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Algebraic solutions of relativistic Coulomb problems

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Abstract The symmetries of relativistic Coulomb problems are surveyed from a conceptual viewpoint exploiting analogies between the classical and the quantum mechanical case. The symmetry of the nonrelativistic problem described by the $so(4)$ Lie algebra is used as guiding idea. The properties of the Dirac Coulomb problem are discussed in detail. Relations between various algebraic approaches to these problems are pointed out. The origin of relativistic equivalent or Dirac oscillators is traced back to the fine-structure Hamiltonian of the Dirac Coulomb problem.

1 Introduction

The Coulomb problem can be treated by algebraic means on the classical and the quantum mechanical level. It is commonly believed that on both levels the high symmetry of the non-relativistic case is lost when switching to a relativistic description. It is shown here that the symmetry reduction is in fact very limited; classically as well as quantum mechanically, one can define for the relativistic problem additional integrals of motion allowing an algebraic solution. In this approach one has to exploit the existence of transformations reducing effectively the relativistic problem to nonrelativistic form.
The analogous symmetry structures in the nonrelativistic and the relativistic regime are demonstrated via some apparently not widely known strategies. Thus the techniques sketched below can be used for a comparison between conventional analytic and algebraic approaches to basic physical problems on various levels of sophistication. Many problems arising within the topics discussed here have already been answered in the literature; we thus omit, where appropriate, technical details and refer to the references for further information. (Since the discussion is based on the Lie-algebra $so(4)$ as guiding idea allowing derivations of bound orbits, eigenvalues and bound state wavefunctions, results obtained in the frame of larger

groups like $SO(4,1)$ or $SO(4,2)$ are not included here. Straightforward extensions to the scattering problem involving $SO(3,1)$ are pointed out whenever possible.)

The paper is organized as follows: After reviewing the Coulomb problems of classical mechanics the Dirac-Coulomb problem is discussed. The presence of spin ignored in the classical case provides now additional degrees of freedom leading to a special symmetry structure elucidated below. Then the analogies between the nonrelativistic and the relativistic Coulomb problem are pointed out and relations between algebraic approaches to these problems based on symmetry groups, spectrum generating algebras and supersymmetry are shown. In the next section it is demonstrated how the concept of relativistic equivalent or Dirac oscillators introduced recently originates in the Dirac Coulomb problem. The paper is concluded by a short summary including some open questions.

2 The classical Kepler problems

The equation of motion for the classical nonrelativistic Kepler problem reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -Ze^2\frac{1}{r^2}\hat{r}, \tag{2.1}$$

where $\vec{p} = m\vec{v}$ defines the linear momentum and $\hat{r} = \vec{r}/r$. The vectorial integrals of motion, i.e., the angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.2}$$

and the Runge-Lenz vector

$$\vec{A} = (Ze^2m)^{-1}\vec{L} \times \vec{p} + \hat{r} \tag{2.3}$$

generate after appropriate normalization of $\vec{A}$ via Poisson-brackets the algebra of the symmetry group $SO(4)$ for the closed bound and $SO(3,1)$ for the unbound hyperbolic orbits. (These are easily derived by evaluating the scalar product $\vec{A} \cdot \vec{r}$.)

These symmetries are often ascribed to Bertrand's theorem stating that the Kepler problem and the harmonic oscillator are the only central potentials having closed nonprecessing orbits. The restrictions of this theorem emphasizing the special role of the Runge-Lenz vector as defined in (2.3) can be overcome by the observation, that it is based on a *global* symmetry generated by vectorial integrals of motion which are continous single-valued vectors in space. When switching to a *local* viewpoint considering only local symmetries of a problem one can allow multi-valued constants of motion.

This switch is motivated by the fact, that (up to isomorphisms) local symmetry algebras of a Hamiltonian vector field on a $2n$-dimensional phase space M (i. e. on a $2n$-dimensional symplectic manifold) depend *only* on the dimension $2n$ of M. A result of this type has already been stated in [1]; it is used here in the form that $so(4)$ is a (local) symmetry algebra for all classical problems with three degrees of freedom as formulated in [2]. (The question, which Lie subalgebras of the Poisson algebra on M are (up to isomorphisms) local symmetry algebra for a given dimension $2n$ of M has been addressed in [3].) A switch from global to local constants of motion accepting (minimally) broken $O(4)$-symmetry thus eliminates the

special role of the aforementioned central force problems.
As examples for studies exploiting the (minimally broken) higher symmetry we mention here (i) lists of classical dynamical systems with the symmetry of the Kepler problem [4], (ii) the observation that for every bounded trajectory of a particle in an arbitrary central field of force a uniformly rotating reference frame can be chosen in which this trajectory becomes closed [5] and (iii) calculations of asteroide orbits based on locally constant Runge-Lenz vectors which are numerically superior to conventional derivations of these orbits [6].
Such algebraic approaches to nonrelativistic classical motion in central potentials based on $O(4)$ as local symmetry group established generalized Runge-Lenz vectors in the form

