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Autor: Hoppe, Jens

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On M-Algebras, the Quantisation of Nambu-Mechanics, and Volume Preserving Diffeomorphisms¹

Jens Hoppe ²
Theoretische Physik
ETH-Hönggerberg
8093 Zürich
Schweiz

Dedicated to K. Hepp and W. Hunziker on the occasion of their 60th birthdays.

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Abstract: M-branes are related to theories on function spaces \mathcal{A} involving M-linear non-commutative maps from $\mathcal{A} \times \cdots \times \mathcal{A}$ to \mathcal{A} . While the Lie-symmetry-algebra of volume preserving diffeomorphisms of T^M cannot be deformed when M > 2, the arising M-algebras naturally relate to Nambu's generalisation of Hamiltonian mechanics, e.g. by providing a representation of the canonical M-commutation relations, $[J_1, \cdots, J_M] = i\hbar$. Concerning multidimensional integrability, an important generalisation of Lax-pairs is given.

¹ETH-TH/95-33

²Heisenberg Fellow. On leave of absence from Karlsruhe University.

1. Introduction

Generalizing fundamental concepts, such as Lie algebras or Hamiltonian dynamics, may have quite divers merits; it can lead to new, interesting possibilities, – or reassure oneself of our present notions. While the result that volume preserving diffeomorphisms of toroidal M-branes, as a Lie-symmetry algebra, cannot be deformed (if M>2) is of the latter nature – the following ideas appear to be worthwhile persueing:

- Using a *M-deformation of the algebra of functions on some M-dimensional manifold for representing the M-linear analogue to Heisenberg's commutation relations that Nambu [1] encountered in multi-Hamiltonian dynamics.
- Generalizing the Jacobi identity for Lie algebras to a (2-bracket) identity involving 2M-1 elements of a vectorspace V for which an antisymmetric M-linear map (M-commutator) from $V \times \cdots \times V$ to V is defined (in a dynamical context, an identity involving M, rather than 2, of the M-commutators, may be preferred).
- A potential relevance of M-algebras to the quantisation of space-time.

Perhaps most importantly (on a concrete, practical level), an explicit example is given (the multidimensional diffeomorphism-invariant integrable field theories found in [2]) for the usefulness (envisaged some time ago [3]) of generalizing Lax-pairs to -triples,

2. M-algebras from M-branes

A relativistic M-brane moving in D-dimensional space time may be described, in a light-cone gauge, by the VDiff Σ -invariant sector of ([4])

$$H = \frac{1}{2} \int_{\Sigma} \frac{d^{M} \varphi}{\rho(\varphi)} \left(\vec{p}^{2} + g \right) \tag{1}$$

where g is the determinant of the M×M matrix $(g_{rs}) := (\nabla_r x^i \nabla_s x_i)_{r,s=1\cdots M}$, x^i and p_i $(i=1,\cdots,D-2=:d)$ are canonically conjugate fields, and ρ is a fixed non-dynamical density on the M-dimensional parameter-manifold Σ (M=1 for strings, M=2 for membranes,...). Generators of VDiff Σ , the group of volume-preserving diffeomorphisms of Σ (resp. the component connected to the identity), are represented by

$$K := \int_{\Sigma} f^r p_i \, \partial_r \, x^i \, d^M \, \varphi \tag{2}$$

with $\nabla_r f^r = 0$. g may be written as

$$g = \sum_{i_1 < i_2 < \dots < i_M} \{x_{i_1}, \dots, x_{i_M}\} \{x^{i_1}, \dots, x^{i_M}\}, \tag{3}$$

where the 'Nambu-bracket' $\{\cdots\}$ is defined for functions f_1, \cdots, f_M on Σ as

$$\{f_1, \cdots, f_M\} := \epsilon^{r_1 \cdots r_M} \partial_{r_1} f_1 \cdots \partial_{r_M} f_M.$$
 (4)

This trivial, but important observation suggests to consider Hamiltonians

$$H_{\lambda} := \frac{1}{2} Tr \Big(\vec{P}^{\,2} \pm \sum_{i_1 < \dots < i_M} [X_{i_1}, \dots, X_{i_M}]_{\lambda}^2 \Big), \tag{5}$$

resp.

$$H_{\lambda} = \frac{1}{2} \sum_{i=1}^{d} \beta (P_{i}, P_{i}) + \frac{1}{2} \sum_{i_{1} < \dots < i_{M}} \beta ([X_{i_{1}}, \dots, X_{i_{M}}]_{\lambda}, [X_{i_{1}}, \dots X_{i_{M}}]_{\lambda}),$$
 (6)

where X^i and P_i are elements of (possibly finite dimensional, λ -dependent) vectorspaces V on which antisymmetric M-linear maps $[, \dots,]_{\lambda} : V \times \dots \times V \to V$ are defined, and β a positive definite hermitean form, preferably invariant with respect to some analogue of volume preserving diffeomorphisms (cp. (2)).

