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On Newton-Cartan Cosmology

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Abstract. After a brief summary of the Newton-Cartan theory in a form which emphasizes its close analogy to general relativity, we illustrate the theory with selective applications in cosmology. The geometrical formulation of this nonrelativistic theory of gravity, pioneered by Cartan and further developed by various workers, leads to a conceptually sound basis of Newtonian cosmology. In our discussion of homogeneous models and cosmological perturbation theory, we stress the close relationship with their general relativistic treatments. Spatially compact flat models also fit into this framework.

1 Introduction

We hope that Klaus and Walter will accept this modest note as a tribute to their outstanding role as teachers of theoretical physics. In their courses they present not only elegant techniques and formal developements, but always emphasize the importance of basic concepts. "Rechnen kann jeder", as Heitler used to say. In this spirit, we devote this article to a theme which is mainly of conceptual nature, and – as we hope – also of some pedagogical interest.

We shall try to make it apparent that Newton's theory of gravity is much closer to general relativity (GR) than commonly appreciated. This has often been stressed in private conversations and letters by our inspiring teacher and colleague Markus Fierz. Here an example from a letter (Nov. 22, 1993):

„Es war um 1953, als ich meinen Newton-Aufsatz schrieb, dass ich Pauli sagte, auch in der allgemeinen Relativitätstheorie seien Raum-Zeit ‚absolut', wie bei Newton. Darauf antwortete Pauli zu meinem Staunen: ‚Sie verraten damit, das Grundprinzip der allgemeinen Relativitätstheorie: dass nämlich Raum-Zeit-Materie nicht unabhängig voneinander gedacht werden können'. Wir konnten uns dann aber einigen, indem ich zugab, dass hier das ‚sine mutua actione' Newtons nicht gilt, obwohl im Ganzen diese Wechselwirkung klein ist. Was ich im Sinne hatte ist dies: Leibniz erklärte, der Raum sei nichts Wirkliches, sondern entspringe der Ordnung der Monaden (Kraft prästabilisierter Harmonie). Newton erwiderte hierauf: der Raum sei mehr als eine blosse Ordnung. Denn der Abstand zweier Punkte habe einen Sinn ganz unabhängig davon, ob der Raum von etwas erfüllt oder leer sei. Newton legte also grosses Gewicht auf den metrischen Charakter des Raumes: dieser macht ihn zum Gegenstand der Physik, zu etwas Wirklichem ... "

Every honest teacher of theoretical physics is confronted at a very early stage of a classical mechanics course with the following difficulty: After having introduced – in the spirit of L. Lange – the operational definition of an inertial frame, the question arises how to proceed when gravitational fields are present. In the traditional presentation of Newtons theory one maintains the fiction of an integrable (flat) affine connection, and puts gravity on the side of the forces, described by vector fields. A much more satisfactory formulation was given by Cartan [1] and Friedrichs [2]. This denies the separate existence of a flat affine connection of space-time and a vector field describing gravitation, but puts gravity on the side of a more general dynamical connection which represents both inertia and gravitation.

Historically, this important step of course was made first by Einstein when he created his general theory of relativity, but it is clearly independent of the relativization of time. Following Cartan and Friedrichs, numerous authors have elaborated on this idea. Here we mention only a selective list of contributions by Havas [3], Trautman [4], Ehlers [5, 6, 7] and Künzle [8, 9].

In the first part of the present paper we give a brief summary of the Newton-Cartan theory, following mainly the work of H.P. Künzle, a former diploma student of Fierz at the ETH. Künzle's presentation, which uses the language of fibre bundles, appears to us as the most natural one, because it just replaces the role of the Lorentz group in GR by the Galilei group. Following this route, one arrives at a theory which is slightly more general than Newton's theory. The latter is only obtained after imposing a somewhat strange looking nonlinear condition for the Riemann tensor. The structural analogy of GR and the Newton-Cartan theory is, however, striking. In particular, the field equations look identical.

In later sections we shall illustrate this also in more concrete terms with some selective applications in cosmology (homogeneous cosmological models and cosmological perturbation theory). This is perhaps not only an academic exercise, because much of the activity in cosmology, especially in connection with large scale structure formation, relies on the Newtonian approximation. We take this as a motivation for putting Newtonian cosmology on a conceptually firm basis. This has to be regarded as an extension of classic works by Heckmann [10] and Heckmann and Schücking [11]. One advantage which results is the possibility to choose the spatial sections as flat tori and thus describe compact cosmologies.

All this confirms in more technical terms the remarks by Markus Fierz, quoted earlier. Einstein was wrong when he believed that his theory of gravitation incorporated the principle of Mach which is entirely in the spirit of Leibniz. This became already quite clear with the famous solution of Gödel, but some relativists, notably Einstein himself, maintained the belief that Mach's principle might have something to do with the finiteness of space [23]. That this is not the case was once and for all demonstrated by the "finite rotating universe" solution found by Ozvàth and Schücking [12]. Space-time has really an independent existence and we are in fact still much closer to Newton than to Leibniz.

A more detailed account of the material treated in this paper can be found in the diploma thesis by one of us [13].

2 Galilei spacetimes and their connections

In what follows, M will always denote the space-time manifold and $L(M)$ the principle bundle of linear frames with the structure group $GL(4, !!R)$. In GR space-time is endowed with a Lorentz metric g which defines a bundle reduction of $L(M)$ to the orthonormal frame bundle $O(M)$ with the homogeneous Lorentz group as the structure group. Conversely, each reduction of the structure group $GL(4, !!R)$ to the homogeneous Lorentz group gives rise to a Lorentz metric, because any element $u \in L(M)$ over $x \in M$ can be regarded as a linear isomorphism of $!!R^4$ onto T_xM, which maps the standard basis $\{e_\mu\}$ of $!!R^4$ to the linear frame u.