$$\vec{A} = f_1(r)\hat{r} + f_2(r)\vec{L} \times \vec{r} \tag{2.4}$$

as suitable instruments to describe the motion. The unknown functions $f_i(r)$, $i = 1, 2$ have to be adapted to the problem at hand by requiring for (2.4) analogous properties as exhibited by the original vector (2.3); they depend in general on the energy, angular momentum and the mass as parameters. (Examples and further references can be found in [7].)
The symmetry is called local since the orbit can be considered locally as a conic section whose eccentricity is determined by the magnitude $A = |\vec{A}|$ of the time-dependent Runge-Lenz vector [6]; the symmetry is minimally broken in the sense that (i) the coefficients $f_i(r)$ in (2.4) can become singular at the turning points or (ii) the vector $\vec{A}$ keeps its length constant while rotating with constant angular speed [5].
In the relativistic Coulomb problem the Runge-Lenz vector (2.3) ceases to a constant of motion and a precession of the orbits known as "advance of perihelion" occurs. Adapting the strategies of the nonrelativistic case to the relativistic regime allows to restore the symmetry (within the limitations just sketched) and an algebraic solution of the problem via a relativistic generalization of the Runge-Lenz vector is feasible as shown now.
For definiteness, we write the equation of motion of the relativistic Kepler problem as [8]

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vec{v}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}\right) = -Ze^2 \frac{1}{r}\hat{r}; \tag{2.5}$$

a more general relativistic two-body system with a unique "post-Newtonian" Runge-Lenz vector allows even to include electromagnetism and gravitation [9]. (The orbital second post-Newtonian equations of motion can be solved by employing the time variation of a generalized Runge-Lenz vector; the results are applicable for a description of higher order angular advance per orbit in binary pulsars [10].)
According to the general strategy the relativistic Runge-Lenz vector $\vec{A}_{\text{rel}}$ is assumed to have the form

$$\vec{A}_{\text{rel}} = A_{\text{rel}} \left\{ \cos\Theta\, \hat{r} - \sin\Theta \frac{\vec{L}_{\text{rel}}}{L_{\text{rel}}} \times \hat{r} \right\}. \tag{2.6}$$

Here A_{rel} and L_{rel} denote the magnitudes of the corresponding vectorial integrals of motion and the "Lenz-angle" Θ is essentially a function of r and the energy; the choice of signs in (2.6) turns out to be convenient [8].
If eq. (2.5) is rewritten in a frame defined by a transition from the time t to the "Eigenzeit"

τ accomplished by

$$d\tau := \sqrt{1 - \frac{v^2}{c^2}} dt, \tag{2.7}$$

it reads

$$m_0 \frac{d^2\vec{r}}{d\tau^2} = -\frac{1}{m_0 c^2} \left\{ E_{\rm rel} Z e^2 \frac{1}{r^2} - Z^2 e^4 \frac{1}{r^3} \right\} \hat{r}. \tag{2.8}$$

Using τ formally as time, one can interpret this equation of motion as a nonrelativistic Kepler problem modified by an inverse cube force, i.e. as a perturbed Kepler problem with precessing orbits. As noted before, this problem is most easily solved by a transition to a rotating coordinate system, choosen such that the orbits appear closed. The effect of this transition is to balance the inverse cube term by the so called fictitious forces in the rotating system. Thus eq. (2.8) is reduced to a formally nonrelativistic Kepler problem; the Runge-Lenz vector of the form (2.3) - defined in the rotating system - allows again a determination of the orbit via the scalar product with $\vec{r}$; the straightforward transformation to the inertial system leads to the precessing orbit. It has been verified that the solution of the nonrelativistic Kepler problem modified by an inverse cube force [11](Heintz) can indeed be adapted to the relativistic problem (2.8) [11](Yoshida).
The concept of solving a relativistic problem like (2.8) by reducing it to nonrelativistic form via a transition to a rotating coordinate system - in other words: the use of a local O(4) symmetry - has apparently been anticipated by Sommerfeld in his derivation of the fine structure formula [12]: the equation of the orbit of the relativistic Kepler motion has the form

$$\frac{1}{r} = C + A \cos \gamma\varphi; \tag{2.9}$$

a transition to a rotating frame with $r = r, \psi = \gamma\varphi$ changes (2.9) to the equation of a "nonrelativistic" elliptic orbit.
The quantum analogue of this transformation [13] discussed below reduces the Dirac-Coulomb problem to nonrelativistic form thus allowing again an algebraic solution.