With

$$[T_{a_1}, \cdots, T_{a_M}]_{\lambda} = f_{a_1 \cdots a_M}^a(\lambda) T_a \tag{7}$$

and

$$\beta(T_a, T_b) = \delta_b^a \tag{8}$$

for some (possibly λ -dependent) basis $\{T_a\}_{a=1}^{\dim V}$ of V, i.e.

$$f_{a_1 \cdots a_M}^a(\lambda) = \beta(T_a, [T_{a_1}, \cdots, T_{a_M}]_{\lambda}), \qquad (9)$$

(6) reads

$$H_{\lambda} = \frac{1}{2} p_{ia}^{*} p_{ia} + \frac{1}{2} (f_{a_{1} \cdots a_{M}}^{a}(\lambda))^{*} f_{b_{1} \cdots b_{M}}^{a}(\lambda)$$

$$\frac{1}{M!} x_{i_{1}a_{1}}^{*} \cdots x_{i_{M}a_{M}}^{*} x_{i_{1}b_{1}} \cdots x_{i_{M}b_{M}},$$

$$(10)$$

while (1) may be written as

$$H = \frac{1}{2} p_{i\alpha}^* p_{i\alpha} + \frac{1}{2} (g_{\alpha_1 \cdots \alpha_M}^{\alpha})^* g_{\beta_1 \cdots \beta_M}^{\alpha} + \frac{1}{M!} x_{i_1 \alpha_1}^* \cdots x_{i_M \beta_M};$$
(11)

$$g^{\alpha}_{\alpha_1 \cdots \alpha_M} := \int_{\Sigma} Y^*_{\alpha} \{ Y_{\alpha_1}, \cdots, Y_{\alpha_M} \} \rho \ d^M \varphi \tag{12}$$

is defined with respect to some orthonormal basis of functions (on Σ) satisfying

$$\int Y_{\alpha}^{*} Y_{\beta} \rho d^{M} \varphi = \delta_{\beta}^{\alpha}$$

$$\alpha, \beta = 1 \cdots \infty$$
(13)

(even for real x_i , it is often convenient to take a complex basis). Obvious questions are:

1) Does there exist a 'natural' sequence of finite dimensional vectorspaces V_n with basis $\{T_a^{(n)}\}$ and antisymmetric maps $F_n: V_n \times \cdots \times V_n \to V_n$ such that for each (M+1)-tuple $(a\ a_1 \cdots a_M)$

$$\lim_{n \to \infty} f_{a_1 \cdots a_M}^a (\lambda_n) \stackrel{?}{=} g_{a_1 \cdots a_M}^a . \tag{14}$$

- 2) For which M do there exist finite dimensional analogues of (2), K(n), leaving $(10)_{\lambda_n}$ invariant, such that, as $n \to \infty$, the full invariance under volume-preserving diffeomorphisms is recovered?
- 3) What about λ -deformations with infinite dimensional V's ?

Let us look at the case of a M-torus, $\Sigma = T^M$: Choosing

$$Y_{\vec{m}} = e^{i\vec{m}\vec{\varphi}}, \ \vec{m} = (m_1, \dots, m_M) \in \mathbb{Z}^M, \ \rho \equiv 1,$$
 (15)

one gets

$$g_{\vec{m}_1 \cdots \vec{m}_M}^{\vec{m}} = i^M(\vec{m}_1, \cdots, \vec{m}_M) \, \delta_{\vec{m}_1 + \cdots + \vec{m}_M}^{\vec{m}}$$
 (16)

where $(\vec{m}_1, \dots, \vec{m}_M) \in \mathbb{Z}$ denotes the determinant of the corresponding $M \times M$ Matrix (an element of $GL(M, \mathbb{Z})$).

Consider now the following '*M-product' (a deformation of the ordinary commutative product of M functions f_1, \dots, f_M on Σ):

$$(f_{1}\cdots f_{M})_{*} := f_{1}\cdots f_{M} + \sum_{m=1}^{\infty} \frac{\left(\frac{(-i)^{M+1}\lambda}{M!}\right)^{m}}{\frac{m!}{m!}\cdots \frac{\epsilon^{r_{1}r'_{1}\cdots r'_{m}}}{\epsilon^{r_{m}r'_{m}\cdots r'_{m}}} \frac{\partial^{m}f_{M}}{\partial\varphi^{r_{1}}\cdots\partial\varphi^{r_{m}}} \cdots \frac{\partial^{m}f_{M}}{\partial\varphi^{r_{1}^{(M)}}\cdots\partial\varphi^{r_{m}^{(M)}}}.$$
(17)

One then finds that

$$(Y_{\vec{m}_1} \cdots Y_{\vec{m}_M})_* = \sqrt{\omega}^{-(\vec{m}_1, \cdots, \vec{m}_M)} Y_{\vec{m}_1 + \cdots \vec{m}_M}$$
$$\sqrt{\omega} = e^{i\frac{\lambda}{M!}}. \tag{18}$$

Defining

$$[f_1, \cdots, f_M]_* := \sum_{\sigma \in S_M} (\operatorname{sign} \sigma) (f_{\sigma 1} \cdots f_{\sigma M})_*$$
(19)

to simply be the antisymmetrized *M-product, one gets

$$[T_{\vec{m}_1}, \dots, T_{\vec{m}_M}] = \frac{-i}{2\pi\Lambda} \sin(2\pi\Lambda (\vec{m}_1, \dots, \vec{m}_M)) T_{\vec{m}_1 + \dots + \vec{m}_M}$$
 (20)

with
$$\Lambda := \frac{\lambda}{2\pi M!}$$
 and $T_{\vec{m}} := \lambda^{-\frac{1}{M-1}} Y_{\vec{m}}$.

For M > 1 arbitrary (but fixed), let V denote the vectorspace (over \mathbb{C}) generated by $\{T_{\vec{m}}\}_{\vec{m} \in \mathbb{Z}^M}$, \mathbb{M}^{Λ} denote (V, *) and \mathbb{A}^{Λ} denote $(V, [\cdots]_*)$.

