definition, where we introduce a parameterization of a set of strictly classical measurements with common sets of states and outcomes. In this definition we denote P_{p,e_λ} as $P_{p,\lambda}$ and the set of all subsets of the set Λ as $\mathcal{P}_\Lambda$.

Definition 3 *Let $\mathcal{E} = \{e_\lambda | \lambda \in \Lambda\}$ and let $O_{\mathcal{E}}$ be the outcomes of e_λ for all $\lambda \in \Lambda$. $\Sigma, \mathcal{E} \in \mathbf{MS}$ is called a 'Λ-m.s.' if there exists a set*

$$\Phi_\Lambda = \{\varphi_\lambda : \Sigma \rightarrow O_{\mathcal{E}} | \lambda \in \Lambda\} \tag{3.4}$$

which is such that $\forall p \in \Sigma, \forall \lambda \in \Lambda, \forall B \in \mathcal{B}_{\mathcal{E}} : P_{p,\lambda}(B) = \mathbf{1}_B[\varphi_\lambda(p)]$. We introduce a map $\Delta\Lambda : \Sigma \times \mathcal{P}_{\mathcal{E}} \rightarrow \mathcal{P}_\Lambda$ such that $\forall p \in \Sigma, \forall o \in O_{\mathcal{E}}, \forall B \in \mathcal{P}_{\mathcal{E}} : \Delta\Lambda_p^o = \{\lambda \in \Lambda | \varphi_\lambda(p) = o\}$ and $\Delta\Lambda_p^B = \cup_{o \in B} \Delta\Lambda_p^o$ ($\Delta\Lambda_p^o$ is the image of $(p, \{o\})$ and $\Delta\Lambda_p^B$ the image of (p, B)).

One easily verifies that we are able to restrict the domain of $\Delta\Lambda$ to $\Sigma \times \mathcal{B}_{\mathcal{E}}$. To avoid notational overkill, we apply the same notations for the map $\Delta\Lambda$ when defined on $\Sigma \times \mathcal{B}_{\mathcal{E}}$ as

[6] The indicator $\mathbf{1}_B : O_e \rightarrow \{0, 1\}$ is such that $\forall o \in B : \mathbf{1}_B(o) = 1$ and $\forall o \in O_e \backslash B : \mathbf{1}_B(o) = 0$.

when defined on $\Sigma \times \mathcal{P}_\mathcal{E}$ (which of the two domains we consider will follow from the context, or will be specified). For a fixed state $p \in \Sigma$, we can consider $\Delta\Lambda_p : \mathcal{B}_\mathcal{E} \to \mathcal{P}_\Lambda$, i.e., $\Delta\Lambda$ with the domain restricted to $\{p\} \times \mathcal{B}_\mathcal{E}$. For every fixed state $p \in \Sigma$ we can introduce $\varphi_p : \Lambda \to O_\mathcal{E}$ which is such that $\forall\lambda \in \Lambda : \varphi_p(\lambda) = \varphi_\lambda(p)$. Let $\Delta\Lambda(\Sigma \times \mathcal{B}_\mathcal{E}) = \{\Delta\Lambda_p^B | p \in \Sigma, \forall B \in \mathcal{B}_\mathcal{E}\}$.

Proposition 1 *Let $\mathcal{B}_\Lambda$ be a sub-σ-field of $\mathcal{P}_\Lambda$ and let $\Delta\Lambda(\Sigma \times \mathcal{B}_\mathcal{E}) \subseteq \mathcal{B}_\Lambda$. For all $p \in \Sigma$, $\Delta\Lambda : \Sigma \times \mathcal{B}_\mathcal{E} \to \mathcal{B}_\Lambda$ defines a σ-morphism, namely $\Delta\Lambda_p : \mathcal{B}_\mathcal{E} \to \mathcal{B}_\Lambda$, and $\forall p \in \Sigma$, $\varphi_p : \Lambda \to O_\mathcal{E}$ is a measurable function.*

The proof of this proposition is straightforward and therefore omitted.

In the following definition we introduce a probability measure on a collection of strictly classical observables in the following sense: we consider a new (in general non-classical) measurement by supposing that one of the strictly classical measurements corresponding with the strictly classical observables occurs with a given probability. The idea of defining new measurements by performing one measurement in a collection has been introduced by Piron (see [21] and [22]). The idea of creating non-classical measurements by considering classical measurements, equipped with a relative frequency of occurrence, has been introduced by Aerts in his model system for a spin-$\frac{1}{2}$ quantum entity (see [1] and [3]).

Definition 4 *A 'Λ-hidden measurement model' $\Sigma, \mathcal{E}, \mu_\Lambda$ consists in:*

i) a Λ-measurement system $\Sigma, \mathcal{E}$

ii) a probability measure $\mu_\Lambda : \mathcal{B}_\Lambda \to [0,1]$ that fulfills $\Delta\Lambda(\Sigma \times \mathcal{B}_\mathcal{E}) \subseteq \mathcal{B}_\Lambda$

Define e_μ as the measurement which is such that a strictly classical measurement $e_\lambda \in \mathcal{E}$ occurs with the probability determined by μ, i.e., $\forall B \in \mathcal{B}_\Lambda$, the probability that $\lambda \in B$ is $\mu_\Lambda(B)$. The 1m.s. Σ, e_μ related to $\Sigma, \mathcal{E}, \mu_\Lambda$ is called a 'μ_Λ-h.m.s.'. If μ_Λ is not specified, but Λ is, we call it a 'Λ-h.m.s.'. If μ_Λ nor Λ are specified, we call it a 'h.m.s.'

Thus, every Λ-hidden measurement model defines a new one measurement system if we suppose that μ_Λ expresses a lack of knowledge concerning which $e_\lambda \in \mathcal{E}$ actually takes place. Since in general, the measurements e_μ are not strictly classical, they are related to non-classical observables. In this definition one easily sees that Λ can indeed be interpreted as the set of states of the measurement context in the sense that for every given $\lambda \in \Lambda$, e_λ determines an interaction between the entity under study and the measurement context.

Proposition 2 *Let Σ, e_μ be the 1m.s. related to a Λ-hidden measurement model $\Sigma, \mathcal{E}, \mu_\Lambda$ and let P_{p,e_μ} be the trace of P_{Σ,e_μ} for a restricted domain $\{p\} \times \{e_\mu\} \times \mathcal{B}_\mathcal{E}$. $\forall p \in \Sigma, \forall B \in \mathcal{B}_\mathcal{E}$:*

$$P_{p,e_\mu}(B) = \mu_\Lambda(\Delta\Lambda_p^B) \tag{3.5}$$

Proof: Since $\Delta\Lambda(\Sigma \times \mathcal{B}_{\mathcal{E}}) \subseteq \mathcal{B}_\Lambda$, P_{Σ,e_μ} is well defined:

$$\begin{array}{ccc} \Sigma \times \mathcal{B}_{\mathcal{E}} & \stackrel{P_{\Sigma,e_\mu}}{\longrightarrow} & [0,1] \\ {}_{\Delta\Lambda}\searrow & & \nearrow_{\mu_\Lambda} \\ & \mathcal{B}_\Lambda & \end{array}$$

$\forall p \in \Sigma, \forall B \in \mathcal{B}_{\mathcal{E}}$: $P_{p,e_\mu}(B) = \mu_\Lambda(\{\lambda | \varphi_\lambda(p) \in B\}) = \mu_\Lambda(\Delta\Lambda_p^B)$. •

Define the set of all h.m.s. in $\mathbf{MS}_0$ as $\mathbf{HMS}_0$, the set of all Λ-h.m.s. in $\mathbf{MS}_0$ as $\mathbf{HMS}_0(\Lambda)$, and the set of all μ_Λ-h.m.s. in $\mathbf{MS}_0$ as $\mathbf{HMS}_0(\mu_\Lambda)$. In the following definition we extend Definition 4 to h.m.s. with multiple non-classical measurements, all of them defined in the same way as we defined e_μ in Definition 4, i.e., we suppose that $\forall e \in \mathcal{E}$, there exists a set of classical observables, paramertized by a set Λ of states of the measurement context.