In a "nonrelativistic" gravity theory M has to be endowed with a **Galilei metric**, which consists of a one-form τ and a symmetric semi-definite contravariant tensor field h of rank 3, satisfying $h(\cdot, \tau) = 0$ $(h^{\mu\nu}\tau_\nu = 0)$. The pair (h, τ) defines again a bundle reduction of $L(M)$, this time with the homogeneous Galilei group as structure group. The reduced bundle consists of all frames $\{e_\nu\}$ in $L(M)$, satisfaying

$$\tau(e_0) = 1, \qquad h(\theta^\mu, \theta^0) = 0, \qquad h(\theta^i, \theta^j) = \delta^{ij} \qquad (i, j = 1, 2, 3), \tag{2.1}$$

where $\{\theta^\mu\}$ denotes the dual frames, $\langle \theta^\mu, e_\nu \rangle = \delta^\mu_\nu$. Since $h(\theta^\mu, \tau) = 0$, these equations imply $\tau = \theta^0$.

Conversely, a reduction of the structure group $GL(4, !!R)$ to the homogeneous Galilei group gives rise to a Galilei metric (h, τ). This just reflects the fact that the homogeneous Galilei group (without time reflections) is the subgroup of $GL(4, !!R)$ which leaves the standard Galilei metric of $!!R^4$ invariant. The latter is defined by equations (2.1) for the standard basis $\{e_\mu\}$ of $!!R^4$ and its dual. This defines the **flat** Galilei spacetime. With this notion it is also clear what is a **locally flat** Galilei spacetimes. These can be characterized as follows.

Proposition 1. *A Galilei spacetime (M, h, τ) is locally flat iff the following two conditions are satisfied:*

(i) $d\tau = 0$;

(ii) *the induced Riemannian metrics on the integral manifolds defined by τ are locally flat.*

From now on we shall only consider bundle reductions to the identity components $G_+^\uparrow$ of the homogeneous Galilei group (orthochronous Galilei group), which we shall denote by $\mathcal{G}(M, G_+^\uparrow)$. The corresponding frames are then space and time oriented.

It is now clear, how to define a **Galilei connection** on (M, h, τ). This is a connection in the corresponding principal bundle $\mathcal{G}(M, G_+^\uparrow)$, which we describe by a connection form ω, satisfying the usual conditions. There is a natural characterization of Galilei connections:

Proposition 2. *A linear connection Γ on a Galilei manifold (M, h, τ) is a Galilei connection iff*

$$\nabla h = 0, \qquad \nabla \tau = 0, \tag{2.2}$$

where ∇ denotes the covariant derivative with respect to Γ.

We consider only symmetric connections. For these the second equation in (2.2) implies $d\tau = 0$. Thus the distribution defined by the 1-form τ is integrable. The corresponding maximal integral manifolds are the spatial **sections of constant time**. Vectors tangent to these sections are annihilated by τ and are called **spacelike** (or horizontal). Tangent vectors which are not annihilated by τ are called **timelike**. If $\tau(V) = 1$ we say that V is a **timelike unit vector**.

In contrast to Lorentz manifolds there is *no unique* symmetric Galilei connection on a Galilei manifold. It is instructive to see this in the light of a famous theorem by Weyl [14] and Cartan [15]. Since this is not so well-known (even among relativists) we state it here:

Theorem. (Weyl, Cartan) *For a closed subgroup G of $GL(n, !!R)$, $n \geq 3$, the following two conditions are equivalent:*

(i) *G consists of all elements of $GL(n, !!R)$ which preserve a certain non-degenerate quadratic form of any signature;*

(ii) *For every n-dimensional manifold M and for every reduced subbundle P of $L(M)$ with group G, there exists a unique torsion-free connection in P.*

It turns out that the set of symmetric Galilei connections is in 1:1 correspondence with the set $\Lambda^2(M)$ of 2-forms on M. The two equations (2.2) imply that the difference of two connection forms (Christoffel symbols) is given by a tensor field of the following type

$$S^\mu_{\alpha\beta} = 2\tau_{(\alpha}\kappa_{\beta)\lambda}h^{\lambda\mu}, \tag{2.3}$$

where $\kappa_{\alpha\beta}$ are the components of a 2-form κ. Special symmetric Galilei connections can be described as follows. Choose a timelike unit vector field V and define the covariant metric $h^\flat$ (relative to V) such that its components $h_{\mu\nu}$ satisfy

$$h_{\mu\nu}V^\nu = 0, \qquad h_{\mu\lambda}h^{\lambda\nu} = \delta^\nu_\mu - \tau_\mu V^\nu, \tag{2.4}$$

then

$${}^V\Gamma^\mu_{\alpha\beta} = h^{\mu\rho}(V^\sigma{}_{,\rho}h_{\sigma(\alpha}\tau_{\beta)} + \frac{1}{2}h^{\lambda\sigma}{}_{,\rho}h_{\lambda\alpha}h_{\sigma\beta}) - h_{\rho(\alpha}h^{\rho\mu}{}_{,\beta)} - \tau_{(\alpha}V^\mu{}_{,\beta)} \tag{2.5}$$

defines a symmetric Galilei connection. This is actually the unique symmetric Galilei connection which satisfies also

$$V^\rho V^\alpha{}_{;\rho} = 0, \qquad h^{\rho\alpha}V^\beta{}_{;\rho} - h^{\rho\beta}V^\alpha{}_{;\rho} = 0. \tag{2.6}$$

With respect to (2.5) the vector field V is then geodesic and rotation free.

Note that relative to a Galilei frame with $V = e_0$, equations (2.4) reduce to $h_{\mu 0} = 0$ and $h_{ij} = \delta_{ij}$; thus $h^\flat = \delta_{ij}\theta^i \otimes \theta^j$. This is a Riemannian metric on the leaves of the foliation defined by τ. Clearly, the restriction of $h^\flat$ on an integral manifold is independent of V, because this is just the inverse of the restriction of the metric h.