The hermitean form β (cp. (8),(9)),

$$\beta(T_{\vec{m}}, T_{\vec{n}}) = \delta_{\vec{n}}^{\vec{m}}, \quad \beta(c_i X_i, d_j X_j) = c_i^* d_j \beta(X_i, X_j),$$

will have the important property ('invariance') that (for $X_i = x_{i\vec{m}}T_{\vec{m}}$ with $x_{i\vec{m}}^* = x_{i-\vec{m}}$)

$$\beta(X, [X_{i_1}, \cdots X_{i_M}]) = -\beta(X_{i_r}, [X_{i_1}, \cdots, X_{i_{r-1}}, X, X_{i_{r+1}}, \cdots, X_{i_M}]),$$

as

$$\beta \ (T_{\vec{m}}, [T_{\vec{m}_1}, \cdots, T_{\vec{m}_M}]) \ = \ \frac{-i}{2\pi\Lambda} \ \delta_{\vec{m}_1, + \cdots + \vec{m}_M}^{\vec{m}} \ \sin \left(2\pi\Lambda(\vec{m}_1, \cdots, \vec{m}_M)\right).$$

For rational $\Lambda = \frac{\tilde{N}}{N}$ (assuming N and $\tilde{N} < N$ having no common divisor > 1) both \mathbb{A}^{Λ} and \mathbb{M}^{Λ} may be divided by an ideal of finite codimension, namely (using the periodicity of the structure-constants) the vectorspace I generated by all elements of the form $T_{\vec{m}} - T_{\vec{m}+N}$ (anything). One thus arrives at considering (for arbitrary odd N)

$$V^{(N)} := \left\langle T_{\vec{m}} | m_r = -\frac{N-1}{2}, \dots, +\frac{N-1}{2} \right\rangle_{\mathbb{C}} \quad r = 1 \dots M$$
 (21)

with a $*_M$ product on $V^{(N)}$ defined just as in (18):

$$(T_{\vec{m}_1} \cdots T_{\vec{m}_M})_* := \frac{-i N}{2\pi \widetilde{N} M!} \omega^{-\frac{1}{2} (\vec{m}_1, \cdots, \vec{m}_M)} T_{\vec{m}_1 + \cdots + \vec{m}_M \pmod{N}}$$

$$\omega = e^{4\pi i \frac{\widetilde{N}}{N}}, \qquad (22)$$

and a corresponding alternating product,

$$[T_{\vec{m}_1}, \cdots, T_{\vec{m}_M}]_* = \frac{-iN}{2\pi\widetilde{N}} \sin\left(2\pi \frac{\widetilde{N}}{N} \left(\vec{m}_1, \cdots, \vec{m}_M\right)\right) T_{\vec{m}_1 + \cdots + \vec{m}_M \pmod{N}}$$

$$\vec{m}_r \in (\mathbb{Z}_N)^M . \tag{23}$$

The 'structure constants' of the alternating finite dimensional M-algebras

$$\mathbb{A}_{N} := (V^{(N)}, [, \cdots,]_{\star}),
f_{\vec{m}_{1} \cdots \vec{m}_{M}}^{(N) \vec{m}} := \frac{-i N}{2\pi \widetilde{N}} \sin \left(2\pi \frac{\widetilde{N}}{N} \left(\vec{m}_{1}, \cdots, \vec{m}_{M}\right)\right) \cdot \delta_{\vec{m}_{1} + \cdots + \vec{m}_{M} \pmod{N}}^{\vec{m}} \tag{24}$$

satisfy (14) (up to an N and \mathbb{Z}_N^M -independent rescaling of the generators, resp. factors of i, which anyway drop out in (10) and (11); $n = N^M$, $f^{(N)} \stackrel{\triangle}{=} f(\lambda_n)$, $\vec{m} \in \mathbb{Z}_N^M$, $V^{(N)} = V_{n=N^3}$, and $\lim_{N \to \infty} V^{(N)} = V$.

$$H_{N} = \frac{1}{2} p_{i-\vec{m}} p_{i\vec{m}}$$

$$+ \frac{1}{2} \frac{N^{2}}{4\pi^{2} \tilde{N}^{2}} \sin \left(2\pi \frac{\tilde{N}}{N} \left(\vec{m}_{1} \cdots \vec{m}_{M}\right)\right) \cdot \sin \left(2\pi \frac{\tilde{N}}{N} \left(\vec{n}_{1}, \cdots \vec{n}_{M}\right)\right)$$

$$\frac{1}{M!} \cdot x_{i_{1}-\vec{m}_{1}} \cdots x_{i_{M}-\vec{m}_{M}} x_{i_{1}\vec{n}_{1}} \cdots x_{i_{M}\vec{n}_{M}} \delta_{\vec{n}_{1}+\cdots+\vec{n}_{M}}^{\vec{m}_{1}+\cdots+\vec{n}_{M}} \pmod{N}$$
(25)

could therefore be considered as a finite-dimensional analogue of (1).