Definition 5 *Let $\Sigma, \mathcal{E} \in$ **MS**. If $\forall e \in \mathcal{E} : \Sigma, e \in \mathbf{HMS}_0$ we call $\Sigma, \mathcal{E}$ a h.m.s.*[7]*. If $\forall e \in \mathcal{E} : \Sigma, e \in \mathbf{HMS}_0(\Lambda)$ we call $\Sigma, \mathcal{E}$ a Λ-h.m.s. If $\forall e \in \mathcal{E} : \Sigma, e \in \mathbf{HMS}_0(\mu_\Lambda)$ we call $\Sigma, \mathcal{E}$ a μ_Λ-h.m.s.*

The set of all h.m.s. is denoted as **HMS**. For a fixed set Λ, we denote the set of all Λ-h.m.s. as $\mathbf{HMS}(\Lambda)$. For a fixed probability measure μ_Λ, we denote the set of all μ_Λ-h.m.s. as $\mathbf{HMS}(\mu_\Lambda)$ (when the specification of Λ is not relevant, we will also use the simplified notation $\mathbf{HMS}(\mu)$). Clearly we have $\mathbf{HMS}(\mu_\Lambda) \subset \mathbf{HMS}(\Lambda) \subset \mathbf{HMS} \subset \mathbf{MS}$. For a fixed set of states Σ and a fixed set of outcomes O we denote the set of all h.m.s. in $\mathbf{MS}(\Sigma, O)$ as $\mathbf{HMS}(\Sigma, O)$. Again for fixed sets Σ and O we denote the set of all Λ-h.m.s. in $\mathbf{MS}(\Sigma, O)$ as $\mathbf{HMS}(\Sigma, O, \Lambda)$ and the set of all μ_Λ-h.m.s. in $\mathbf{MS}(\Sigma, O)$ as $\mathbf{HMS}(\Sigma, O, \mu_\Lambda)$. For every $\Sigma, \mathcal{E} \in \mathbf{HMS}$ we can define a map $\Delta\Lambda : \Sigma \times \mathcal{E} \times \mathcal{B}_{\mathcal{E}} \to \mathcal{P}_\Lambda$, such that $\forall e \in \mathcal{E}$, the restriction of this new map to $\Sigma \times \{e\} \times \mathcal{B}_e$ corresponds with the map introduced in Definition 3 and, such that $\forall B \in \mathcal{B}_{\mathcal{E}} : \Delta\Lambda_{p,e}^B = \Delta\Lambda_{p,e}^{B \cap O_e}$ (we denote the restriction of this new map to $\{p\} \times \{e\} \times \mathcal{B}_{\mathcal{E}}$ as $\Delta\Lambda_{p,e}$). The results of this section remain valid for this new map if $\forall e \in \mathcal{E}$, we replace $\Delta\Lambda$ by the map $\Delta\Lambda_e : \Sigma \times \mathcal{B}_{\mathcal{E}} \to \mathcal{P}_\Lambda$ (which is obtained by restriction of the domain of $\Delta\Lambda : \Sigma \times \mathcal{E} \times \mathcal{B}_{\mathcal{E}} \to \mathcal{P}_\Lambda$), if we replace $\Delta\Lambda_p$ by $\Delta\Lambda_{p,e} : \mathcal{B}_{\mathcal{E}} \to \mathcal{P}_\Lambda$ and if we replace φ_p by $\varphi_{p,e} : \Lambda \to O_{\mathcal{E}}$.

[7] We remark that the symbol $\mathcal{E}$ which appears in Definition 4 (i.e., a Λ-set of strictly classical measurements) is from a conceptual point of view completely different from the one which appears in Definition 5 (any set of measurements on an entity with Σ as set of states such that all $e \in \mathcal{E}$ are defined in the same way as we defined e_μ in Definition 4), i.e., for every $e \in \mathcal{E}$ of Definition 5 there exists a set of strictly classical measurements $\mathcal{E}_e$.

3.3 Compatibility of the definition of a h.m.s. with the geometric structure of quantum mechanics

If there exists an additional structure on the set of all possible outcomes of a measurement system[8], one could demand that this additional structure induces a structure on Λ. In this section we show how the additional structure in the description of a physical entity can be implemented in a straightforward way within this framework. We consider the case of a quantum entity submitted to measurements with a finite number of outcomes. We will show that it suffices to have a h.m.s.-representations for only one of the measurements to obtain a representation for all measurements. If $\mathcal{E}$ consists of all measurements with n outcomes, we can represent such a measurement by n eigenvectors $p_{e,1},\ldots,p_{e,n}$ and n corresponding eigenvalues $o_{e,1},\ldots,o_{e,n}$. Consider one given measurement e_0 (with $p_{0,1},\ldots,p_{0,n}$ as eigenvectors and $o_{0,1},\ldots,o_{0,n}$ as respective eigenvalues) for which we have a h.m.s.-representation, i.e., there exist:

$$\Phi_{\Lambda,0} = \{\varphi_{0,\lambda} : \Sigma \rightarrow \{p_{0,1},\ldots,p_{0,n}\} | \lambda \in \Lambda\} \tag{3.6}$$

and

$$\mu_{\Lambda,0} : \mathcal{B}_\Lambda \rightarrow [0,1] \tag{3.7}$$

that characterize this h.m.s.-representation. Then, we can define a representation for every $e \in \mathcal{E}$ in the following way:

$$\Phi_{\Lambda,e} = \{\varphi_{e,\lambda} : \Sigma \rightarrow \{p_{0,1},\ldots,p_{0,n}\} : p \mapsto U_e \circ \varphi_{0,\lambda} \circ U_e^{-1}(p) | \lambda \in \Lambda\} \tag{3.8}$$

and

$$\mu_{\Lambda,e} = \mu_{\Lambda,0} \tag{3.9}$$

where U_e is the unitary transformation defined by $\forall i : p_{e,i} = U_e(p_{0,i})$. In this way, the h.m.s.-representation clearly 'respects' the structure that characterizes this quantum entity. For an example of the application of eq.3.8 and eq.3.9 we refer to Aerts' model system which can be found in [1], [2], [4], [6], [8] and [10], and which is also discussed within the formalism of this approach in [11].

4 On the existence of h.m.s.-representations

Before we proceed we need to introduce some measure theoretical notations and lemma's. Nonetheless, to avoid a notational overkill in the main section of this paper, we have collected all lemma's and proofs in an appendix at the end of this paper.

[8]For example, a partial ordering of the subsets of all outcomes and/or the implementation of spatial symmetries.

4.1 Some mathematical preliminaries and notations

First we will introduce and study a collection of mathematical objects that'll play a crucial role in the characterization of the h.m.s. in **HMS**, and thus, also in the criterion for the existence of h.m.s.-representations which will be presented at the end of this section.

Definition 6 *Let $\mathcal{B}$ be a Borel algebra, and let $\mu : \mathcal{B} \to [0,1]$ be a probability measure. Define $\mathcal{B}/\mu$ as the set of equivalence classes for the relation $\sim$ on $\mathcal{B}$, which is defined by: $B \sim B' \Leftrightarrow \mu(B \triangle B') = 0$. We call $(\mathcal{B}, \mu)$ a measure space if $\mathcal{B} \cong \mathcal{B}/\mu$, i.e.:*

$$\{B | B \in \mathcal{B}, \mu(B) = 0\} = \{\emptyset\} \tag{4.1}$$

Two measure spaces $(\mathcal{B}, \mu)$ and $(\mathcal{B}', \mu')$ are isomorphic (denoted as $(\mathcal{B}, \mu) \cong (\mathcal{B}', \mu')$), if there exists a σ-isomorphism $H : \mathcal{B} \to \mathcal{B}'$ which is such that $\forall B \in \mathcal{B} : \mu(B) = \mu'(H(B))$.