One can show that the integral manifolds (sections of constant time) are totally geodesic for any symmetric Galilei connection and that the induced connection on a leave coincides with the Levi-Cività connection corresponding to $h^\flat$.

The Newton-Cartan theory of gravity involves special symmetric Galilei connections of the form

$$\Gamma^\mu_{\alpha\beta} = {}^V\Gamma^\mu_{\alpha\beta} + S^\mu_{\alpha\beta}, \tag{2.7}$$

where $S^\mu_{\alpha\beta}$ is given by (2.3) with $d\kappa = 0$. Such connections will be called **Newtonian**.

We need also a characterization of locally flat Galilei spacetimes.

Proposition 3. *For a Galilei manifold (M, h, τ) with symmetric Galilei connection Γ the following statements are equivalent:*

(i) *the Galilei manifold is locally flat;*

(ii) $R^{\mu\nu} := h^{\mu\rho}h^{\nu\sigma}R_{\rho\sigma} = 0$;

(iii) $R_{\mu\nu} = \alpha_{(\mu}\tau_{\nu)}$ *for some 1-form α.*

Recalling that any Newtonian connection can be expressed in terms of the Galilei metric (h, τ), a timelike unit vector field V and a closed 2-form κ, the question arises, when – for a given Galilei metric – a change of κ and V does not affect the Newtonian connection. One can show that this **Newtonian gauge group** is given by

$$
\begin{aligned}
V^\mu &\longmapsto V^\mu + h^{\mu\nu} w_\nu, \\
A_\mu &\longmapsto A_\mu + f_{,\mu} + w_\mu - (V^\nu w_\nu + \frac{1}{2} h^{\nu\lambda} w_\nu w_\lambda)\tau_\mu,
\end{aligned}
\tag{2.8}
$$

where f is a smooth function and w, A are 1-forms with $\kappa = \frac{1}{2} dA$. (For an elegant proof see [16].)

For many purposes it is useful to work in **adapted coordinates**: As a consequence of the Frobenius theorem for the integrable distribution defined by τ, we can introduce local coordinates (t, x^1, x^2, x^3) in the neighborhood of any spacetime point such that $\tau = dt$ and $\tau(\partial_i) = 0$. The integral manifolds are then the slices of constant t (**absolute time**). Furthermore, the condition $h(\cdot, \tau) = 0$ implies that $h = h^{ij} \partial_i \otimes \partial_j$. In adapted coordinates $\{x^\mu\}$ with $(x^0 \equiv t)$

$$
\tau = dx^0, \qquad h = h^{ij} \partial_i \otimes \partial_j \tag{2.9}
$$

and the timelike unit vector field

$$
V = \partial_0, \tag{2.10}
$$

the expressions for the Christoffel symbols of a symmetric Galilei connection become

$$
\begin{aligned}
\Gamma^0_{\alpha\beta} &= 0, & \Gamma^a_{0b} &= h^{ac}(\kappa_{bc} + \tfrac{1}{2} h_{bc,0}) \\
\Gamma^a_{00} &= 2h^{ab}\kappa_{0b}, & \Gamma^a_{bc} &= \tfrac{1}{2} h^{ad}(h_{db,c} + h_{dc,b} - h_{bc,d}).
\end{aligned}
\tag{2.11}
$$

Here (h_{ij}) is the inverse matrix of (h^{ij}), in other words $h^\flat = h_{ij} dx^i \otimes dx^j$. The last equation in (2.11) proves our previous statement, that the induced connection on the slices of constant time is the Levi-Cività connection for the restrictions of $h^\flat$.

In addition to the **space metric** h we introduce the **time metric** $g = \tau \otimes \tau$. Clearly,

$$
g_{\alpha\beta} h^{\beta\gamma} = 0. \tag{2.12}
$$

In contrast to GR, the two (degenerate) metrics h and g are not the inverses of each other.

3 The Newton-Cartan theory

After these geometrical preparations we can now formulate the Newton-Cartan theory in a form which emphasizes its close analogy with GR. The theory consists of three parts:

I Spacetime is a Galilei manifold (M, h, τ), with a Newtonian connection Γ.

II Matter is described in part by a symmetric contravariant energy momentum tensor $T^{\alpha\beta}$ with vanishing covariant divergence (relative to Γ):

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{3.1}$$

III The field equations are

$$R_{\alpha\beta} = 8\pi G(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T) - \Lambda g_{\alpha\beta}, \tag{3.2}$$

where $T_{\alpha\beta} := g_{\alpha\sigma}g_{\beta\rho}T^{\sigma\rho}$, $T := g_{\sigma\rho}T^{\sigma\rho}$.

In this formulation we have basically replaced the Lorentz group by the Galilei group. Several remarks are in order.

First, it has to be emphasized, that (3.1) is *not* a consequence of the field equations. This is related to the fact that a Galilei metric does not fix the connection.

The specialization to a Newtonian connection lookes somewhat mysterious. There is an equivalent formulation of this in terms of a symmetry of the Riemann tensor [7]:

$$\alpha(R(\beta^\sharp, X)Y) = \beta(R(\alpha^\sharp, Y)X) \tag{3.3}$$

for any covectors α, β and vectors X, Y; $\sharp$ denotes the map $\alpha \mapsto \alpha^\sharp = h(\cdot, \alpha)$. In index notation (3.3) reads

$$h^{\gamma\rho}R^\alpha_{\beta\rho\delta} = h^{\alpha\rho}R^\gamma_{\delta\rho\beta}. \tag{3.4}$$

In GR, where $h^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$, this symmetry is automatically satisfied. Since the Galilei metric does not fix the connection, we have the freedom to impose (3.4) as a further restriction.