3. Multidimensional Commutation Relations

Before turning to questions of symmetry, let me discuss in a little more detail the *M-algebras \mathbb{M}^{Λ} , with defining relations (cp. (18); note the slight change of notation/normalisation)

$$(T_{\vec{m}_1}\cdots T_{\vec{m}_M})_* = \omega^{-\frac{1}{2}(\vec{m}_1,\cdots,\vec{m}_M)} T_{\vec{m}_1+\cdots+\vec{m}_M}(*),$$

and as vectorspaces generated by basis-elements $T_{\vec{m}}$, $\vec{m} \in S$ (where $S = \mathbb{Z}^M$, $S = (\mathbb{Z}_N)^M$, or any combination thereof – in the M-brane context, depending on whether $\Sigma = T^M$, resp. a fully, or partially, discretized M-torus).

First of all note, that for any M elements, $A_1, \dots A_M \in V$, any even permutation $\sigma \in S_M$ (the symmetric group in M objects), and any choice of S (even \mathbb{R}^M),

$$(A_1 \cdots A_M)_* = (A_{\sigma(1)} \cdots A_{\sigma(M)}) \quad (\text{sign } \sigma = +) , \tag{26}$$

and that $E := T_{\vec{0}}$ acts as a 'unity' in the sense that if one of the A_r is equal to $T_{\vec{0}}$, the *M-product becomes commutative (i.e. independent of the order of its M entries).

Using E, one may identify $T_{(m=\pm|m|,0,\cdots,0)}$ with the |m|-th power of $E_{\pm 1}:=T_{(\pm 1,0,\cdots,0)}$,

$$T_{(m,0,\cdots,0)} = ((((E \cdots EE_{\pm 1})_* \cdots EE_{\pm 1})_* \cdots EE_{\pm 1})_* , \qquad (27)$$

$$\uparrow \qquad \qquad |m| \text{ brackets}$$

so that one may wonder whether \mathbb{M}^{Λ} can be thought of as being generated by

$$E = T_{\vec{0}}, E_{\pm 1} = T_{(\pm 1 \ 0 \cdots 0)}, \cdots, E_{\pm M} = T_{(0 \cdots 0 \ \pm 1)}.$$

This is indeed the case: Let \mathbb{F}^M be the free (non associative) M-algebra generated by 2M+1 elements $E, E_{\pm 1}, \dots, E_{\pm M}$; define arbitrary powers $(E_r)^m$ of the generating elements as in (27) (from now on $E_{-r}^{|m|} =: E_r^{-|m|}$, a notation that will be justified via (29)), and let

$$E_{\vec{m}} := E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M} . \tag{28}$$

Divide \mathbb{F}^M by the ideal generated by elements

$$E_{\vec{m}'} E_{\vec{m}''} \cdots E_{\vec{m}(M)} = \omega^{\gamma(\vec{m}', \vec{m}'', \dots, \vec{m}^{(M)})} \cdot E_{\vec{m}' + \dots + \vec{m}^{(M)}}$$
 (29)

where $\omega = e^{4\pi i \Lambda}$ and

$$2\gamma(\vec{m}', \dots, \vec{m}^{(M)}) := (m_1 \cdot m_2 \cdot \dots \cdot m_M) - (\vec{m}', \vec{m}'', \dots, \vec{m}^{(M)})$$

$$- \sum_{r=1}^{M} \left(\prod_{s=1}^{M} m_s^{(r)} \right)$$

$$(\vec{m} := \vec{m}' + \vec{m}'' + \dots + \vec{m}^{(M)}).$$
(30)

This quotient then is isomorphic to \mathbb{M}^{Λ} , as can be seen by defining

$$T_{\vec{m}} := \omega^{\frac{1}{2} m_1 m_2 \cdots m_M} E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}, \qquad (31)$$

which (due to (29) being zero in \mathbb{F}^{Λ}/I) satisfies (18) (with Y standing for T). Note that

$$E_2^{m_2} E_1^{m_1} E_3^{m_3} \cdots E_M^{m_M} = \omega^{m_1 m_2 \cdots m_M} \cdot E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}, \tag{32}$$

in particular:

$$E_2 E_1 E_3 \cdots E_M = \omega E_1 E_2 \cdots E_M \tag{33}$$

(while any even permutation does not alter the product, cp. (26)).

In order to get a feeling for (29)/(30) it may be instructive to consider M=3: due to (29),

$$(E_{1}^{n_{1}} E_{2}^{n_{2}} E_{3}^{n_{3}})(E_{1}^{l_{1}} E_{2}^{l_{2}} E_{3}^{l_{3}})(E_{1}^{k_{1}} E_{2}^{k_{2}} E_{3}^{k_{3}})$$

$$= E_{1}^{n_{1}+l_{1}+k_{1}} E_{2}^{n_{2}+l_{2}+k_{2}} E_{3}^{n_{3}+l_{3}+k_{3}}$$

$$\cdot \omega^{n_{1}l_{3}k_{2}+n_{2}l_{1}k_{3}+n_{3}l_{2}k_{1}}$$

$$\cdot \sqrt{\omega}^{n_{1}(l_{2}l_{3}+k_{2}k_{3})+n_{2}(l_{1}l_{3}+k_{1}k_{3})+n_{3}(l_{1}l_{2}+k_{1}k_{2})}$$

$$\cdot \sqrt{\omega}^{n_{1}n_{2}(l_{3}+k_{3})+n_{1}n_{3}(l_{2}+k_{2})+n_{2}n_{3}(l_{1}+k_{1})}.$$

$$(34)$$

The general rule (30) can hence be stated as follows:

Consider all possible triples (resp. M-tuples) containing powers of each of the $E_r(r=1\cdots M)$ exactly once. If the 'contraction' picks out exactly one factor from each of the 3 (resp. M) factors in (34) it does <u>not</u> contribute if they are already in the correct order, modulo even permutations (cp. 26), (like $E_1^{n_1} E_2^{l_2} E_3^{k_3}$, or $E_2^{n_2} E_3^{l_3} E_2^{k_1}$), while they contribute a factor $\omega^{\text{(product of the }E-\text{powers)}}$, when they are <u>not</u> in the correct (modulo even permutation) order (like $E_2^{n_2} E_1^{l_1} E_3^{k_3}$). Contractions entirely within one of the factors don't contribute, while mixed contractions (involving at least 2, but not all, of the factors in (34)), all contribute a factor $\sqrt{\omega}^{\text{(product of the }E-\text{powers)}}$, irrespective of their order.