One can verify that $\mathcal{B}/\mu$ is again a Borel algebra, and that μ induces a probability measure on $\mathcal{B}/\mu$. For a proof we refer to [7]. The Borel sets of $[0,1]$ will be denoted by $\mathcal{B}_{[0,1]}$ and the Lebesgue measure by $\mu_{[0,1]}$. The quotient $\mathcal{B}_{[0,1]}/\mu_{[0,1]}$ is denoted by $\mathcal{B}_{\mathbb{R}}$ and the probability measure introduced on $\mathcal{B}_{\mathbb{R}}$ by $\mu_{[0,1]}$ as $\mu_{\mathbb{R}}$. If we consider the measure space $(\mathcal{B}_{\mathbb{R}}, \mu_{\mathbb{R}})$, we omit the index $\mathbb{R}$ in $\mu_{\mathbb{R}}$ (in Lemma 1 we will see that that this cannot lead to any confusion). To characterize 'not to big' Borel algebras we have the following definition:

Definition 7 *We call a Borel algebra $\mathcal{B}$ separable if there exists a countable dense subset, i.e., if there exists a set $\mathcal{D} = \{B_i | i \in \mathbb{N}\}$ which is such that the smallest Borel subalgebra of $\mathcal{B}$ containing $\mathcal{D}$ is $\mathcal{B}$ itself. We call a measure space $(\mathcal{B}, \mu)$ separable if $\mathcal{B}$ is separable.*

Let $\mathbf{M}$ be the collection of all classes consisting of isomorphic separable measure spaces, i.e., every $\mathcal{M}$ in $\mathbf{M}$ is a class of isomorphic separable measure spaces. In the appendix at the end of this paper, we characterize $\mathbf{M}$ in an explicit way. On $\mathbf{M}$ we introduce the following relation[9].

Definition 8 *Define a binary relation $\leq$ on $\mathbf{M}$ by: $\mathcal{M} \leq \mathcal{M}'$ if $\forall (\mathcal{B}, \mu) \in \mathcal{M}$ and $\forall (\mathcal{B}', \mu') \in \mathcal{M}'$, there exists a σ-morphism $F : \mathcal{B} \to \mathcal{B}'$ such that $\forall B \in \mathcal{B} : \mu'(F(B)) = \mu(B)$.*

Clearly, it suffices to have one σ-morphism F such that $\forall B \in \mathcal{B} : \mu'(F(B)) = \mu(B)$.

Proposition 3 *The σ-morphism F in Definition 8 is one to one.*

The proof of this proposition is straightforward and omitted. Denote the set of all integers, smaller or equal then a given $n \in \mathbb{N}$ as $\mathbb{X}_n$. Let $\mathcal{B}_n$ be the Borel algebra of all subsets

[9] In [11] we prove that $\mathbf{M}, \leq$ is a poset, i.e., $\leq$ is a partial order relation.

of $\mathbb{X}_n$ and let $\mathcal{B}_{\mathbb{N}}$ be the Borel algebra of all subsets of $\mathbb{N}$. Denote the class of all sets isomorphic with $\mathbb{X}_n$ as $\mathbf{X}_n$, the class of all sets isomorphic with $\mathbb{N}$ as $\mathbf{X}_{\mathbb{N}}$, and the class of all sets isomorphic with $\mathbb{R}$ as $\mathbf{X}_{\mathbb{R}}$. Let $\mathbf{X} = \cup_{n\in\mathbb{N}}\mathbf{X}_n \cup \mathbf{X}_{\mathbb{N}} \cup \mathbf{X}_{\mathbb{R}}$. For a given set $X \in \mathbf{X}$, denote the set of all subsets of X as $\mathcal{P}_X$. There exists a one-to-one map $h_X : X \to [0,1]$, and thus, we can consider $\mathcal{B}_{X,\mathbb{R}} = \{\{x|h_X(x) \in B\} \mid B \in \mathcal{B}_{[0,1]}\} \subseteq \mathcal{P}_X$. Clearly, h_X is a measurable function, i.e., we can consider the σ-morphism $H_X : \mathcal{B}_{[0,1]} \to \mathcal{B}_{X,\mathbb{R}}$ induced by this measurable function. Let $\mathbf{MX}$ be the collection of all triples $(X, \mathcal{B}_X, \mu_X)$, where $X \in \mathbf{X}$, $\mathcal{B}_X = \mathcal{B}_{X,\mathbb{R}}$ and $\mu_X : \mathcal{B}_X \to [0,1]$ is a probability measure. In the following proposition we prove a connection between the relation $\leq$ on $\mathbf{M}$ and the existence of measurable functions for objects in $\mathbf{MX}$.

Proposition 4 *Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ in $\mathbf{MX}$, and suppose that the measure space related to $\mathcal{B}_X$ and μ_X belongs to $\mathcal{M}_X$, and the one related to $\mathcal{B}_Y$ and μ_Y belongs to $\mathcal{M}_Y$. If $\mathcal{M}_X \leq \mathcal{M}_Y$, there exists a measurable function $f : Y \to X$ such that the related σ-morphism $F : \mathcal{B}_X \to \mathcal{B}_Y$ fulfills $\forall B \in \mathcal{B}_X : \mu_X(B) = \mu_Y(F(B))$.*

For the proof of this proposition we refer to the appendix at the end of this paper.

4.2 A criterion on the existence of h.m.s.-representations

In this section we identify an explicit criterion that enables us to verify whether a given h.m.s. is a representation of a given m.s. This criterion will be the main key in the proof on the existence for a h.m.s.-representation for every m.s. Moreover, as it has been shown in [10] and [11], this criterion also enables us to build a complete classification of all possible h.m.s.-representations for a given quantum-like m.s. Nonetheless, in this paper we only want to show that our definition for context dependence can be imposed on every m.s.

If no confusion is possible, we write $\mu \in \mathbf{MX}$ (or $\mu_\Lambda \in \mathbf{MX}$) in stead of $(\Lambda, \mathcal{B}_\mu, \mu_\Lambda) \in \mathbf{MX}$. Consider $\Sigma, \mathcal{E} \in \mathbf{MS}$ with an event probability $P_{\Sigma,\mathcal{E}} : \Sigma \times \mathcal{E} \times \mathcal{B}_{\mathcal{E}} \to [0,1]$. $\forall p \in \Sigma, \forall e \in \mathcal{E}$ we denote $\mathcal{B}_e/P_{p,e}$ as $\mathcal{B}_{p,e}$, and the induced probability measure on $\mathcal{B}_{p,e}$ as $\mu_{p,e}$. $\forall \Sigma, \mathcal{E} \in \mathbf{MS}$, $(\mathcal{B}_{p,e}, \mu_{p,e})$ is a separable measure space for all $p \in \Sigma$ and for all $e \in \mathcal{E}$, and thus, $(O_e, \mathcal{B}_e, P_{p,e}) \in \mathbf{MX}$.

- Let $\mathcal{M}_{p,e}$ be the unique class in $\mathbf{M}$ such that $(\mathcal{B}_{p,e}, \mu_{p,e}) \in \mathcal{M}_{p,e}$.
- $\forall \Sigma, \mathcal{E} \in \mathbf{MS}$ we introduce: $\Delta\mathbf{M}(\Sigma, \mathcal{E}) = \{\mathcal{M}_{p,e} | p \in \Sigma, e \in \mathcal{E}\}$

For every $\Sigma, e \in \mathbf{HMS}_0$ there exists μ_Λ such that $\Sigma, e \in \mathbf{HMS}_0(\mu_\Lambda)$. Denote $\mathcal{B}_\Lambda/\mu_\Lambda$ as $\mathcal{B}_\mu$, and the induced probability measure on $\mathcal{B}_\mu$ as μ. Analogously, if $\Sigma, \mathcal{E} \in \mathbf{HMS}$, we can define $\mathcal{B}_\mu, \mu$ for all $e \in \mathcal{E}$. For $\Sigma, \mathcal{E} \in \mathbf{HMS}(\mu_\Lambda)$, there exists one unique measure space $(\mathcal{B}_\mu, \mu)$, which is called 'the measure space related to the μ_Λ-h.m.s. $\Sigma, \mathcal{E}$'. For $\Sigma, \mathcal{E} \in \mathbf{HMS}(\Lambda)$, we have to consider a measure space $(\mathcal{B}_\mu, \mu)$ for all $e \in \mathcal{E}$.