The field equations, which can also be written in the form ($g_{\alpha\beta} = \tau_\alpha\tau_\beta$)

$$R_{\alpha\beta} = 4\pi G\rho\tau_\alpha\tau_\beta - \Lambda\tau_\alpha\tau_\beta, \qquad \rho := T = \tau_\alpha\tau_\beta T^{\alpha\beta}, \tag{3.5}$$

allow us to introduce **Galilei coordinates**: Clearly (3.5) implies $R^{\alpha\beta} = 0$ and thus by Proposition 3 the Galilei manifold is locally flat. We can therefore specialize the adapted coordinate conditions (2.9) even further such that

$$\tau = dx^0, \qquad h = \delta^{ij}\partial_i \otimes \partial_j. \tag{3.6}$$

In adapted coordinates we have $R_{ij} = 0$ as a consequence of the field equations, which also implies that the threedimensional time slices are locally flat.

In Galilei coordinates the Christoffel symbols (2.11) simplify to

$$\Gamma^0_{\alpha\beta} = 0, \qquad \Gamma^a_{00} = 2h^{ac}\kappa_{0c}, \qquad \Gamma^a_{0b} = h^{ac}\kappa_{bc}, \qquad \Gamma^a_{bc} = 0. \tag{3.7}$$

The Newton-Cartan theory is slightly more general than Newtons theory of gravitation. This can be seen by writing the field equations (3.5) for $\Lambda = 0$ in Galilei coordinates. Inserting (3.7) one finds

$$2\kappa_{0\ ,j}^{\ j} - \kappa_{ij}\kappa^{ij} = 4\pi G\rho, \tag{3.8}$$

and

$$\kappa_{i\ ,j}^{\ j} = 0. \tag{3.9}$$

In addition to this we also have $d\kappa = 0$. We would obtain Newton's theory if the Galilei coordinates could be choosen such that $\kappa_{ij} = 0$. (Note that we can still perform time dependent rotations and translations.) Now, one can show [6] that this is possible if and only if the following nonlinear condition for the Riemann tensor is imposed

$$h^{\gamma\rho} R^{\alpha}_{\beta\gamma\delta} R^{\beta}_{\alpha\rho\lambda} = 0. \tag{3.10}$$

Relative to Galilei coordinates which satisfy also $\kappa_{ij} = 0$, we obtain for $\vec{g} = -2(\kappa_{01}, \kappa_{02}, \kappa_{03})$ from (3.8) and $d\kappa = 0$ the basic equations of the Newtonian theory:

$$\text{div}\vec{g} = -4\pi G\rho, \qquad \text{curl}\vec{g} = 0. \tag{3.11}$$

Ehlers has shown [6], that the strange condition (3.10) can be deduced from a spatial boundary condition at infinity which can naturally be imposed for the description of isolated systems.

One advantage of the geometrical formulation of the Newton-Cartan theory is that the spatial sections can also be chosen as flat tori. This enables us to describe spatially compact cosmological models. Some cosmological aspects will be presented later.

Finally note that equation (3.8) reads (including the cosmological term)

$$\text{div}\vec{g} = -4\pi G\rho + \Lambda + \kappa_{ij}\kappa^{ij}. \tag{3.12}$$

This shows that $\kappa_{ij}\kappa^{ij}$ acts (like a positive Λ) as a repulsive source.

4 Fluid models in the Newton-Cartan theory

This section serves mainly as a preparation for our later discussion of Newtonian cosmology.

We introduce again a distinguished timelike unit vector field V on the Galilei manifold (M, h, τ) with time metric $g = \tau \otimes \tau$. The integral curves of V define a family of fundamental observers. Note that $\tau(V) = 1$ translates into $\tau_\alpha = g_{\alpha\beta} V^\beta$. The matter model is assumed to be an ideal fluid with four velocity u, which is also a timelike unit vector field. We begin with some kinematical considerations which are familiar in GR.

It is useful to introduce the projection operator $P : T_xM \to S_x := \ker\tau_x$ from the tangent spaces onto the horizontal (i.e., spacelike) subspaces definied by

$$P(X) = X - g(X, V)V. \tag{4.1}$$

Clearly, $P(V) = 0$ and $\tau(P(X)) = 0$. The components of P are

$$P^\nu_\mu = \delta^\nu_\mu - g_{\mu\lambda}V^\lambda V^\nu. \tag{4.2}$$

As before, $h_{\mu\nu}$ denotes the components of $h^\flat$. We have the identities

$$\begin{aligned} &P^\nu_\mu = h_{\mu\lambda}h^{\lambda\nu}, \quad P^\mu_\lambda h^{\lambda\nu} = h^{\mu\nu}, \quad P^\lambda_\mu h_{\lambda\nu} = h_{\mu\nu}, \\ &P^\lambda_\mu P^\nu_\lambda = P^\nu_\mu, \quad P^\sigma_\rho P^\rho_\sigma = 3. \end{aligned} \tag{4.3}$$

For the covariant derivatives of u and V one verifies readily the following facts:

$$\begin{aligned} &g_{\mu\lambda}V^\lambda_{;\nu} = 0, \qquad g_{\mu\lambda}u^\lambda_{;\nu} = 0, \\ &\nabla_X V \text{ and } \nabla_X u \text{ are horizontal}, \\ &P^\mu_\lambda V^\lambda_{;\nu} = V^\mu_{;\nu}, \qquad P^\mu_\lambda u^\lambda_{;\nu} = u^\mu_{;\nu}. \end{aligned} \tag{4.4}$$

The **vorticity** (relative to V) is the skew symmetric bilinear form

$$\Omega(X, Y) = \frac{1}{2}[h(\nabla_{P(Y)}u, P(X)) - h(\nabla_{P(X)}u, P(Y))] \tag{4.5}$$

and the (rate of) **strain** is

$$\Theta(X, Y) = \frac{1}{2}[h(\nabla_{P(Y)}u, P(X)) + h(\nabla_{P(X)}u, P(Y))]. \tag{4.6}$$