Due to (32), all 'monomials' are proportional to one of the elements $E_{\vec{m}}$ (cp. (28)) – which therefore form a basis (with the convention $E_{\vec{0}} \equiv E$). Note that $2\pi M! \Lambda = \lambda \to 0$ is a 'classical limit' (resp. $\lambda \neq 0$ a 'quantisation' of the classical Nambu-structure) as, formally,

$$[\ln E_1, \ln E_2, \cdots, \ln E_M] = i \lambda E.$$
(35)

Having obtained this relation, one could of course start with objects $\ln E_r =: J_r$, $[J_1, J_2, \cdots, J_M] = i \lambda E$, and derive generalized 'Hausdorff-formulae' for products involving the $e^{i m_r J_r}$.

Of course, (35) cannot be true in any M-algebra containing only finite linear combinations of the basis-elements $E_{\vec{m}}$, as $T_{\vec{0}} = E$ never appears on the r.h.s. of (20); this is similar to the fact that the canonical commutation relations of ordinary quantum mechanics, $[q, p] = i \hbar \mathbf{1}$, cannot hold for trace-class operators. (35) may be justified by formally

expanding
$$\ln E_r = -\sum_{n_r=1}^{\infty} \sum_{k_r=0}^{n_r} \binom{n_r}{k_r} \frac{(-)^{k_r}}{n_r} E_r^k$$
, using

$$[E_1^{k_1}, E_2^{k_2}, \cdots, E_M^{k_M}] = \frac{M!}{2} (1 - \omega^{k_1 \cdots k_M}) E_1^{k_1} \cdots E_M^{k_M}$$

and then resumming recursively, after the first step obtaining

$$\frac{M!}{2} \ln E_1 \cdots \ln E_M - \frac{M!}{2} \sum_{\substack{n_r, k_r \\ r > 1}} ' \cdots \ln(E_1 \omega^{k_2 \cdots k_M}) E_2^{k_2} \cdots E_M^{k_M} = \frac{M!}{2} (\ln \omega) \cdot E , \quad (36)$$

as formally,

$$\sum_{n_r=1}^{\infty} \sum_{k_r=1}^{n_r} \binom{n_r}{k_r} \frac{(-)^{k_r}}{n_r} k_r E_r^k = E_r \cdot \sum_{n'=0}^{\infty} (E - E_r)^{n'} = E.$$

4. Breakdown of Conventional Symmetries

Let us now discuss the question, whether theories like (5) or (6) can have symmetries reminiscent of volume preserving diffeomorphisms; in particular whether the generators (2) may be 'translated' to finite dimensional analogues. * For simplicity, consider again $\Sigma = T^M$.

As $f^r = \partial_s \omega^{rs} = \epsilon^{rsr_1\cdots r_{M-2}} \partial_s \omega_{r_1\cdots r_{M-2}}$ for non-constant (divergence-free) vector-fields on T^M , (2) may be written in the form

$$K_{r_1 \cdots r_{M-2}} = \int d^M \varphi \ \omega_{r_1 \cdots r_{M-2}} \left\{ p_i, x^i, \varphi^{r_1}, \cdots, \varphi^{r_{M-2}} \right\} ,$$
 (37)

resp., in Fourier-components,

$$K_{r_1\cdots r_{M-2}}^{\vec{l}} = \sum_{\substack{\vec{m},\vec{n}\\ \in \mathbb{Z}^M}} \delta_{\vec{m}+\vec{n}}^{\vec{l}} p_{i\vec{m}} x_{i\vec{n}} (\vec{m}, \vec{n}, \vec{e}_{r_1}, \cdots, \vec{e}_{r_{M-2}})$$
(38)

(where \vec{e}_r denotes the unit vector in the r-direction).

Suppose the deformed theory was invariant under transformations that are still generated in a conventional way by phase-space functions of the form

$$K^{\vec{l}} = \sum_{\vec{m}, \vec{n} \in S} p_{i\vec{m}} x_{i\vec{n}} \, \delta^{\vec{l}}_{\vec{m} + \vec{n}} \, c_{\vec{m}\vec{n}} \,. \tag{39}$$