- Let $\mathcal{M}_\mu$ be the unique class in $\mathbf{M}$ such that $(\mathcal{B}_\mu, \mu) \in \mathcal{M}_\mu$.

For a h.m.s. in $\Sigma, \mathcal{E} \in \mathbf{HMS}(\Lambda)$ we have to consider one measure space $\mathcal{B}_\mu, \mu$ for all $e \in \mathcal{E}$. For every $\Lambda \in \mathbf{X}$ we introduce the following subset of $\mathbf{M}$:

- $\mathbf{M}_\Lambda = \{\mathcal{M}_\mu | \mu_\Lambda \in \mathbf{MX}\}$

We also introduce the following relation on subsets of $\mathbf{M}$.

Definition 9 $\forall \mathbf{N}, \mathbf{N}' \subseteq \mathbf{M}$*:*

$$\mathbf{N} \leq \mathbf{N}' \Longleftrightarrow \forall \mathcal{M} \in \mathbf{N}, \exists \mathcal{M}' \in \mathbf{N}' : \mathcal{M} \leq \mathcal{M}'$$

We'll denote $\mathbf{N} \leq \{\mathcal{M}\}$ as $\mathbf{N} \leq \mathcal{M}$ and $\{\mathcal{M}\} \leq \mathbf{N}$ as $\mathcal{M} \leq \mathbf{N}$. In the following definition we introduce a subcollection of $\mathbf{HMS}$ that contains these h.m.s. in which appear only separable measure spaces.

Definition 10 *Let* $\mathbf{HMS}_0^S$ *be the collection of all* $\Sigma, e \in \mathbf{HMS}_0$ *such that* $(\mathcal{B}_\mu, \mu)$ *is a separable measure space and let* $\mathbf{HMS}^S$ *be the collection of all* $\Sigma, \mathcal{E} \in \mathbf{HMS}$ *such that* $\forall e \in \mathcal{E}$*:* $\Sigma, e \in \mathbf{HMS}_0^S$.

In the following section, we will prove that it suffices to consider measure spaces contained in classes in $\mathbf{M}$, and this automatically allows us to limit ourselves to h.m.s. in $\mathbf{HMS}^S$.

Now we identify the necessary and sufficient condition for the existence of a μ_Λ-h.m.s.-representation in $\mathbf{HMS}(\Sigma, O_\mathcal{E}, \mu_\Lambda)$, for a given m.s. in $\mathbf{MS}$.

Theorem 1 *Let* $\Sigma, \mathcal{E} \in \mathbf{MS}$ *and* $\mu_\Lambda \in \mathbf{MX}$*:*

$$\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\Sigma, O_\mathcal{E}, \mu_\Lambda) \Leftrightarrow \Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}_\mu \tag{4.2}$$

Proof: $\Longrightarrow$ Let $e \in \mathcal{E}$. According to Definition 5, there exists $\Sigma, \mathcal{E}', \mu_\Lambda$ such that $\Sigma, e \sim \Sigma, e_\mu$. Thus, there exists a σ-morphism $\nu : \mathcal{B}_e \to \mathcal{B}_{\mathcal{E}'}$ which is such that $\forall B \in \mathcal{B}_e : P_{p,e_\mu}(\nu(B)) = P_{p,e}(B)$ ($\zeta : \Sigma \to \Sigma$ is the identity, $\eta : \{e\} \to \{e_\mu\}$ is trivial). Moreover, there exists $\Delta\Lambda_{p,e} : \mathcal{B}_\mathcal{E} \to \mathcal{B}_\Lambda$ (see Proposition 1) which is such that $\forall B \in \mathcal{B}_\mathcal{E} : \mu_\Lambda(\Delta\Lambda_{p,e}(B)) = P_{p,e_\mu}(B \cap O_e)$ (see Proposition 2). Since $\mathcal{B}_{\mathcal{E}'} \subseteq \mathcal{B}_\mathcal{E}$, we can consider the map $[\Delta\Lambda_{p,e} \circ \nu] : \mathcal{B}_e \to \mathcal{B}_\Lambda$. Clearly, $[\Delta\Lambda_{p,e} \circ \nu]$ is also a σ-morphism and fulfills $\forall B \in \mathcal{B}_e : \mu_\Lambda([\Delta\Lambda_{p,e} \circ \nu](B)) = P_{p,e}(B)$. Define $F_p : \mathcal{B}_e \to \mathcal{B}_{p,e}$ and $F_\mu : \mathcal{B}_\Lambda \to \mathcal{B}_\mu$ by the following scheme:

$$\begin{array}{ccccccc}
\mathcal{B}_e & \overset{\nu}{\longrightarrow} & \mathcal{B}_{\mathcal{E}'} & \overset{\Delta\Lambda_{p,e}}{\longrightarrow} & \mathcal{B}_\Lambda \\
{\scriptstyle F_p}\downarrow & {\scriptstyle P_{p,e}}\searrow & {\scriptstyle P_{p,e_\mu}}\downarrow & \swarrow{\scriptstyle \mu_\Lambda} & \downarrow{\scriptstyle F_\mu} \\
\mathcal{B}_{p,e} & \underset{\mu_{p,e}}{\longrightarrow} & [0,1] & \underset{\mu}{\longleftarrow} & \mathcal{B}_\mu
\end{array}$$

Thus, $\forall B \in \mathcal{B}_e : \mu([F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B)) = P_{p,e}(B)$. For all $B \in \mathcal{B}_{p,e}$, there exists at least one $B_1 \in \mathcal{B}_e$ such that $F_p(B_1) = B$. Let $B_1' = [F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B_1) \in \mathcal{B}_\mu$. If $B_2 \neq B_1$ and $F_p(B_2) = B$, then $P_{p,e}(B_1 \triangle B_2) = 0$, and thus

$$\begin{aligned}
\mu([F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B_1)\triangle[F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B_2)) &= \\
\mu([F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B_1 \triangle B_2)) &= \\
P_{p,e}(B_1 \triangle B_2) &= 0
\end{aligned}$$

By definition of F_μ there exists only one $B_1' = [F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B_2) = [F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B_1)$. Thus, we can define $F_\nu : \mathcal{B}_{p,e} \to \mathcal{B}_\mu$ such that $\forall B \in \mathcal{B}_{p,e} : F_\nu(B) = [F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B') \Leftrightarrow B = F_p(B')$.

$$\begin{tikzcd}
\mathcal{B}_e \arrow[r, "\nu"] \arrow[dr, "F_p"'] & \mathcal{B}_{\mathcal{E}'} \arrow[r, "\Delta\Lambda_{p,e}"] & \mathcal{B}_\Lambda \arrow[d, "F_\mu"] \\
& \mathcal{B}_{p,e} \arrow[r, "F_\nu"] & \mathcal{B}_\mu
\end{tikzcd}$$

Let $B' \in \mathcal{B}_e$ be such that $F_\nu(B) = [F_\nu \circ F_p](B')$. We have, $\mu(F_\nu(B)) = \mu([F_\nu \circ F_p](B')) = \mu([F_\mu \circ \Delta\Lambda_{p,e} \circ \nu](B')) = \mu_\Lambda([\Delta\Lambda_{p,e} \circ \nu](B')) = P_{p,e_\mu}(\nu(B')) = P_{p,e}(B') = \mu_{p,e}(B)$, and thus, Definition 8 is fulfilled. As a consequence, $\mathcal{M}_{p,e} \leq \mathcal{M}_\mu$, and thus, $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}_\mu$.