The **expansion** rate is

$$\theta = h^{\alpha\beta}\Theta_{\alpha\beta} \tag{4.7}$$

and the (rate of) **shear** is the trace-free part of the strain

$$\sigma(X, Y) = \Theta(X, Y) - \frac{1}{3}\theta h(X, Y). \tag{4.8}$$

While these quantities have the usual interpretation for the fluid motion relative to V, they are, unfortunatly, *not tensoriel.* They are, however, simply related to the contravariant tensor fields $\Omega^\sharp, \Theta^\sharp$ with components

$$\Omega^{\alpha\beta} = \frac{1}{2}(u^\alpha_{;\lambda}h^{\lambda\beta} - u^\beta_{;\lambda}h^{\lambda\alpha}), \tag{4.9}$$

$$\Theta^{\alpha\beta} = \frac{1}{2}(u^\alpha_{;\lambda}h^{\lambda\beta} + u^\beta_{;\lambda}h^{\lambda\alpha}). \tag{4.10}$$

Indeed, the components of (4.5) and (4.6) are given by

$$\Omega_{\alpha\beta} = h_{\alpha\rho} h_{\beta\sigma} \Omega^{\rho\sigma}, \qquad \Theta_{\alpha\beta} = h_{\alpha\rho} h_{\beta\sigma} \Theta^{\rho\sigma}. \tag{4.11}$$

With (4.3) and (4.4) one finds that θ is simply given by

$$\theta = u^{\sigma}_{;\sigma} \tag{4.12}$$

and the covariant derivative of the velocity field can be decomposed as follows

$$h_{\alpha\lambda} u^{\lambda}_{;\beta} = \Theta_{\alpha\beta} + \Omega_{\alpha\beta} + h_{\alpha\rho} V^{\lambda} u^{\rho}_{;\lambda} g_{\beta\sigma} V^{\sigma} \tag{4.13}$$

With the help of (4.13) we can now derive a Raychaudhuri equation in the Newton-Cartan theory. As in GR we start from the identity

$$u^{\alpha}_{;\beta;\gamma} - u^{\alpha}_{;\gamma;\beta} = R^{\alpha}_{\sigma\gamma\beta} u^{\sigma}$$

which gives

$$u^{\beta} u^{\alpha}_{;\alpha;\beta} = (u^{\beta} u^{\alpha}_{;\beta})_{;\beta} - u^{\beta}_{;\alpha} u^{\alpha}_{;\beta} - R_{\alpha\beta} u^{\alpha} u^{\beta}. \tag{4.14}$$

With the help of (4.13) and the identities collected in (4.3) and (4.4) one can write the second term on the right as follows

$$u^{\alpha}_{;\beta} u^{\beta}_{;\alpha} = h^{\rho\alpha} h^{\sigma\beta} (\Theta_{\rho\beta} \Theta_{\sigma\alpha} + \Omega_{\rho\beta} \Omega_{\sigma\alpha}). \tag{4.15}$$

The first term on the right in (4.14) is

$$(u^{\beta} u^{\alpha}_{;\beta})_{;\alpha} = \mathrm{div}(\nabla_u u). \tag{4.16}$$

After a few steps (see [13]), we arrive at the following two equivalent forms of the **Raychaudhuri equation**

$$\begin{aligned} \mathrm{div}(\nabla_u u) &= \nabla_u \theta + \frac{1}{3}\theta^2 + h^{\alpha\rho} h^{\beta\sigma} (\sigma_{\rho\sigma}\sigma_{\alpha\beta} - \Omega_{\rho\sigma}\Omega_{\alpha\beta}) + \mathrm{Ric}(u, u) \\ &= \nabla_u u + \frac{1}{3}\theta^2 + h_{\alpha\rho} h_{\beta\sigma} (\sigma^{\rho\sigma}\sigma^{\alpha\beta} - \Omega^{\rho\sigma}\Omega^{\alpha\beta}) + \mathrm{Ric}(u, u). \end{aligned} \tag{4.17}$$

Note that these equations hold for any Galilei manifold with a symmetric Galilei connection.

At this point we use the field equations (3.5) and obtain (with $\tau_\alpha \tau_\beta = g_{\alpha\sigma} g_{\beta\rho} u^{\sigma} u^{\rho}$)

$$\mathrm{div}(\nabla_u u) = \nabla_u \theta + \frac{1}{3}\theta^2 + h^{\alpha\rho} h^{\beta\sigma} (\sigma_{\rho\sigma}\sigma_{\alpha\beta} - \Omega_{\rho\sigma}\Omega_{\alpha\beta}) + 4\pi G \rho - \Lambda. \tag{4.18}$$

This equation will play an important role.

Now we consider an ideal fluid with the energy momentum tensor

$$T = \rho u \otimes u + p h. \tag{4.19}$$

From $\nabla T = 0$ one obtains the continuity equation

$$\operatorname{div}(\rho u) = 0 \tag{4.20}$$

and the Euler equation

$$\nabla_u u = -\frac{1}{\rho}\operatorname{div}(ph). \tag{4.21}$$

In contrast to GR, equation (4.20) is a conservation law, because it is for a symmetric Galilei connection equivalent to (L_u denotes the Lie derivative with respect to u)

$$\mathrm{L}_u(\rho \mathrm{vol}) = 0, \tag{4.22}$$

where vol is the standard volume $\tau \wedge \mathrm{vol}_3$, vol_3 being the Riemannian volume form of the spatial slices. (This equivalence can easily be verified in adapted coordinates.) Thus the integral of ρvol over a comoving domain remains constant.

We mention that it is possible to derive the Raychaudhuri equation (4.18) also from the Euler equation and the field equation [13]. (This is closer to what one does in nonrelativistic fluid dynamics.) The two quite different derivations reflect some kind of consistency between field and matter equations.