3 The Dirac-Coulomb problem

The Dirac equation for the Coulomb problem reads

$$H_D \psi = (c\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3 mc^2 - \frac{\alpha c}{r})\psi = E\psi, \tag{3.1}$$

where Dirac's original notation and the convention $\hbar = 1$ have been used [13, 14, 15]. Like for any other central potential $V(r)$, this Hamiltonian commutes with the total angular momentum

$$\vec{J} := \vec{L} + \frac{1}{2}\vec{\sigma}, \tag{3.2}$$

the Dirac operator

$$K = \rho_3(\vec{\sigma} \cdot \vec{L} + 1) \tag{3.3}$$

and the inversion operator

$$P = \rho_3 I. \tag{3.4}$$

Since eqs. (3.1-3.4) describe a complete set of commuting operators, the eigenkets of the Coulomb problem can be labelled accordingly as $|E, j^2, m_j, \kappa>$ where κ is the eigenvalue of K. The well-known twofold degeneracy of states with respect to the sign of κ is accounted for by the existence of the Biedenharn-Johnson-Lippman (BJL) operator

$$R = \vec{\sigma} \cdot \hat{r} - \frac{i}{\alpha mc^2} K \rho_1 (H - \rho_3 mc^2), \tag{3.5}$$

which commutes with the Hamiltonian H_D of (3.1) but anti-commutes with the Dirac-operator K of (3.3) (and the parity-operator P of (3.4)).
The symmetries $SO(4)$ ($SO(3,1)$) of the bound (scattering) states of the nonrelativistic Coulomb problem, generated by the angular momentum operator and the appropriately normalized operator corresponding to the Runge-Lenz vector, do not hold in the Dirac case. The symmetry group generated the constants of motion is no longer uniquely defined. In the frame defined by the quantum analogue of the Sommerfeld transformation, for instance, an O(4) symmetry generated by K (eq. (3.3)) and R (eq. (3.5)) can immediately be written down: the operators

$$M_1 := \frac{1}{2}\frac{K}{(K^2)^{\frac{1}{2}}}, \; M_2 := \frac{1}{2}\frac{R}{(R^2)^{\frac{1}{2}}}, \; M_3 := -2iM_1M_2 \tag{3.6}$$

satisfy the characteristic SU(2) commutation relations

$$[M_i, M_j] := \epsilon_{ijk} M_k \tag{3.7}$$

and generate together with the total angular momentum $\vec{J}$ the symmetry group $SU(2) \times SU(2) \sim O(4)$. (The vector $\vec{M} := (M_1, M_2, M_3)$ has been coined "Lenz-spin" [16].)
The characteristic anti-commutation relations

$$[R, K]_+ = RK + KR = 0, \; [R, P]_+ = 0, \tag{3.8}$$

however, indicate that the underlying symmetry is not a conventional Lie-symmetry like in the nonrelativistic case but generated by a superalgebra.
We address the question of symmetry by first sketching the defining relations of supersymmetric quantum mechanics (susy qm) employed below. A supersymmetric Hamiltonian $\tilde{H}$ is defined via the anticommutator of "supercharges" $Q_\pm$ as

$$\{Q_+, Q_-\} := \tilde{H}; \tag{3.9}$$

the supercharges satisfy

$$Q_+^2 = Q_-^2 = 0 \tag{3.10}$$

and commute with the Hamiltonian $\tilde{H}$. A simple representation of this algebra has the form

$$Q_\pm := (p \pm i\phi(x))a^\pm := h_\pm a^\pm, \tag{3.11}$$

where the fermionic operators $a^{\pm}$ are defined via

$$[a^+, a^-] = 1, \quad (a^+)^2 = (a^-)^2 = 0. \tag{3.12}$$

The supercharges $Q_\pm$ act on eigenstates of

$$\tilde{H}\psi = \begin{pmatrix} h_+h_- & 0 \\ 0 & h_-h_+ \end{pmatrix} \begin{pmatrix} \psi_-^n \\ \psi_+^n \end{pmatrix} = E\psi \tag{3.13}$$

as

$$Q_\pm \psi_\mp^n = \pm i\sqrt{E_n}\psi_\pm^n, \tag{3.14}$$

where the index n needs not to be confined to discrete values.
When trying to apply the relations (3.9-3.13) to the Dirac-Coulomb problem, one has to choose a representation leading upon appropriate choice of the operators $h_\pm$ to self-adjoint Hamiltonians $H_\pm := h_\pm h_\mp$ defined in (3.13). These Hamiltonians, in turn, have to allow separation in polar coordinates in order to determine the eigenvalues and eigenfunctions of (3.1).
In the next step we have to check if the conserved operators constitute a realization of this superalgebra in the full functional space or if we have to separate (3.1) first in polar coordinates before trying to find the symmetries of the Dirac Coulomb problem. A superalgebra acting in the full functional space would be the analogon to the primary supersymmetry of the nonrelativistic Coulomb problem with spin [17]. For this purpose we try to identify the operators $h_\pm$ of (3.11) with the BJL operator (3.4). This leads to two copies of the eigenvalue problem [15]

$$R^2\phi = \mu\phi \tag{3.15}$$

which has been rediscovered recently [14]. When inserting (3.5) in (3.15) and evaluating the resulting expression, some problems arise : the eigenvalue problem (3.15) is not self-adjoint and can not be separated in polar coordinates. The origin of these problems, unavoidable in the representation set up by eqs. (3.1-3.4), can be identified in the Temple operator

$$\Gamma := \rho_3 K + i\alpha\rho_1(\vec{\sigma}\cdot\hat{r}). \tag{3.16}$$

The BJL operator contains the non-selfadjoint operator Γ via [13]

$$\alpha R = -i\rho_1\left(\Gamma - \frac{KH_{\mathrm{D}}}{mc^2}\right). \tag{3.17}$$