$\Longleftarrow$ Let $p \in \Sigma$ and $e \in \mathcal{E}$. Since $\mathcal{M}_{p,e} \leq \mathcal{M}_\mu$, and, since both $(O_e, \mathcal{B}_e, P_{p,e})$ and $(\Lambda, \mathcal{B}_\Lambda, \mu_\Lambda)$ are in $\mathbf{MX}$, we can apply Proposition 4. Thus, there exists a measurable function $f_p : \Lambda \to O_e$ such that the related σ-morphism $F_p : \mathcal{B}_e \to \mathcal{B}_\Lambda$ fulfills $\forall B \in \mathcal{B}_e : P_{p,e}(B) = \mu_\Lambda(F_p(B))$. Define $\Delta\Lambda_e : \Sigma \times \mathcal{B}_\mathcal{E} \to \mathcal{B}_\Lambda$ such that $\forall B \in \mathcal{B}_\mathcal{E} : \Delta\Lambda^B_{p,e} = F_p(O_e \cap B)$. Define $\varphi_\lambda : \Sigma \to X$ such that $\forall p \in \Sigma$: $\varphi_\lambda(p) = f_p(\lambda)$. We have $\forall p \in \Sigma$: $\Delta\Lambda^B_p = \{\lambda | \lambda \in \Lambda, f_p(\lambda)\} = \{\lambda | \lambda \in \Lambda, \varphi_\lambda(p)\}$. Thus, there exists a set of strictly classical observables $\mathcal{E}_e$. Thus, $\Delta\Lambda_e$ defines a Λ-m.s. Still following Proposition 4, $\forall B \in \mathcal{B}_e : P_{p,e}(B) = \mu_\Lambda(F(B))$, and thus, $\forall B \in \mathcal{B}_\mathcal{E} : P_{p,e}(B) = \mu_\Lambda(F(O_e \cap B)) = \mu_\Lambda(\Delta\Lambda^B_{p,e})$ (see eq.3.1). If we identify e with e_μ, the measurement related to $\Sigma, \mathcal{E}_e, \mu_\Lambda$, we obtain $\Sigma, e \underset{\sim}{\in} \mathbf{HMS}_0(\Sigma, O_\mathcal{E}, \mu_\Lambda)$, and thus, $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\Sigma, O_\mathcal{E}, \mu_\Lambda)$. •

An alternative version of this theorem expresses the sufficient and necessary condition for the existence of at least one representation in $\mathbf{HMS}^S(\Sigma, O_\mathcal{E}, \Lambda)$:

Theorem 2 *Let* $\Sigma, \mathcal{E} \in \mathbf{MS}$ *and* $\Lambda \in \mathbf{X}$*:*

$$\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}^S(\Sigma, O_\mathcal{E}, \Lambda) \Leftrightarrow \Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathbf{M}_\Lambda \tag{4.3}$$

Proof: $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}^S(\Sigma, O_\mathcal{E}, \Lambda) \Leftrightarrow \forall e \in \mathcal{E} : \Sigma, e \underset{\sim}{\in} \mathbf{HMS}^S_0(\Sigma, O_\mathcal{E}, \Lambda) \Leftrightarrow \forall e \in \mathcal{E}, \exists \mu_\Lambda : \Sigma, e \underset{\sim}{\in} \mathbf{HMS}_0(\Sigma, O_\mathcal{E}, \mu_\Lambda) \Leftrightarrow \forall e \in \mathcal{E}, \exists \mu_\Lambda : \Delta\mathbf{M}(\Sigma, e) \leq \mathcal{M}_\mu \Leftrightarrow \forall e \in \mathcal{E}, \exists \mathcal{M}_\mu \in \mathbf{M}_\Lambda : \Delta\mathbf{M}(\Sigma, e) \leq \mathcal{M}_\mu \Leftrightarrow \forall e \in \mathcal{E} : \Delta\mathbf{M}(\Sigma, e) \leq \mathbf{M}_\Lambda \Leftrightarrow \Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathbf{M}_\Lambda$. •

4.3 A proof for the existence of h.m.s.-representations for all m.s.

In the following theorem we prove that the axiomatics for the dependence on the measurement context imposed by the definition of a h.m.s. implies no restriction for a general m.s., i.e., every m.s. can be represented as a h.m.s., with $[0,1]$ as set of states of the measurement context.

Theorem 3 $\forall \Sigma, \mathcal{E} \in \mathbf{MS}$: $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}^S(\Sigma, O_{\mathcal{E}}, [0,1])$.

Proof: According to Lemma 6 we know that $\mathbf{M} \leq \mathcal{M}_{\mathbb{R}}$. For all $\Sigma, \mathcal{E} \in \mathbf{MS}$ we have $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathbf{M}$. Thus, $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq \mathcal{M}_{\mathbb{R}}$, and thus, $\Sigma, \mathcal{E} \underset{\sim}{\in} \mathbf{HMS}(\Sigma, O_{\mathcal{E}}, \mu_{[0,1]}) \subseteq \mathbf{HMS}^S(\Sigma, O$
•

5 Conclusion.

Every m.s. in **MS** has a representation as a h.m.s. in **HMS**, and thus, also quantum mechanics can be represented in this way. As a consequence, the h.m.s.-formalism that is presented in this paper can be seen as *an axiomatics for general physical entities for context dependence that leads to a classical representation of non-classical systems.* We also identified the general condition for the existence of a h.m.s.-representation with Λ as set of 'states of the measurement context', or with μ_Λ as relative frequency of occurence of these states of the measurement context. If no further restrictions or assumptions are made on Λ, we only obtain restrictions on the ordinality of Λ, and on the specific probability measure μ_Λ that we consider. A lot of problems are still to be solved, for example, how precisely should this h.m.s.-formalism be fitted in the more general operational formalisms for quantum mechanics like Piron's approach (see [21] and [22]) or the Foulis-Randall approach (see [14] and [15])[10]. Still, we think that the approach presented in this paper certainly leads to a successful extension of the contemporary quantum framework as well from a philosophical as from a mathematical point of view.

6 Appendix: some measure theoretical lemma's

Let $\mathcal{B}$ and $\mathcal{B}'$ be two Borel Algebras. Denote their direct union[11] by $\mathcal{B} \bigcirc\!\!\!\!\!\cup\, \mathcal{B}'$, i.e., $\mathcal{B} \bigcirc\!\!\!\!\!\cup\, \mathcal{B}' = \{(B, B') | B \in \mathcal{B}, B' \in \mathcal{B}'\}$ equipped with three relations:

$$
\begin{aligned}
(B_1, B_1') \cup (B_2, B_2') &= (B_1 \cup B_2, B_1' \cup B_2') \\
(B_1, B_1') \cap (B_2, B_2') &= (B_1 \cap B_2, B_1' \cap B_2') \\
{}^c(B_1, B_1') &= ({}^cB_1, {}^cB_1')
\end{aligned}
$$

[10]More recently, Aerts introduced an operational approach, namely the closure structure approach (see [3], [6] and [25]), which is intrinsic compatible with the general idea of hidden measurements.

[11]A more general construction, and also more details, can be found in [24].

In the following definition we introduce an extension of this notion of direct union of Borel algebras to the collection of measure spaces, i.e., we introduce a way to 'compose' measure spaces.

Definition 11 *Let* $(\mathcal{B},\mu)$ *and* $(\mathcal{B}',\mu')$ *be measure spaces,* $a \in]0,1[$ *and* $\mu \underset{a}{\bigoplus} \mu' : \mathcal{B} \bigcirc \mathcal{B}' \to [0,1]$ *such that* $\forall (B,B') \in \mathcal{B} \bigcirc \mathcal{B}' : \mu \underset{a}{\bigoplus} \mu'(B,B') = (1-a)\mu(B) + a\mu'(B')$. *Define the weighted direct union* $(\mathcal{B},\mu) \underset{a}{\bigcirc} (\mathcal{B}',\mu')$ *of* $(\mathcal{B},\mu)$ *and* $(\mathcal{B}',\mu')$ *as the measure space*[12] $(\mathcal{B} \bigcirc \mathcal{B}', \mu \underset{a}{\bigoplus} \mu')$.