As an application of (4.18) we now show, that there are *no static dust solutions* in the Newton-Cartan theory for $\Lambda = 0$ and that for $\Lambda > 0$ there is just one static solution, which corresponds to the **Einstein universe**.

By definition a **static** velocity field u is one with vanishing vorticity,

$$\Omega^\sharp \; (= \Omega^{\alpha\beta}\partial_\alpha \otimes \partial_\beta) = 0, \tag{4.23}$$

and for which the Lie derivatives of the expansion and the strain vanish:

$$\mathrm{L}_u\theta = 0, \qquad \mathrm{L}_u\Theta^\sharp = 0 \qquad (\Theta^\sharp = \Theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta). \tag{4.24}$$

Indeed, assume that there is no pressure term in (4.19), then (4.21) reduces to $\nabla_u u = 0$. Using also the staticity conditions in the Raychaudhuri equation (4.18), we find

$$4\pi G\rho = \Lambda - h^{\alpha\rho}h^{\beta\sigma}\sigma_{\rho\sigma}\sigma_{\alpha\beta} - \frac{1}{3}\theta^2. \tag{4.25}$$

This equation has for $\Lambda = 0$ obviously no solution with $\rho > 0$. (Note, we have not used the second equation of (4.24) to arrive at this conclusion.)

Consider next the case $\Lambda > 0$. If we write (4.25) in terms of Galilei coordinates, we obtain

$$4\pi G\rho = \Lambda - \frac{1}{2}[u^i_{,j}u^j_{,i} + \sum_{i,j}(u^i_{,j})^2]. \tag{4.26}$$

In such coordinates one has (with the first equation in (4.24))

$$h^{ij}(L_u\Theta^\sharp)_{ij} = u^i_{,j}u^j_{,i} + \sum_{i,j}(u^i_{,j})^2,$$

and this vanishes by the second equation of (4.24). Thus the density ρ satisfies the relation

$$\rho = \frac{\Lambda}{4\pi G} \tag{4.27}$$

of the Einstein universe.

These conclusions hold in particular for Newtonian cosmological dust models. It has to be emphasized that we have not made any symmetry assumptions (apart from staticity). A very similar argument works also in GR [13].

5 Newton-Cartan cosmology

It is very fortunate that the post-recombination universe can be described largely in the Newtonian approximation. This brings enormous simplifications in treating the problems of structure formation, in particular in the nonlinear regime. Thanks to this circumstance, we can for instance use N-body simulations.

We consider this as a motivation (beside others) to put Newtonian cosmology on a conceptually firm basis. This can readily be achieved in the framework of the geometrical formulation of the Newton-Cartan theory that we have described in the previous sections. Again, the analogy to GR is striking. To illustrate this, we consider first homogeneous cosmological models and then develop the cosmological perturbation theory of Friedmann-Lemaitre models.

5.1 Homogeneous cosmological models

In analogy to the discussion of homogeneous cosmological models in GR (for an introduction see [17]) we consider first the geometrical aspect, without imposing the field equations. Spacetime is then described by a Galilei manifold (M, h, τ) with a symmetric Galilei connection Γ. We introduce adapted coordinates (see equations (2.9)). The spatial coordinates $\{x^i\}$ parametrize the slices $\sum_t$ of constant time on which h induces the Riemannian metric $h^\flat = h_{ij}dx^i \otimes dx^j$. We choose again $V = \partial_t$.

Let us assume now that there is a free isometric left action of a 3-dimensional Lie group G on the slices $\sum_t$ with G on which $h^\flat$ defines a time-dependent family of Riemannian metrics. Relative to a left invariant basis $\{\theta^a\}$ of G this family is of the form $h^\flat = h_{ab}(t)\theta^a \otimes \theta^b$.

Using $\nabla g = \nabla h = 0$ ($g = \tau \otimes \tau = dt \otimes dt$), Cartans structure equations for the connection and the Maurer-Cartan equations for the Lie group G, one can then work out the Ricci tensor for all Bianchi types, with the result given in [13]. Here, we consider only the Bianchi type I, because the field equations imply that the $\sum_t$ are flat. The metric homogeneity is thus not an additional restriction in the Newton-Cartan theory.

For the choice $\theta^a = dx^a$ we can compute the Ricci tensor also directly with the help of (2.11) and set up the field equations (3.5). The result is

$$\begin{aligned} R_{00} &= -\frac{1}{2}(h^{ij}\dot{h}_{ij})_{,0} - \frac{1}{4}h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li} + 2h^{ij}\kappa_{0j,i} + \kappa^{ij}\kappa_{ij} \\ &= 4\pi G\rho - \Lambda, && (5.1) \\ R_{0i} &= h^{jk}\kappa_{ik,j} = 0. && (5.2) \end{aligned}$$

R_{ij} vanishes identically. Equation (5.1) is obviously equivalent to the Raychaudhuri equation (4.18) for $u = V$. The latter reads in adapted coordinates for any velocity field u

$$\mathrm{div}\nabla_u u = \nabla_u \theta + \frac{1}{3}\theta^2 + \sigma^{ab}\sigma_{ab} - \Omega^{ab}\Omega_{ab} + 4\pi G\rho - \Lambda. \tag{5.3}$$

The other field equation (5.2) is equivalent to $\Omega^{ij}{}_{,j} = 0$ for $u = V$, since for any u

$$\Omega_{ab} = \frac{1}{2}[h_{ac}u^c_{,b} - h_{bc}u^c_{,a} - 2\kappa_{ab}]. \tag{5.4}$$

We give also the expressions for the other kinematical quantities:

$$\begin{aligned} \Theta_{ab} &= \frac{1}{2}[h_{ac}u^c_{,b} + h_{bc}u^c_{,a} + \dot{h}_{ab}], && (5.5) \\ \theta &= u^a_{,a} + \frac{1}{2}h^{ab}\dot{h}_{ab}, && (5.6) \\ \mathrm{div}(\nabla_u u) &= \dot{u}^a_{,a} + 2h^{ab}\kappa_{0a,b} + 2h^{bc}\kappa_{ac}u^a_{,b} + h^{bc}u^a_{,b}\dot{h}_{ac} + u^a u^b_{,ab} + u^a_{,b}u^b_{,a}. && (5.7) \end{aligned}$$

Beside this the 2-form κ is assumed to be closed (Newtonian connection).