Using $[x_{i\vec{m}}, p_{j\vec{n}}] = \delta_{ij}\delta_{\vec{m}}^{-\vec{n}}$, while leaving open whether $S = \mathbb{Z}^M$ or $S = (\mathbb{Z}_N)^M$ as well as (independently) whether δ is defined mod N, or not, one has

$$[K^{\vec{l}}, \widetilde{K}^{\vec{l}'}] = \sum_{\substack{\vec{m}_1,\vec{n} \\ \in S}} p_{i\vec{m}} x_{i\vec{n}} \, \delta_{\vec{m}+\vec{n}}^{\vec{l}+\vec{l}'} \, \widetilde{c}_{\vec{m}\vec{n}}$$
with
$$\widetilde{c}_{\vec{m}\vec{n}} = \sum_{\vec{k} \in S} \left(\delta_{\vec{k}}^{\vec{l}-\vec{n}} \, \delta_{-\vec{k}}^{\vec{l}'-\vec{n}} \, c_{\vec{m}\vec{k}} \, \widetilde{c}_{-\vec{k}\vec{n}} - \begin{pmatrix} \vec{l} \leftrightarrow \vec{l}' \\ c \leftrightarrow \widetilde{c} \end{pmatrix} \right), \tag{40}$$

^{*}For M=2, this question was already considered in [4] and answered positively.

while $K^{\vec{l}} = 0$ would require $c_{\vec{m}\vec{n}} = - - - c_{\vec{n}\vec{m}}$ and

$$\sin(2\pi\Lambda(\vec{a}_{1},\dots,\vec{a}_{M})) \sin(2\pi\Lambda(\vec{a}_{1}+\dots+\vec{a}_{M},\vec{a}_{2}',\dots,\vec{a}_{M}'))$$

$$\cdot c_{\vec{a}_{1}+\dots \vec{a}_{1}'+\dots \vec{a}_{M}',\vec{a}_{1}'} \cdot x_{i_{1}\vec{a}_{1}} x_{i_{1}\vec{a}_{1}'} \cdot x_{i_{M}\vec{a}_{M}} x_{i_{M}\vec{a}_{M}'} = 0$$
(41)

(where for (41) consistency of the δ -functions used in (39) and (25) $_{\Lambda}$ with the index set S was assumed).

The effect of the $x_{i\vec{m}}$ -factors in (41) is to make the product $\sin \cdot \sin \cdot c$, symmetric under any interchange $\vec{a}_r \leftrightarrow \vec{a}_r'$, as well as any simultaneous interchange $\vec{a}_r \leftrightarrow \vec{a}_s$, $\vec{a}_r' \leftrightarrow \vec{a}_s'$. Choosing, e.g., $\vec{a}_r' = \vec{a}_r(r = 1 \cdots M)$, with $\sin(2\pi\Lambda(\vec{a}_1 \cdots \vec{a}_M)) \neq 0$, (41) requires that

$$\sum_{\sigma \in S_M} c_{\vec{a}_{\sigma 1} + 2(\vec{a}_{\sigma 2} + \dots + \vec{a}_{\sigma M}), \vec{a}_{\sigma 1}} = 0.$$

$$(42)$$

This condition is insensitive to any alteration of the deformation: replacing the sine-function in (41) (resp. $(25)_{\Lambda}, \cdots$) by any other function of the determinant will still result in (42) as a necessary condition. Apart from M=2 ($c_{\vec{a}_1+2\vec{a}_2,\vec{a}_1}+c_{\vec{a}_2+2\vec{a}_1,\vec{a}_2}=0$ is trivially satisfied by any odd function) (42) is <u>not</u> satisfied by

$$c_{\vec{m}\vec{n}} = \sin(2\pi\Lambda(\vec{m}, \vec{n}, \vec{k}_1, \dots, \vec{k}_{M-2})),$$
 (43)

--- nor would (40) be a linear combination of the generators (39), for such a $c_{\vec{m}\vec{n}}$; for M=3, e.g., one would obtain

$$\widetilde{c}_{\vec{m}\vec{n}}(\vec{l}\,\vec{l}';\vec{k}\,\vec{k}') = \sin\left(2\pi\Lambda\left(\vec{l},\vec{l}',\frac{\vec{k}+\vec{k}'}{2}\right)\right)
\cdot \sin\left(2\pi\Lambda\left(\left(\vec{m},\vec{n},\frac{\vec{k}+\vec{k}'}{2}\right) + \left(\vec{m}-\vec{n},\frac{\vec{l}-\vec{l}'}{2},\frac{\vec{k}-\vec{k}'}{2}\right)\right)\right)
- \sin\left(2\pi\Lambda\left(\vec{l},\vec{l}',\frac{\vec{k}-\vec{k}'}{2}\right)
\cdot \sin\left(2\pi\Lambda\left(\vec{m},\vec{n},\frac{\vec{k}-\vec{k}'}{2}\right) + \left(\vec{m}-\vec{n},\frac{\vec{l}-\vec{l}'}{2},\frac{\vec{k}+\vec{k}'}{2}\right)\right)$$
(44)

--- which means that the algebra closes only for $\vec{k}' = \vec{k}$ (for $\Lambda = \frac{1}{N}$ this would give N^3 closed Lie algebras, each of dimension N^3 ; in fact, each consisting of N copies of gl(N)).

- In any case, if $c_{\vec{m}\vec{n}}$ was a function of $(\vec{m}_1\vec{n}_1\vec{k}_1,\cdots,\vec{k}_{M-2})$, one could let $\vec{a}_2,\vec{a}_3,\cdots\vec{a}_M$ differ only in the ('irrelevant') $\vec{k}_1,\cdots\vec{k}_{M-2}$ directions and obtain

$$f(((2M-2)\vec{a}_2, \vec{a}_1, \cdots)) + (M-1)f((2\vec{a}_1, \vec{a}_2, \cdots)) = 0,$$
 (45)

which eliminates all $c_{m\bar{n}}$ that are non-linear functions of the determinant.