To resolve these problems one has to switch to a representation where Γ (and R) become self-adjoint and eigenvalues can be assigned. This is achieved by the quantum analogue of Sommerfeld's transformation mentioned above; it reads explicitly [13]

$$S := \exp\left(-\frac{1}{2}\rho_2\vec{\sigma}\cdot\hat{r}\tanh^{-1}(\frac{\alpha}{K})\right). \tag{3.18}$$

Switching to the frame defined by S, the Temple operator Γ transforms like

$$\Gamma \to \tilde{\Gamma} = S\Gamma S^{-1} = \rho_3 K|(1-(\frac{\alpha}{K})^2)^{1/2}|, \tag{3.19}$$

with the obvious eigenvalues

$$\tilde{\Gamma} \to \gamma := \pm|(\kappa^2 - \alpha^2)^{1/2}| \tag{3.20}$$

upon the identification $\mathrm{sign}(\gamma) = \mathrm{sign}(\kappa)$. In the frame S the BJL operator can be written as

$$\tilde{R} = \frac{1}{k}\rho_3\vec{\sigma}\cdot\hat{r}\tilde{\Gamma}\left(i\hat{r}\vec{p} + \frac{1+\tilde{\Gamma}}{r} + \frac{\alpha E}{c^2\tilde{\Gamma}}\right), \tag{3.21}$$

where the Hamiltonian has been replaced by its eigenvalues and the normalization factor $k^{-1} = \left(\frac{E^2}{c^2} - m^2c^2\right)^{-1/2}$ has been introduced. (This normalization is analogous to the normalization of the Runge-Lenz vector in the nonrelativistic case; it allows a simple derivation of the eigenvalues for the present relativistic problem as shown below.)
The eigenvalue problem (3.15) takes a well defined form when the BJL operator R is replaced by $\tilde{R}$ defined in the frame S. To demonstrate this form we normalize $\tilde{R}$ to unity using

$$\tilde{R}^2 = \gamma^2 + \frac{\alpha^2 E^2}{k^2}. \tag{3.22}$$

Thus $\tilde{R}$ can be written as

$$\tilde{R} = \frac{1}{k}\left(\gamma^2 + \frac{\alpha^2 E^2}{k^2}\right)^{-1/2}\rho_3\vec{\sigma}\cdot\hat{r}\tilde{\Gamma}\left(i\hat{r}\vec{p} + \frac{1+\tilde{\Gamma}}{r} + \frac{\alpha E}{c^2\tilde{\Gamma}}\right). \tag{3.23}$$

Now it is obvious that this operator generates the symmetry group Z_2. Inserting (3.23) in the eigenvalue problem (3.15) results in

$$\tilde{R}^2\psi = \frac{\tilde{\Gamma}^2}{k^2 + \frac{\alpha^2E^2}{\gamma^2}}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\tilde{\Gamma}(\tilde{\Gamma}+1)}{r^2} - \frac{2\alpha E}{cr} - \frac{\alpha^2E^2}{c^4\tilde{\Gamma}^2}\right)\psi = \mu\psi. \tag{3.24}$$

The eigenvalue problem (3.24) associated with the normalized BJL operator $\tilde{R}$ in the frame S is a well defined expression which could be used to define a "primary supersymmetry" of the Dirac Coulomb problem. Replacing, however, $\tilde{\Gamma}$ by its eigenvalues – which is equivalent to separate (3.24) in polar coordinates – reduces this eigenvalue problem into a well known form: it turns into the iterated Dirac equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(\gamma)(l(\gamma)+1)}{r^2} - \frac{2\alpha E}{cr}\right)\phi = \left(E^2 - m^2c^4\right)\phi, \tag{3.25}$$

where the non-integer angular momentum

$$l(\gamma) = |\gamma| + \frac{1}{2}(\mathrm{sign}\,(\gamma) - 1) \tag{3.26}$$

has been introduced. Thus we see that for the present problem a primary supersymmetry can not be defined; only the system of radial equations has a supersymmetric structure.
The relation to the supersymmetry of the radial equations of the Dirac-Coulomb problem is easily established by the observation that the BJL operator $\tilde{R}$ in (3.23) factorizes into

operators acting separately on the angular and radial components of the eigenkets. The radial operators of $\tilde{R}$ define non-normalized supercharges via

$$Q_{\pm} := (i\hat{r}\cdot\vec{p} + \frac{1+\tilde{\Gamma}}{r} + \frac{\alpha E}{c\tilde{\Gamma}})(\frac{1}{2}(\rho_1 \pm i\rho_2)). \tag{3.27}$$

Invoking in addition the normalization of the BJL operator $\tilde{R}$ in the frame S gives the iterated Dirac-Coulomb Hamiltonians via the supercharges $Q_{\pm}$ defined in (3.27) using

$$\begin{aligned} \{Q_+, Q_-\}\phi &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(\gamma)(l(\gamma)+1)}{r^2} - \frac{\alpha^2 E^2}{c^2\tilde{\Gamma}^2} - \frac{2\alpha E}{cr}\right)\phi \\ &= \left(E^2 - m^2c^4 - \frac{\alpha^2 E^2}{c^2\gamma^2}\right)\phi. \end{aligned} \tag{3.28}$$

The supersymmetry generated by the BJL operator is nothing else but the known radial supersymmetry [18].