Matter is assumed to be an ideal fluid with energy momentum tensor (4.19). In adapted coordinates the continuity equation (4.20) and the Euler equation (4.21) become

$$\dot{\rho} + (\rho u^i)_{,i} + \frac{1}{2}h^{ij}\dot{h}_{ij}\rho = 0 \tag{5.8}$$

and

$$\dot{u}^i + u^j u^i_{,j} + 2h^{ij}\kappa_{0j} + 2u^j(\frac{1}{2}h^{ik}\dot{h}_{kj} + h^{ik}\kappa_{jk}) + \frac{1}{\rho}h^{ij}p_{,j} = 0. \tag{5.9}$$

Unlike as in GR, we cannot conclude from our basic equations that the physical quantities like ρ and p are only functions of time. The reason is clear: As already emphasized, the field equations imply that spacetime has to be of Bianchi type I.

Let us specialize the field and matter equations to Newtonian gravity, characterized by condition (3.10). We can then introduce Galilei coordinates such that $\kappa_{ij} = 0$. Relative to these equations (5.8), (5.9) and (5.1) reduce to

$$\dot{\rho} + (\rho u^i)_{,i} = 0,$$
$$\dot{u}^i + u^j u^i_{,j} = -\tfrac{1}{\rho} p_{,i} + g^i, \quad (5.10)$$
$$-g^i_{,i} = 4\pi G\rho - \Lambda,$$

where $g^i = -2\kappa_{oi}$ (as in (3.11)). We thus arrive at the traditional equations for Newtonian gravity, coupled to an ideal fluid.

Let us go back to the Newton-Cartan theory and assume now that ρ and p are – in adapted coordinates – only functions of t. The Euler equation (4.21) implies then $\nabla_u u = 0$ and the continuity equation (5.8) shows that $u^i_{,i}$ depends only on t. This leads to the following simplification of (5.3)

$$\dot{\theta} + \frac{1}{3}\theta^2 + \sigma^{ab}\sigma_{ab} - \Omega^{ab}\Omega_{ab} + 4\pi G\rho - \Lambda = 0. \quad (5.11)$$

Here we have used that θ is also only a function of t, because (5.8) and (5.6) imply

$$\dot{\rho} + \theta\rho = 0. \quad (5.12)$$

Specializing again to Newtonian gravity, we can reach stronger conclusions.

Lemma. *If $\Omega^\sharp + \Theta^\sharp$ is also translation invariant, then there exist for Newtonian gravity Galilei coordinates relative to which the spatial components of $u = \partial_0 + u^i\partial_i$ are linear functions of the x^j:*

$$u^i = a^i_j(t)x^j + b^i(t). \quad (5.13)$$

Proof. We know that we can introduce Galilei coordinates such that $\kappa_{ij} = 0$. Since $L_{\partial_k}(\Omega^\sharp + \Theta^\sharp) = 0$ implies that $(\Omega_{ij} + \Theta_{ij})_{,k} = 0$, equations (5.5) and (5.6) show that $u^i_{,jk} = 0$. □

With a time dependent translation we can pass to Galilei coordinates for which the inhomogeneity in (5.13) disappears. We are now in a situation which has been discussed in classic papers by Heckmann and Schücking [10, 11].

We consider finally homogeneous and isotropic Friedmann-Lemaitre models in the Newton-Cartan theory. We find these with the ansatz

$$V = u, \qquad h_{ij} = a^2(t)\delta_{ij}, \qquad \Omega^\sharp = 0. \quad (5.14)$$

From the remark connected to equation (2.6) it is clear, that the symmetric Galilei connection is now fixed and given by ${}^V\Gamma^\mu_{\alpha\beta}$. Furthermore, we find

$$\theta = 3\frac{\dot{a}}{a}, \qquad \sigma_{ab} = 0, \qquad \dot{\theta} = 3\frac{\ddot{a}a - \dot{a}^2}{a^2}, \quad (5.15)$$

and the basic equations (5.11), (5.12) reduce to

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}\rho, \quad (5.16)$$

$$\ddot{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}. \quad (5.17)$$

Equation (5.16) implies

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3, \quad (5.18)$$

and when this is used in (5.17) we obtain the Friedmann equation

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{3}a^2, \quad (5.19)$$

in which the integration constant k can be choosen to be $k = 0, \pm 1$.

In the next section we discuss the perturbation theory of these homogeneous and isotropic solutions in the framework of the Newton-Cartan theory.

5.2 Cosmological perturbation analysis in the Newton-Cartan theory

We consider cosmological models deviating only by a small amount from a Friedmann-Lemaitre universe, which is defined to be the background. Correspondingly we split all geometric and matter variables into their background values, indexed by $^{(0)}$, and small deviations δp, $\delta\rho$, $\delta\kappa$, etc.