Interestingly, $c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{ something})_{\text{if } M>2}$ is suggested by yet another consideration: replacing $\{p_i, x_i, \varphi^3, \cdots, \varphi^M\}$ (cp. (37); for notational simplicity taking $r_1 = 3, \cdots, r_{M-2} = M$) by

$$[P_i, X_i, \ln E_3, \cdots, \ln E_M], \qquad (46)$$

(with $P_i = p_{i\vec{m}}T_{\vec{m}}, X_i = x_{i\vec{m}}T_{\vec{m}}$) formally expanding the logarithms in a power series, using (20), and then (formally) summing again, one obtains something proportional to

$$p_{i\vec{m}} x_{i\vec{n}} T_{\vec{m}+\vec{n}} \cdot (m_1 n_2 - m_2 n_1) . \tag{47}$$

$$[P_{i}, X_{i}, \ln E_{3}, \cdots, \ln E_{M}]$$

$$= p_{i\vec{m}} x_{i\vec{n}} (-)^{M-2} \sum_{n_{3}=1}^{\infty} \sum_{k_{3}=0}^{n_{3}} \cdots \sum_{n_{M}=1}^{\infty} \sum_{k_{M}=0}^{n_{M}} \binom{n_{3}}{k_{3}} \cdots \binom{n_{M}}{k_{M}} \frac{(-)^{k_{3}+\cdots+k_{M}}}{n_{3}\cdots n_{M}}$$

$$\cdot [T_{\vec{m}}, T_{\vec{n}}, E_{3}^{k_{3}}, \cdots, E_{M}^{k_{M}}]$$

$$\sim \sum \cdots \sin (2\pi \Lambda (\vec{m}, \vec{n}, k_{3} \vec{e}_{3}, \cdots, k_{M} \vec{e}_{M})) \cdot T_{\vec{m}+\vec{n}+\vec{k}}$$

$$\sim \sum \cdots \left(\sqrt{\omega}^{k_{3}\cdots k_{M}z} - \sqrt{\omega}^{-k_{3}\cdots k_{M}z} \right) (\sqrt{\omega})^{\prod_{r=1}^{M}(m_{r}+n_{r}+k_{r})} \cdot$$

$$\cdot E_{1}^{m_{1}+n_{1}} E_{2}^{m_{2}+n_{2}} E_{3}^{m_{3}+n_{3}+k_{3}} \cdots E_{M}^{m_{M}+n_{M}+k_{M}}$$

$$\sim \sum \cdots \left(\ln \left(\sqrt{\omega}^{k_{4}\cdots k_{M}z+\prod_{r\neq 3}(m_{r}+n_{r}+k_{r})} E_{3} \right) - \ln \left(\sqrt{\omega}^{-k_{4}\cdots k_{M}z+\prod_{r\neq 3}(m_{r}+n_{r}+k_{r})} E_{3} \right) \right)$$

$$- \ln \left(\sqrt{\omega}^{-k_{4}\cdots k_{M}z+\prod_{r\neq 3}(m_{r}+n_{r}+k_{r})} E_{3} \right)$$

$$- \ln \left(\sqrt{\omega}^{-k_{4}\cdots k_{M}z+\prod_{r\neq 3}(m_{r}+n_{r}+k_{r})} E_{3} \right)$$

$$\sim E_{1}^{m_{1}+n_{1}} E_{2}^{m_{2}+n_{2}} E_{3}^{m_{3}+n_{3}} E_{4}^{m_{4}+n_{4}+k_{4}} \cdots E_{M}^{m_{M}+n_{M}+k_{M}}$$

$$\left(z := (\vec{m}, \vec{n}, \vec{e}_{3}, \cdots, \vec{e}_{M}) = m_{1} n_{2} - m_{2} n_{1} \right)$$

$$\leq (\ln \omega) p_{i\vec{m}} x_{i\vec{n}} z (\vec{m}, \vec{n}) \sqrt{\omega} \prod_{1}^{M} (m_{r}+n_{r}) E_{1}^{m_{1}+n_{1}} \cdots E_{M}^{m_{M}+n_{M}}$$

$$= (m_{1} n_{2} - m_{2} n_{1}) p_{i\vec{m}} x_{i\vec{n}} (\ln \omega) \cdot T_{\vec{m}+\vec{n}}$$

$$= (m_{1} n_{2} - m_{2} n_{1}) p_{i\vec{m}} x_{i\vec{n}} (\ln \omega) \cdot T_{\vec{m}+\vec{n}}$$

$$\text{where (for } r > 3) - \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-)^{k}}{n} k \cdot E_{r}^{k} \cdot (\omega^{\cdots})^{k} = E \text{ was used.}$$

However,

$$c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{ anything}) \tag{48}$$

does <u>not</u> satisfy (41). Moreover, even if one considers more general deformations of the Hamiltonian, i.e. replacing the sine-function in (41) by an arbitrary odd (power-series) function f of the determinant, the corresponding condition,

$$f(\vec{a}_1, \dots, \vec{a}_M) f(\vec{a}_1 + \dots + \vec{a}_M, \vec{a}'_2, \dots, \vec{a}'_M) \cdot (\vec{e}, \vec{a}'_1, \dots) = 0$$

 $+ (M \cdot 2^M - 1) \text{ permutations},$ (49)

 $\vec{e} = \sum_{r=1}^{M} (\vec{a}_r + \vec{a}'_r)$, can never be satisfied by any non-linear function f – as on can see, e.g., by choosing $\vec{a}'_r = \mu_r \vec{a}_r$. Supposing $f(x) = \alpha x + \beta x^{2n+1} = \cdots$, and denoting $(\vec{a}_1, \dots, \vec{a}_M)$ by z, $\prod_{r=1}^{M} \mu_r$ by μ , the terms $\mu_1, \alpha z \beta (\mu z)^{2n+1}$, e.g., (occurring only twice, with the same sign) could never cancel.