As in section 4.1, we denote the set of all integers, smaller or equal then a given $n \in \mathbb{N}$ as $\mathbb{X}_n$. Let $\mathcal{B}_n$ be the Borel algebra of all subsets of $\mathbb{X}_n$ and let $\mathcal{B}_\mathbb{N}$ be the Borel algebra of all subsets of $\mathbb{N}$. We introduce the following sets of monotonous decreasing strictly positive functions:

$$
\begin{aligned}
M_n &= \{m : \mathbb{X}_n \to [0,1] \mid \sum_{i=1}^{i=n} m(i) = 1, i \le j \Rightarrow m(j) \le m(i)\} \\
M_\mathbb{N} &= \{m : \mathbb{N} \to [0,1] \mid \sum_{i \in \mathbb{N}} m(i) = 1, i \le j \Rightarrow m(j) \le m(i)\}
\end{aligned}
$$

For all $m \in M_n \cup M_\mathbb{N}$ we define a probability measure $\mu_m : \mathcal{B}_N \to [0,1]$ by $\forall i : \mu_m(\{i\}) = m(i)$. We also introduce the following notations for some classes of measure spaces:

$$\mathcal{M}_\mathbb{R} = \{(\mathcal{B},\mu) \mid (\mathcal{B},\mu) \cong (\mathcal{B}_\mathbb{R},\mu)\}$$

$\forall N \in \mathbb{N} \cup \{\mathbb{N}\}, \forall m \in M_N$:

$$\mathcal{M}_N^m = \{(\mathcal{B},\mu) \mid (\mathcal{B},\mu) \cong (\mathcal{B}_N,\mu_m)\}$$

$\forall N \in \mathbb{N} \cup \{\mathbb{N}\}, \forall m \in M_N, \forall a \in]0,1[$:

$$\mathcal{M}_{N,a}^m = \{(\mathcal{B},\mu) \mid (\mathcal{B},\mu) \cong (\mathcal{B}_\mathbb{R},\mu) \underset{a}{\bigcirc} (\mathcal{B}_N,\mu_m)\}$$

and also the following notations for sets of such classes:

$$
\begin{aligned}
&\mathbf{M}_N = \{\mathcal{M}_N^m \mid m \in M_N\} \\
&\mathbf{M}_{\mathbb{R},a} = \{\mathcal{M}_{N,a}^m \mid N \in \mathbb{N} \cup \{\mathbb{N}\}, m \in M_N\} \\
&\mathbf{M} = \cup_{N \in \mathbb{N} \cup \{\mathbb{N}\}} \mathbf{M}_N \cup_{a \in]0,1[} \mathbf{M}_{\mathbb{R},a} \cup \{\mathcal{M}_\mathbb{R}\}
\end{aligned}
$$

The use of this symbol $\mathbf{M}$ (which we used in section 4.1 as a notation for the collection of all classes consisting of isomorphic separable measure spaces) is justified by the following lemma.

[12] One easily verifies that this weighted direct union is indeed a measure space.

Lemma 1 *The collection of all separable measure spaces is:*

$$\mathcal{M}_{\mathbb{R}} \cup_{N\in\mathbb{N}\cup\{\mathbb{N}\}} \cup_{m\in M_N} \mathcal{M}_N^m \cup_{a\in]0,1[} \mathcal{M}_{N,a}^m \tag{6.1}$$

Moreover, for every separable measure space $(\mathcal{B},\mu)$, $\exists! \mathcal{M} \in \mathbf{M}$ *such that* $(\mathcal{B},\mu) \in \mathcal{M}$.

The proof is a rather long construction that uses Lemma 2, Lemma 3, Lemma 4 (see further) and the Loomis-Sikorki theorem (see [20] and [23]). Since the content if the theorem agrees with our intuition, and the proof of it doesn't contribute in an essential way to the understanding of the subject of this paper, this proof is omitted. An explicit proof with the notations of this paper can be found in [10].

Lemma 2 *If* $\mathcal{B}$ *is a separable Borel algebra with* $\{B \in \mathcal{B} | B' \subset B \Rightarrow B' = \emptyset\} = \{\emptyset\}$, *then* $\mathcal{B} \cong \mathcal{B}_{\mathbb{R}}$. *Moreover, for every probability measure* $\mu : \mathcal{B} \to [0,1]$, *there exists a* σ*-isomorphism* $F_\mu : \mathcal{B} \to \mathcal{B}_{\mathbb{R}}$ *such that* $\forall B \in \mathcal{B} : \mu(B) = \mu_{\mathbb{R}}(F_\mu(B))$.

Proof: This lemma is proved by Marczewski. For an outline of it we refer to [7] or [18]. •

Lemma 3 *Let* $(\mathcal{B},\mu)$ *be a measure space,* $B_0 \in \mathcal{B}$, $a = \mu(B_0)$, $\mathcal{B}_l = \{B \in \mathcal{B} | B \cap B_0 = \emptyset\}$ *and* $\mathcal{B}_r = \{B \in \mathcal{B} | B \cap B_0 = B\}$. *Define two maps,* $\mu_l : \mathcal{B}_l \to [0,1]$ *and* $\mu_r : \mathcal{B}_r \to [0,1]$ *such that* $\forall B \in \mathcal{B}_l : \mu_l(B) = \frac{\mu(B)}{1-a}$ *and* $\forall B \in \mathcal{B}_r : \mu_r(B) = \frac{\mu(B)}{a}$. *Then, both* $(\mathcal{B}_l,\mu_l)$ *and* $(\mathcal{B}_r,\mu_r)$ *are measure spaces. Moreover we have* $(\mathcal{B},\mu) \cong (\mathcal{B}_l,\mu_l) \underset{a}{\textcircled{\cup}} (\mathcal{B}_r,\mu_r)$.

Proof: One easily sees that $\mathcal{B}_r$ (resp. $\mathcal{B}_l$) are Borel algebras, with B_0 (resp. B_0^c) as greatest element. By definition, μ_l and μ_r are σ-additive. Since $\mu(B_0) = a$ and $\mu(B_0^c) = 1 - a$, both μ_l and μ_r are normalized. Thus, μ_l and μ_r are probability measures, and thus, $(\mathcal{B}_l,\mu_l)$ and $(\mathcal{B}_r,\mu_r)$ are measure spaces. We have to show that there exists a σ-isomorphism $H : \mathcal{B} \to \mathcal{B}_l \textcircled{\cup} \mathcal{B}_r$ such that $\forall B \in \mathcal{B}, \forall (B_l, B_r) \in \mathcal{B}_l \textcircled{\cup} \mathcal{B}_r : (B_l, B_r) = H(B) \Rightarrow \mu(B) = \mu_l \underset{a}{\oplus} \mu_r(B_l, B_r)$. Since $\forall B \in \mathcal{B}$ we have: $\mu_l \underset{a}{\oplus} \mu_r(B \cap B_0^c, B \cap B_0) = (1-a)\mu_l(B \cap B_0^c) + a\mu_r(B \cap B_0) = \mu((B \cap B_0^c) \cup (B \cap B_0)) = \mu(B)$, we can define H by $\forall B \in \mathcal{B} : H(B) = (B \cap B_0^c, B \cap B_0)$. •

Lemma 4 *A measure space cannot have an uncountable subset of disjoint elements with a nonzero probability.*

Proof: Suppose that there exists such a set $\mathcal{D}$. Let $\mathcal{D}_i = \{B | B \in \mathcal{D}, \mu(B) > \frac{1}{i}\}$. Clearly, $\mathcal{D} = \cup_{i\in\mathbb{N}} \mathcal{D}_i$. Since $\mathcal{D}$ is uncountable, there exists $n \in \mathbb{N}$ such that $\mathcal{D}_n$ contains an infinite set of elements. Let $\mathcal{D}'_n = \{B_i | i \in \mathbb{N}\}$ be a countable subset of $\mathcal{D}_n$. We have $\mu(\cup_{B\in\mathcal{D}_n}) \geq \mu(\cup_{B\in\mathcal{D}'_n}) = \sum_{i\in\mathbb{N}} \mu(B_i) \geq \sum_{i\in\mathbb{N}} \frac{1}{n} = \infty$. •

Lemma 5 *Let* $\mu_1 : \mathcal{B}_{[0,1]} \to [0,1]$ *and* $\mu_2 : \mathcal{B}_{[0,1]} \to [0,1]$ *be two probability measures such that* $\mathcal{B}_{[0,1]}/\mu_1 \cong \mathcal{B}_{[0,1]}/\mu_2 \cong \mathcal{B}_{\mathbb{R}}$. *There exists a measurable function* $f : [0,1] \to [0,1]$, *which is such that the related* σ*-morphism* $F : \mathcal{B}_{[0,1]} \to \mathcal{B}_{[0,1]}$ *fulfills* $\forall B \in \mathcal{B}_{[0,1]} : \mu_1(B) = \mu_2(F(B))$.