4 Analogies between non-relativistic and relativistic Coulomb problems

Similar to the classical case many analogies between the quantum problems exist. These are demonstrated now concentrating on algebraic derivations of the eigenvalues, eigenstates and the solution to the scattering problem for the Schrödinger and the Dirac Coulomb problem while including discussions of the symmetric Dirac and the Klein-Gordon case within a special approach.
The $SO(4)$ symmetry of the nonrelativistic problem, generated by the angular momentum

$$\vec{L} = \vec{r}\times\vec{p} \tag{4.1}$$

and the normalized Runge-Lenz vector

$$\begin{aligned} \vec{\tilde{A}} &= (-2mH)^{-1/2}\vec{A} \\ &= (-2mH)^{-1/2}(\hat{r} + (2m\alpha)^{-1}(\vec{L}\times\vec{p} - \vec{p}\times\vec{L})), \end{aligned} \tag{4.2}$$

can be used as follows:

(a) The eigenvalues follow directly from the invariant operator (Casimir operator) of $SO(4)$ by writing the Hamiltonian as

$$H = \frac{1}{L^2 + \tilde{A}^2 + 1} \rightarrow -\frac{1}{2n^2}; \tag{4.3}$$

these eigenvalues (including degeneracy) are given by standard group theoretical arguments [13].

(b) The eigenstates for a fixed value of n can be constructed by employing shift operators of $SO(4)$. In an appropriate basis classified by the quantum numbers (l, m) these shift operator have the form

$$\vec{S} := \vec{A}(D+1) - \frac{1}{2}[L^2, \vec{A}] \tag{4.4}$$

with

$$D := (L^2 + \frac{1}{4})^{1/2} - \frac{1}{2}. \tag{4.5}$$

The component S_3 of $\vec{S}$ acts on the three-dimensional eigenstates $|nlm>$ of the Coulomb problem with $\psi(r,\Theta,\varphi) = \phi_{nl}(r)Y_{\mathrm{lm}}(\Theta,\varphi) := |nlm>$ as

$$\begin{aligned} S_3\phi_{nl}(r)Y_{\mathrm{lm}}(\Theta,\varphi) & \\ &= Y_{l-1m}(\Theta,\varphi)\left(\frac{\partial}{\partial r} + \frac{l}{r} - \frac{1}{l}\right)\phi_{nl}(r) \\ &= Y_{l-1m}(\Theta,\varphi)\phi_{nl-1}(r). \end{aligned} \tag{4.6}$$

The operators $S_\pm := S_1 \pm iS_2$, by comparison, lead to

$$S_\pm|nlm> = |nl-1m\pm 1> . \tag{4.7}$$

In this form $\vec{A}$ and $\vec{L}$ suffice to derive the bound states of the nonrelativistic Coulomb problem. It can be verified by inspection, that these invariants obviously determine the ladder operators of the factorization method when applied to the corresponding radial equation [19]. The factorization method [20] is thus tied to the group-theoretical interpretation of the eigenkets [21]. (This interpretation, in turn, can be applied to modified Coulomb problems as the example of the Dyon problem shows [22].)
Similar properties hold in the relativistic case. When writing the non-normalized BJL operator (3.5) as

$$R = \vec{\sigma}\cdot\left(\frac{1}{2\alpha mc}\rho_3(\vec{L}\times\vec{p} - \vec{p}\times\vec{L}) + \hat{r}\right) + \frac{1}{c^2 r}\rho_2 K, \tag{4.8}$$

it is easily seen that R reduces in the nonrelativistic limit ($c \to \infty$) to $\vec{\sigma}\cdot\vec{A}$, i.e. to the natural generalization of the Runge-Lenz Vector $\vec{A}$ (cf. (4.2)) when spin is incorporated. Like in the nonrelativistic case, a normalized form of the BJL operator can be used to derive the energy eigenvalues: when normalizing R in the form (3.21), the Dirac-Coulomb Hamiltonian can be written in the frame S as

$$H_{\mathrm{D}} = mc^4\left(1 + \frac{\alpha^2}{(\tilde{R}+\tilde{\Gamma})^2}\right)^{-1/2}, \tag{4.9}$$

which gives the eigenvalues

$$E_d = mc^4\left(1 + \frac{\alpha^2}{(n + (\kappa^2-\alpha^2)^{\frac{1}{2}})^2}\right)^{-1/2}. \tag{4.10}$$

The relation to the solution of the Dirac-Coulomb problem within the factorization method is established by writing the operator $\tilde{R}$ of (3.21) as

$$\tilde{R} = \frac{1}{k}\rho_3\vec{\sigma}\cdot\hat{\rho}\gamma\left(\frac{\partial}{\partial\rho} - \frac{\gamma}{\rho} + \frac{\alpha}{c\gamma}\right) \tag{4.11}$$

with $\rho := E \cdot r$. The factor $\vec{\rho}\cdot\hat{\sigma}$ acts only on the angular parts of the corresponding eigenkets; thus this form of $\tilde{R}$ exhibiting the operator acting on the radial functions shows again the close relation between the integrals of motion and the ladder operators of the factorization scheme.