The Galilei metric (h, τ) is kept fixed. This determines the part ${}^V\Gamma$, given in (2.5), of the symmetric Galilei connection. Because this is just the background connection, we have $\kappa^{(0)} = 0$. The perturbation of the connection is entirely described by $\delta\kappa$. We also note that $g(u,u) = 1$ $(\tau(u) = 1)$ requires that the four velocity field is of the form

$$u = \partial_0 + \delta u^i \partial_i. \quad (5.20)$$

Inserting all this into the field and matter equations leads to a set of perturbation equations for $\delta\rho$, δp, δu^i and $\delta\kappa$ which are still exact. In writing them down, we drop the variational symbol δ and use the notation

$$\rho = \rho^{(0)}(1 + D) \quad (5.21)$$

In [13] the following complete set of perturbation equations is derived:

$$\dot{D} + [u^i(1 + D)]_{,i} = 0, \quad (5.22)$$

$$\dot{u}^i + u^j u^i_{,j} + 2\frac{\dot{a}}{a}u^i = -\frac{1}{\rho}h^{ij}p_{,j} - 2h^{ij}\kappa_{0j} + 2h^{ij}\kappa_{jl}u^l, \quad (5.23)$$

$$2h^{ij}\kappa_{0j,i} = -\kappa^{ij}\kappa_{ij} + 4\pi G\rho^{(0)}D, \tag{5.24}$$

$$d\kappa = 0, \tag{5.25}$$

$$h^{jl}\kappa_{ij,l} = 0. \tag{5.26}$$

These agree for $\kappa_{ij} = 0$ with the usual Newtonian perturbation equations (see, e.g., [18]). (In making this comparison one has to note, that the peculiar velocity field v^i is usually defined by $v^i = a(t)u^i$. The gravitational field is again given by $\kappa_{0i} : g^i = -2\kappa_{0i}$.)

Linearization of the perturbation equations gives

$$\dot{D} + u^i_{,i} = 0, \tag{5.27}$$

$$\dot{u}^i + 2\frac{\dot{a}}{a}u^i = -\frac{1}{\rho^{(0)}}h^{ij}p_{,j} - 2h^{ij}\kappa_{0j}, \tag{5.28}$$

$$2h^{ij}\kappa_{0j,i} = 4\pi G\rho^{(0)}D, \tag{5.29}$$

$$d\kappa = 0, \qquad h^{jl}\kappa_{ij,l} = 0. \tag{5.30}$$

Eliminating u^i we arrive at the well-known perturbation equation for the density fluctuations:

$$\ddot{D} + 2\frac{\dot{a}}{a}D = \frac{1}{\rho^{(0)}}h^{ij}p_{,ij} + 4\pi G\rho^{(0)}D. \tag{5.31}$$

From (5.22) - (5.26) we can derive in a standard manner (exact) perturbation equations for vorticity and shear. One equation agrees with the Raychaudhuri equation for the perturbations:

$$\nabla_u\theta + \Theta^{ij}\Theta_{ij} - \Omega^{ij}\Omega_{ij} + \left(\frac{1}{\rho}h^{ij}p_{,j}\right)_{,i} + 4\pi G\rho - \Lambda = 0. \tag{5.32}$$

For the vorticity one finds [13]

$$(\nabla_u\Omega)_{ij} + h^{\alpha\beta}\Omega_{\beta j}\Theta_{i\alpha} - h^{\alpha\beta}\Omega_{\alpha i}\Theta_{\beta j} = p_{,[j}\left(\frac{1}{\rho}\right)_{,i]}. \tag{5.33}$$

(We have again dropped the variational symbol on Ω, Θ, u, p; but ρ is the total density.)

We conclude this discussion by writing the exact perturbation equations (5.22) - (5.26) in a covariant form:

$$\nabla_V D + \text{div}[(1+D)(\delta^\beta_\alpha - g_{\alpha\lambda}V^\lambda V^\beta)u^\alpha\partial_\beta] = 0, \tag{5.34}$$

$$\nabla_u u = -\frac{1}{\rho}\text{div}ph, \tag{5.35}$$

$$\text{div}\nabla_V V = -h^{\alpha\gamma}h^{\beta\delta}\kappa_{\alpha\beta}\kappa_{\gamma\delta} + 4\pi G\rho^{(0)}D, \tag{5.36}$$

$$d\kappa = 0, \tag{5.37}$$

$$h^{\alpha\beta}\nabla_\alpha[(\delta^\sigma_\gamma - g_{\gamma\delta}V^\delta V^\sigma)\kappa_{\sigma\beta}] = 0. \tag{5.38}$$

6 Concluding remarks

The Newton-Cartan theory can sometimes provide useful insights for problems in GR. An interesting example concerns the cosmic no-hair conjecture, which is not yet settled in sufficient generality within GR. Bauer et al [19] were, however, able to prove satisfactory theorems in the framework of the Newton-Cartan theory. For ideal fluid models they showed that solutions corresponding to nearly homogeneous initial data for a compact time slice exist in the case $\Lambda > 0$ for all positive times and that the difference between the inhomogeneous and homogeneous solutions tends to zero in a strong sense. Perturbations are thus strongly damped. Presumably a corresponding nonlinear stability property holds also in GR, but this appears very difficult to prove.

The geometrical formulation of the Newton-Cartan theory has also played a useful role in rigorous discussions of the Newtonian limit of GR [6]. The starting point is the observation by Ehlers that both theories fit naturally into a larger frame theory with two metrics $h^{\alpha\beta}$, $g_{\alpha\beta}$ related by $g_{\alpha\sigma}h^{\sigma\beta} = -\lambda\delta_{\alpha}^{\beta}$ ($\lambda = 1$ for GR and $\lambda = 0$ for the Newton-Cartan theory).

This frame theory has also played a remarkable role in the work of Heilig [20] for establishing rigorous existence theorems in GR for solutions which describe rotating stars.

Several more formal aspects have been studied by Künzle and collaborators. An example is the generalization of the Galilei invariant spin-$\frac{1}{2}$-wave equation to a curved Newton spacetime [16, 21].

Finally, without being complete, we mention that Duval et al [22] have obtained the Newton-Cartan theory through a dimensional reduction of a Kaluza-Klein theory along a null vector.

All this demonstrates once more the remarkable continuity in the development of theoretical physics. The word "revolution" rarely deserves to be used in this context. To our knowledge, it appears in Einstein's writings only once, namely in connection with his hypothesis of the light quantum [24]. He did not regard its use to be appropriate in all his work on special and general relativity.

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