Proof: Let $b \in [0,1]$. We prove that there exists $x \in [0,1]$ such that $\mu_1([0,x]) = b$. Suppose that x doesn't exist. Let b_- be the supremum of all $b' \in [0, b[$ such that there exists $x' \in [0,1]$ fulfilling $\mu_1([0,x']) = b'$. Then, there exists an increasing sequence $(b_i)_i$ with for all $i \in \mathbb{N}$: $b_i \in [b_- - 1/i, b_-]$ and $\exists x_i \in [0,1]$ such that $\mu_1([0,x_i]) = b_i$. Clearly, b_- is the supremum of $\{b_i | i \in \mathbb{N}\}$ and $(x_i)_i$ is also an increasing sequence. Denote the supremum of $\{x_i | i \in \mathbb{N}\}$ as x_-. There are two possibilities $x_- \in \{x_i | i \in \mathbb{N}\}$ and $x_- \notin \{x_i | i \in \mathbb{N}\}$. If $x_- \in \{x_i | i \in \mathbb{N}\}$ then $\cup_{i \in \mathbb{N}}[0, x_i] = [0, x_-]$, and thus $\mu_1(\cup_{i\in\mathbb{N}}[0,x_i]) = \mu_1([0,x_-])$. If $x_- \notin \{x_i | i \in \mathbb{N}\}$ then $\cup_{i\in\mathbb{N}}[0,x_i] = [0, x_-[$, and again we find $\mu_1([0,x_-]) = \mu_1([0,x_-[) = \mu_1(\cup_{i\in\mathbb{N}}[0,x_i])$, since $\mu_1(\{x_-\}) = 0$. We also have for all $i \in \mathbb{N}$: $\mu_1(]x_i, x_{i+1}]) = \mu_1([0,x_{i+1}]) - \mu_1([0,x_i])$. Thus:

$$\begin{aligned}
\mu_1([0,x_-]) &= \mu_1(\cup_{i\in\mathbb{N}}[0,x_i]) = \mu_1([0,x_1] \cup (\cup_{i\in\mathbb{N}}]x_i, x_{i+1}])) \\
&= \mu_1([0,x_1]) + \sum_{i\in\mathbb{N}} \mu_1(]x_i, x_{i+1}]) \\
&= \mu_1([0,x_1]) + \sum_{i\in\mathbb{N}} \mu_1([0,x_{i+1}]) - \sum_{i\in\mathbb{N}} \mu_1([0,x_i]) \\
&= b_1 + \sum_{i\in\mathbb{N}} (b_{i+1} - b_i) = b_-
\end{aligned}$$

Define b_+ as the infimum of all $b' \in]b, 1]$ such that $\exists x' \in [0,1] : \mu_1([0,x']) = b'$ (there exists at least one such b' since $\mu_1([0,1]) = 1$). Then, there exists an decreasing sequence $(b_i)_i$ with for all $i \in \mathbb{N}$: $b_i \in [b_+, b_+ + 1/i]$ and $\exists x_i \in [0,1]$ such that $\mu_1([0,x_i]) = b_i$. Denote the infimum of $\{x_i | i \in \mathbb{N}\}$ as x_+. Clearly, $\cap_{i\in\mathbb{N}}[0,x_i] = [0,x_+]$ and $(x_i)_i$ is also an decreasing sequence. Thus:

$$\begin{aligned}
\mu_1([0,x_+]) &= \mu_1(\cap_{i\in\mathbb{N}}[0,x_i]) = \mu_1((\cup_{i\in\mathbb{N}}[0,x_i]^c)^c) \\
&= 1 - \mu_1(\cup_{i\in\mathbb{N}}]x_i, 1]) = 1 - \mu_1(]x_1, 1] \cup (\cup_{i\in\mathbb{N}}]x_{i+1}, x_i])) \\
&= 1 - (\mu_1(]x_1, 1]) + \sum_{i\in\mathbb{N}} \mu_1(]x_{i+1}, x_i])) \\
&= 1 - (1 - \mu_1([0,x_1]) + \sum_{i\in\mathbb{N}} (\mu_1([0,x_i]) - \mu_1([0,x_{i+1}]))) \\
&= 1 - (1 - b_1 + \sum_{i\in\mathbb{N}} (b_i - b_{i+1})) = b_+
\end{aligned}$$

For all $x' \in]x_-, x_+[$ we have $\mu_1([0,x']) \geq \mu_1([0,x_-]) = b_-$, $\mu_1([0,x']) \leq \mu_1([0,x_+]) = b_+$, but, as a consequence of the definition of b_- and b_+, there exist no $x' \in]x_-, x_+[$ such that $\mu_1([0,x']) \in [b_-, b_+]$. Thus we obtain a contradiction. As a consequence, x exists. For all $x \in [0,1]$, define f such that $\mu_1([0, f(x)]) = \mu_2([0,x])$. We can define a σ-morphism $F : \mathcal{B}_{[0,1]} \to \mathcal{B}_{[0,1]}$ related to this measurable function. Thus, $F([0,x]) = \{y | f(y) \in [0,x]\} = \{y | f(y) \leq x\} = \{y | \mu_2([0,y]) \leq \mu_1([0,x])\}$ for all $x \in [0,1]$. For all $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$:

$$\begin{aligned}
F(]x_1, x_2]) &= F([0,x_2] \setminus [0,x_1]) = F([0,x_2]) \setminus F([0,x_1]) \\
&= \{y | \mu_2([0,y]) \leq \mu_1([0,x_2])\} \setminus \{y | \mu_2([0,y]) \leq \mu_1([0,x_1])\} \\
&=]y(x_1), y(x_2)]
\end{aligned}$$

where $y(x_1)$ is the smallest real in $[0,1]$ such that $\mu_2([0,y(x_1)]) = \mu_1([0,x])$ and $y(x_2)$ is the largest real in $[0,1]$ such that $\mu_2([0,y(x_2)]) = \mu_1([0,x])$. All this leads us to $\mu_2(F(]x_1,x_2])) = \mu_2(]y(x_1),y(x_2)]) = \mu_2([0,y(x_2)]) - \mu_2([0,y(x_1)]) = \mu_1([0,x_2]) - \mu_1([0,x_1]) = \mu_1(]x_1,x_2])$. By definition, $\mathcal{B}_{[0,1]}$ is the smallest Borel subalgebra of $\mathcal{P}_{[0,1]}$ containing $\{]a,b]|0 \leq a < b \leq 1; a,b \in [0,1]\}$. This completes the proof as a consequence of the σ-additivity of μ_1 and μ_2. •

Lemma 6 $\mathbf{M}, \leq$ *has a greatest ellement, namely* $\mathcal{M}_{\mathbb{R}}$, *i.e.,* $\mathbf{M} \leq \mathcal{M}_{\mathbb{R}}$