(c) The S-matrix of the nonrelativistic Coulomb problem is easily derived via the $SO(3,1)$ symmetry generated by the constants of motion after replacing H by $-H$ in the normalization of the Runge-Lenz vector. This procedure can also be extended to the relativistic regime using again the corresponding constants of motion [19].
The basic idea of this group-theoretic approach to scattering is based on the observation that scattering is essentially a matter of asymptotics where the S-matrix transforms an aymptotic in-state into an asymptotic out-state. This justifies the introduction of a scattering space as the sphere bundle $S^2 \times S^0$ obtained by deleting the origin and contracting the radial space, viz $R^3 - \{0\} \sim S^2$, such that the radial derivative describing the in/-out motion limits to the sphere $S^0(\sim Z_2)$ with generator B, $B^2 = E$. Thus the generators of the symmetry group, when restricted to the scattering space, realize two abstractly equivalent sets of generators realized on $S^2 \times S^0$ and distinguished by the eigenvalue of $B \to \epsilon = \pm 1$. This Ansatz allows a systematic derivation of the S-matrices of the nonrelativistic Coulomb problem, the Dirac Coulomb problem and the symmetric Dirac Coulomb problem (cf. eq. (5.1), below) using in each case the same strategy and exploiting the underlying symmetry [19].

(d) The relations between the shift operators of the factorization method, the invariants generating the symmetry and supersymmetric quantum mechanics motivate an analysis of the radial equations of the Coulomb problems alone which are all accessible to the spectrum algebra approach based on $SU(1,1)$. (The relativistic Coulomb problems now include the symmetric Dirac and the Klein-Gordon case.) In this approach - whose applicability is by no means restricted to the problems considered here but covers all problems allowing a reduction of the radial equation to the confluent hypergeometric equation - the Hamiltonian is written in terms of the generators of the algebra satisfying the commutation relations

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2; \tag{4.12}$$

the eigenvalues then follow from standard group theory.
In the Coulomb problems considered here, the use of $SU(1,1)$ in the non-relativistic and the relativistic cases exploits the fact, that in appropriate frames after iterating the Dirac equations and re-definitions of parameters the radial equations of the

Schrödinger, Klein-Gordon, symmetric Dirac and the full Dirac Coulomb problem can be written as

$$\left(\frac{1}{\rho^2}\frac{d}{d\rho}\rho^2\frac{d}{d\rho}+\frac{w}{4\rho}-\frac{1}{4}-\frac{l(\gamma)(l(\gamma)+1)}{\rho^2}\right)R=0, \tag{4.13}$$

where ρ is the radial variable scaled according to the problem under consideration, w denotes the non-relativistic and the relativistic forms of the Sommerfeld parameter and $l(\gamma)(l(\gamma)+1)$ abreviates the various forms of the angular momentum term. Equation (4.13) is easily recognized as the differential equation of the confluent hypergeometric function, whose role in unitary representations of $SU(1,1)$ has been clarified in [23]. The first derivation of the eigenvalues of the Klein-Gordon and the Dirac Coulomb problem via $SU(1,1)$ by writing the corresponding radial equations in the form (4.13) and subsequently transforming them into the differential equation of the harmonic oscillator with an "angular momentum" term has been given in [24]; the symmetric Dirac Coulomb problem has been treated along the same lines in [25].

5 From an approximate Dirac Coulomb Hamiltonian to relativistic equivalent oscillators

The high symmetry of the nonrelativistic problem is broken in the Dirac case by the fine-structure. When substracting from the Dirac Coulomb Hamiltonian the symmetry-breaking term one obtains the relativistic Hamiltonian [13, 14]

$$H_{sym}=\rho_1\vec{\sigma}\cdot\vec{p}+\rho_3 m-\frac{\alpha Z}{r}-ig\rho_2\frac{\vec{\sigma}\cdot\hat{r}}{r}f(K), \tag{5.1}$$

where

$$f(K)=K[1+\left(\frac{\alpha Z}{K}\right)^2]-1 \tag{5.2}$$

is a function of the Dirac operator K defined above and g is a coupling constant. This approximate Dirac Coulomb Hamiltonian has again two vectorial integrals of motion allowing a straightforward solution of the bound state problem [13, 26]; the scattering problem with SO(3,1) symmetry can also be solved along the same lines as sketched above [19].
These features of the symmetric Hamiltonian H_{sym} suggest a detailed analysis of

$$H_{reo}=\rho_1\left(\vec{\sigma}\cdot\vec{p}+i\lambda^2\vec{\sigma}\cdot\vec{r}\rho_3\frac{K}{|K|}\right)+\rho_3 m, \tag{5.3}$$

viz. a relativistic Hamiltonian with the potential responsible for fine-structure effects [27]. This was the starting point for the collection of relativistic Hamiltonians now coined "relativistic equivalent oscillators" (short: reos) or "Dirac oscillators" leading in the nonrelativistic limit to a harmonic oscillator potential with a spin-orbit coupling term, i.e. to the paradigm of a supersymmetric Hamiltonian.
The analogue of the BJL operator for the reo-Hamiltonian generates together with angular momentum and Dirac's operator K the symmetry group $SO(4)\times SU(2)$ for the bound states

[27]. The symmetry group of the scattering states is not yet completely clear; the group $O(4,2) \times SU(2)$ was suggested in the original study [28] without proof.
If adding an energy shifting term λK ($\lambda = const.$) to H_{reo} one arrives at a Hamiltonian used in nuclear physics; in the nonrelativistic limit it reduces to the Nilsson Hamiltonian [29].
A nice generalization is achieved by considering the set of Dirac equations given by

$$H\psi = (\rho_1(\vec{\sigma} \cdot \vec{p} + i\rho_3\vec{\sigma} \cdot \vec{r} f(j, K)) + \rho_3 m)\psi = E\psi, \tag{5.4}$$

where $f(j, K)$ is an arbitrary function of J^2 and Dirac's operator K. These equations constitute a large set of reos where the degeneracy of eigenstates can be "tuned" [30]. Since the degeneracy of energy-levels can be tuned while using the same basic form of the Hamiltonian, it is no surprise that many different symmetry groups for a classification of energy levels have been suggested [31].

6 Summary

The symmetries of relativistic Coulomb problems have been surveyed emphasizing analogies between the classical and the quantal problems. Sommerfelds derivation of the fine-structure formula has been put in a group-theoretical context. The origin of relativistic equivalent oscillators has been traced back to the fine-structure of the Dirac Coulomb Hamiltonian. The relation between the conserved operators of the quantal problems to the factorization method, whose supersymmetric structure has been studied in many papers, has been established.
The question if the approximate Dirac Coulomb Hamiltonian with $O(4)(O(3,1))$ symmetry has also a supersymmetric structure awaits further elaboration. We note here that within the quasipotential approach the relativistic Coulomb problem has the same symmetries as the approximate Dirac Coulomb Hamiltonian [32].
The present discussion leads to some open to be studied in the future: (i) it is not yet clear, if analogues of the BJL operator for a given central force field in the Dirac equation accounting for the inherent degeneracy in the eigenvalues of the Dirac operator K exist; (ii) the relation between the $SU(1,1)$ spectrum algebra approach and and supersymmetric quantum mechanics as pointed out in [33] indicates a relation between the symmetry algebra approach propagated here and the spectrum algebra approach which is not yet completely understood; (iii) the restriction of spectrum algebra approaches to small Z has to be lifted in order to describe heavy-ion collisions where high values of Z are reached. This has so far only been discussed in the context of the Klein-Gordon Coulomb problem [34].

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References

[1] L. P. Eisenhart, *Continous Groups of Transformations* (Princeton Univ. Press Princeton New Jersey 1933).

[2] H. Bacry, H. Ruegg and J. M. Souriau, *Dynamical groups and spherical potentials in classical mechanics*, Comm. Math. Phys. **3** 323-333 (1966).

[3] N. Mukunda, *Dynamical symmetries and classical mechanics*, Phys. Rev. **155** 1383-1386 (1967); E. Aguirre, H. D. Doebner and J. D. Hennig, *All local classical symmetries in Hamiltonian mechanics*, Lett. Math. Phys. **7** 85-90 (1983).

[4] V. I. Karloukovski, *Classical dynamical systems with the symmetry of the Kepler problem*, JINR-preprint E2-11291 (Dubna 1978).

[5] V. B. Serebrennikov, A. E. Shabad and Y. A. Smorodinsky, *O_4 covariant Hamiltonian formalism for bounded motion in a central field of force*, Nuovo Cim. **47A** 333-357 (1978).

[6] E. A. Bartnik, H. Haberzettl and W. Sandhas, *Equations of motion using the dynamical evolution of the Runge-Lenz vector*, Astrophys. J. **334** 517-526 (1988).

[7] A. Holas and N. A. March, *A generalization of the Runge-Lenz constant of classical motion in a central field*, J. Phys. A **23** 735-749 (1990); T. Iwai and N. Kotoyama, *Multifold Kepler systems - dynamical systems all of whose bounded trajectories are closed*, J. Math. Phys. **36** 1790-1811 (1995).

[8] E. Kern, *Relativistische Verallgemeinerungen des Runge-Lenz Vektors I,II*, Zeitschr. Naturforschung **39a** 720-747 (1984) (in German).

[9] F. Argüeso and J. L. Sanz, *Post-Newtonian extensions of the Runge- Lenz vector*, J. Math. Phys. **25** 2935-2938 (1984); Th. P. Mitchell, *Vector constants and their algebras for classical Hamiltonians*, J. Math. Phys. **19** 2082-2084 (1978).

[10] T. Damour and G. Schäfer, *Higher order relativistic periastron advances and binary pulsars*, Nuovo Cimento B **101**, 127-176 (1988).