Proof: First we prove that $\forall \mathcal{M}^m_{\mathbb{N},a} \in \mathbf{M}_{\mathbb{R},a} : \mathcal{M}^m_{\mathbb{N},a} \leq \mathcal{M}_{\mathbb{R}}$. Consider the Borel algebra[13] $\bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}$, and a probability measure $\mu' : \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}} \to [0,1]$ which is defined by the relations $\forall B \in \mathcal{B}_{\mathbb{R}}$ (μ is defined as in $(\mathcal{B}_{\mathbb{R}},\mu)$): $\mu'(B,\emptyset,\ldots) = (1-a).\mu(B); \mu'(\emptyset,B,\emptyset,\ldots) = a.m(1).\mu(B); \mu'(\emptyset,\emptyset,B,\emptyset,\ldots) = a.m(2).\mu(B);\ldots$. One verifies that $\{B \in \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}|\mu(B) = 0\} = \{\emptyset\}$ and that $\bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}$ is separable, i.e., $\bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}, \mu'$ is a separable measure space. Clearly, there exists no $B \in \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}$ with $\mu'(B) \neq 0$, and such that $B' \in \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}$ and $B' \subset B$ implies $B' = \emptyset$, and thus, $(\bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}},\mu') \cong (\mathcal{B}_{\mathbb{R}},\mu)$ (see Lemma 2), i.e., there exists a σ-isomorphism $H : \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}} \to \mathcal{B}_{\mathbb{R}}$ such that $\forall B \in \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}} : \mu'(B) = \mu(H(B))$. For all $B \in \mathcal{B}_{\mathbb{N}}$, define a map $X_B : \mathbb{N} \to \{\emptyset, I\}$ which is such that $\forall i \in B : X_B(i) = I$ and $\forall i \notin B : X_B(i) = \emptyset$. We define a map $F : \mathcal{B}_{\mathbb{R}}\bigcirc\mathcal{B}_{\mathbb{N}} \to \bigcirc_{i\in\mathbb{N}}\mathcal{B}_{\mathbb{R}}$ by the relations $\forall B \in \mathcal{B}_{\mathbb{R}} : F(B,\emptyset) = (B,\emptyset,\emptyset,\ldots)$ and $\forall B \in \mathcal{B}_{\mathbb{N}} : F(\emptyset,B) = (\emptyset, X_B(1), X_B(2), X_B(3),\ldots)$. One verifies that the σ-morphism $H \circ F : \mathcal{B}_{\mathbb{R}}\bigcirc\mathcal{B}_{\mathbb{N}} \to \mathcal{B}_{\mathbb{R}}$ fulfills the requirements of Definition 8 and thus we have $\mathcal{M}^m_{\mathbb{N},a} \leq \mathcal{M}_{\mathbb{R}}$. Along the same lines one proves that $\forall \mathcal{M}^m_{n,a} \in \mathbf{M}_{\mathbb{R},a} : \mathcal{M}^m_{n,a} \leq \mathcal{M}_{\mathbb{R}}$ and that $\mathbf{M}_{\mathbb{N}} \cup_{n\in\mathbb{N}} \mathbf{M}_n \leq \mathcal{M}_{\mathbb{R}}$. As a consequence $\mathbf{M} \leq \mathcal{M}_{\mathbb{R}}$. •

We end this appendix with the proof of proposition 4.

Proof: Consider two σ-epimorphisms $F_X : \mathcal{B}_X \to \mathcal{B}_X/\mu_X$ and $F_Y : \mathcal{B}_Y \to \mathcal{B}_Y/\mu_Y$, which induce a probability measure $\mu : \mathcal{B}_X/\mu_X \to [0,1]$, respectively $\mu' : \mathcal{B}_Y/\mu_Y \to [0,1]$. Clearly, $(\mathcal{B}_X/\mu_X,\mu)$ and $(\mathcal{B}_Y/\mu_Y,\mu')$ are measure spaces. There also exists $F' : \mathcal{B}_X/\mu_X \to \mathcal{B}_Y/\mu_Y$ which fulfills Definition 8. Let $\mathcal{D}_X = \{B \in \mathcal{B}_X|\mu_X(B) \neq 0, B \supset B' \in \mathcal{B}_X \Rightarrow B' = \emptyset\}$. Since $\mathcal{D}_X$ is at most countable (see Lemma 4), there exists a smallest set $\mathbb{X} \in \cup\{\mathbb{X}_i|i \in \mathbb{N}\}$ of indices such that $\mathcal{D}_X = \{B_i|i \in \mathbb{X}\}$. $\forall i \in \mathbb{N}$: let $B'_i \in \mathcal{B}_Y$ be such that $F_Y(B'_i) = [F' \circ F_X](B_i)$, and $B''_i = B'_i \backslash (\cup_{j=1}^{j=i-1} B'_j)$. Clearly, $\cup_{i\in\mathbb{X}_N} B''_i = \cup_{i\in\mathbb{X}_N} B'_i$ and $\forall i,j \in \mathbb{X} : i \neq j \Rightarrow B''_i \cap B''_j = \emptyset$. Since $\forall i,j \in \mathbb{X} : i \neq j \Rightarrow B_i \cap B_j = \emptyset$, we have $\forall i \in \mathbb{X} : B_i \cap (\cup_{j=1}^{j=i-1} B_j) = \emptyset$, and thus, $F_Y(B'_i) \cap (\cup_{j=1}^{j=i-1} F_Y(B'_j)) = \emptyset$. As a consequence, $\forall i \in \mathbb{X} : \mu_Y(B'_i \cap (\cup_{j=1}^{j=i-1} B'_j)) = \mu'(F_Y(B'_i) \cap (\cup_{j=1}^{j=i-1} F_Y(B'_j))) = 0$, and thus, $\forall i \in \mathbb{X} : F_Y(B''_i) = F_Y(B'_i \backslash (\cup_{j=1}^{j=i-1} B'_j)) = F_Y(B'_i) = [F' \circ F_X](B_i)$, what leads to $\mu_Y(B''_i) = \mu'(F_Y(B''_i)) = \mu'[F' \circ F_X](B''_i)) = \mu_X(B_i)$. Define $X_1 = \cup_{i\in\mathbb{X}} B_i$, $X_2 = X \backslash X_1$, $Y_1 = \cup_{i\in\mathbb{X}} B''_i$ and $Y_2 = Y \backslash Y_1$. Suppose that $\mu_X(X_2) = \mu_Y(Y_2) \neq 0$. Consider $\mathcal{B}'_X = \{X_2 \cap B|B \in \mathcal{B}_X\}$ and $\mathcal{B}'_Y = \{Y_2 \cap B|B \in \mathcal{B}_Y\}$. Following Lemma 2 and Lemma 3, we know that $\mathcal{B}'_X/\mu'_X \cong \mathcal{B}'_Y/\mu'_Y \cong \mathcal{B}_{\mathbb{R}}$ (μ'_X and μ'_Y are the restrictions of μ_X to $\mathcal{B}'_X$, respectively μ_Y to $\mathcal{B}'_Y$, multiplied by $1/\mu_Y(Y_2)$, and thus, they correspond with μ_r in

[13] One can easily prove that it poses no problem to extend the notion of direct union to countable sets of Borel algebras. For more details we refer to [24].

Lemma 3). This observation, together with the definition of **MX**, leads to $\mathcal{B}'_X \cong \mathcal{B}'_Y \cong \mathcal{B}_{[0,1]}$. Let $f : Y \to X$ be such that $\forall i \in \mathbb{X}, \forall y \in B''_i : f(y) \in B_i$. There are two possibilities: $\mu_Y(Y_2) = 0$ or $\mu_Y(Y_2) \neq 0$. If $\mu_Y(Y_2) = 0$, $\forall y \in Y_2$: we can choose $f(y)$ in X_2. If $\mu_Y(Y_2) \neq 0$, we define $f(y)$ for all $y \in Y_2$ by applying Lemma 5 (i.e., we identify $\mathcal{B}'_X$ and $\mathcal{B}'_Y$ with $\mathcal{B}_{[0,1]}$, μ'_X with μ_1 and μ'_Y with μ_2). We can define the related σ-morphism $F : \mathcal{B}_X \to \mathcal{B}_Y$. We find that $\forall i \in \mathbb{X} : F(B_i) = B''_i$, what leads to $\mu_Y(F(B_i)) = \mu_Y(B''_i) = \mu_X(B_i)$. $\forall B \in \mathcal{B}'_Y$: $\mu_Y(F(B)) = \mu_X(B)$, as a consequence of Lemma 5. •

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