[11] W. H. Heintz, *On the Kepler problem modified by an inverse cube force*, Am. J. Phys. **44** 687-694 (1976); T. Yoshida, *Considerations on the precessing orbit via a rotating Laplace-Runge-Lenz vector*, Am. J. Phys. **55** 1133-1136 (1987).

[12] A. Sommerfeld, *Atombau und Spektrallinien* (Vieweg, Braunschweig 1929) 5th ed. The 3rd edition of this book (translated by H. L. Brose) *Atomic structure and spectral lines* (Methuen, London 1923) contains the geometric approach.

[13] L. C. Biedenharn, *Remarks on the relativistic Kepler problem*, Phys. Rev. **126** 845-851 (1961); L. C. Biedenharn, *The Sommerfeld Puzzle revisited and resolved*, Foundations of Physics **13** 13-34 (1983).

[14] J. P. Dahl and Th. Jorgensen, *On the Dirac-Kepler problem: the Johnson-Lippman operator, supersymmetry and normal-mode representations*, Int. J. Quantum Chemistry **53** 161-181 (1995).

[15] C. A. Coulson and A. Joseph, *Self-adjoint ladder operators II*, Rev. Mod. Phys. **39**, 838-849 (1967).

[16] E. H. de Groot, *The virial theorem and the Dirac H-atom*, Am. J. Phys. **50** 1141-1143 (1982).

[17] R. D. Tangerman and J. A. Tjon, *Exact supersymmetry of the nonrelativistic hydrogen problem*, Phys. Rev. A **48** 1089-1092 (1993).

[18] J. D. Jarvis and G. E. Stedman, *Supersymmetry in second order relativistic equations for the hydrogen atom*, J. Phys. A **19** 1373-1385 (1986); C. V. Sukumar, *Supersymmetry and the Dirac equation for a central Coulomb field*, J. Phys. A **18** L697-L701 (1985).

[19] A. Stahlhofen and L. C. Biedenharn, *Group-theoretic approach to scattering: the Dirac Coulomb problem and relativistic supersymmetry*, in: H. D. Doebner, J. D. Hennig and T. D. Palev (eds.): "Proc. XVI Int. Conf. Group-Theoret. Methods in Physics (Varna 1987)", Springer Lecture Notes in Physics **313**, 261-267 (Springer 1988); J. Wu, A. Stahlhofen, L. C. Biedenharn and F. Iachello, *A group-theoretical calculation of the S-matrix for the Dirac-Coulomb problem*, J. Phys. A **20** 4637-4652 (1987).

[20] L. Infeld and T. E. Hull, *The factorization method*, Rev. Mod. Phys. **23** 21-68 (1951).

[21] A. A. Stahlhofen and K. Bleuler, *An algebraic form of the factorization method*, Nuovo Cim. **104 B** 447-465 (1989).

[22] A. A. Stahlhofen, *On the Dyon problem*, Mod. Phys. Lett. A **5** 2007-2012 (1990).

[23] W. Montgomery and L. O. Raifeartaigh, *Noncompact Lie-algebraic approach to the unitary representations of $SU(1,1)$: Role of the confluent hypergeometric equation*, J. Math. Phys. **15** 380-382 (1974).

[24] H. Bacry and J. L. Richard, *Partial group-theoretical treatment for the relativistic hydrogen atom*, J. Math. Phys. **8** 2230-2233 (1967).

[25] J. Cizek and J. Paldus, *An algebraic approach to bound states of simple one-electron systems*, Int. J. Quantum Chem. **XII** 875-896 (1977).

[26] L. C. Biedenharn and N. V. V. J. Swamy, *Remarks on the relativistic Kepler problem. II. Approximate Dirac Coulomb Hamiltonian possessing two vector invariants*, Phys. Rev. **133B** 1353-1360 (1964).

[27] N. V. V. J. Swamy, *Exact solution of the Dirac equation with an equivalent oscillator potential*, Phys. Rev. **180** 1225-1226 (1969).

[28] E. F. Chaffin, *Group structure of the relativistic equivalent oscillator*, J. Math. Phys. **14** 977-979 (1973).

[29] J. Brown and N. V. V. J. Swamy, *Relativistic effects in nuclear motion*, Nucl. Phys. A **160** 641-648 (1971).

[30] Y. M. Cho, *A potential approach to spinor quarks*, Nuovo Cim. **23 A** 550-556 (1974).

[31] R. P. Martinez-y-Romero, H. N. Nunez-Yepez and A. L. Salas-Brito, *Relativistic quantum mechanics of a Dirac oscillator*, Eur. J. Phys. **16** 135-141 (1995).

[32] Sh. M. Nagiyev, *On a hidden symmetry of a relativistic Coulomb problem in the quasipotential approach*, J. Phys. A **21** 2559-2564 (1988).

[33] G. Levai, *Search for $SU(1,1)$ shape invariant potentials: a systematic study*, J. Phys. A **27**, 3809-3828 (1994).

[34] M.Bawin, *$SO(2,1)$ and strong Coulomb coupling*, J. Phys. A **10**, 1-5 (1977)