The preceding arguments possibly suffice to prove that, independent of the above dynamical context, the Lie algebra of volume-preserving diffeomorphisms of $T^{M>2}$ does not possess any non-trivial deformations.*

5. Rigidity of Canonical Nambu-Poisson Manifolds

For the multilinear antisymmetric map (4), and 2M-1 arbitrary functions f_1, \dots, f_{2M-1} , one has (cp. [5]):

$$\{\{f_{M}, f_{1}, \cdots, f_{M-1}\}, f_{M+1}, \cdots, f_{2M-1}\}$$

$$+ \{f_{M}, \{f_{M+1}, f_{1}, \cdots, f_{M-1}\}, f_{M+2}, \cdots, f_{2M-1}\}$$

$$+ \cdots + \{f_{M}, \cdots, f_{2M-2}, \{f_{2M-1}, f_{1}, \cdots, f_{M-1}\}\}$$

$$= \{\{f_{M}, \cdots, f_{2M-1}\}, f_{1}, \cdots, f_{M-1}\}.$$

$$(50)$$

Takhtajan [5], stressing its relevance for time-evolution in Nambu-mechanics [1], named (50) 'Fundamental Identity (FI)', and defined a 'Nambu-Poisson-manifold of order M 'to be a manifold X together with a multilinear antisymmetric map $\{\cdots\}$ satisfying (50) and the Leibniz-rule

$$\{f_1\tilde{f}_1, f_2, \cdots, f_M\} = f_1\{\tilde{f}_1, f_2, \cdots, f_M\} + \{f_1, \cdots, f_M\} \tilde{f}_1$$
 (51)

for functions $f_r: X \to \mathbb{R}$ (or \mathbb{C}).

Without (51), i.e. just demanding (50) for an antisymmetric M linear map: $V \times \cdots \times V \to V$, V some vectorspace, Takhtajan defines a 'Nambu-Lie-gebra' [5], – also called 'Fillipov [6] Lie algebra' [7]). I would like to point out various other identities satisfied by canonical Nambu-Poisson brackets (4), and show that all of them – including (50)! – do not allow deformations (of certain natural type), if M > 2.

At least from a non-dynamical point of view, all identities involving Nambu-brackets obtained from antisymmetrizing the product of two determinants formed from 2M M-vectors,

$$(\vec{a}_1 \cdots \vec{a}_M)(\vec{a}_{M+1} \cdots \vec{a}_{2M}) \tag{52}$$

with respect to M+1 of the $\vec{a}_{\alpha}(\alpha=1\cdots 2M)$ should be treated on an equal footing. For M=3, e.g., one has – apart from

$$(\vec{a} \ \vec{b} \ \vec{c}_1)(\vec{c}_2 \ \vec{c}_3 \ \vec{c}_4) - (\vec{a} \ \vec{b} \ \vec{c}_2)(\vec{c}_3 \ \vec{c}_4 \ \vec{c}_1) + (\vec{a} \ \vec{b} \ \vec{c}_3)(\vec{c}_4 \ \vec{c}_1 \ \vec{c}_2) - (\vec{a} \ \vec{b} \ \vec{c}_4)(\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3) = 0 ,$$
 (53)

which gives rise to $(50)_{M=3}$ for functions $f \in T^3$ – also

$$(a \ \vec{c}_{[1} \ \vec{c}_{2})(\vec{c}_{3} \ \vec{c}_{4]} \ \vec{b}) = 0 , \qquad (54)$$

^{*}M. Bordemann has informed me that apparently an even more general statement of this nature has recently been proven in [19].

yielding the following 6-term identity (FI)₆ (which can of course also be proven by using just the definition (4), $\{f, g, h\} = \epsilon_{\alpha\beta\gamma} \partial_{\alpha} f \partial_{\beta} g \partial_{\gamma} h$, rather than (54); i.e. not necessarily specifying the manifold X):

$$\{\{f, f_{[1}, f_{2}\} f_{3}, f_{4]}\} = 0 \tag{55}$$

as well as the 4-term identity (FI),

$$\{\{f, f_1, f_2\}, g, f_3\}
+ \{\{f, f_2, f_3\}, g, f_1\}
+ \{\{f, f_3, f_1\}, g, f_2\} = -\{f, g, \{f_1, f_2, f_3\}\}$$
(56)

- - - each of which is independent of $(50)_{M=3}$ (while any 2 of the 3 identities yield the $3^{\rm rd}$).

Naively, one would think that (56) should follow from $(50)_3$ alone, as (54) follows from (53) (perhaps one should note that for general M, a theorem concerning vector invariants [8] states, that any (!) vector-bracket identity is an algebraic consequence of

$$(\vec{a}_{[1} \vec{a}_{2} \cdots \vec{a}_{M}) (\vec{a}_{M+1]} \cdots \vec{a}_{2M}) = 0;$$

however, in the proof of (56) via vector-bracket identities, one in particular needs (54) for the special case $\vec{a} = \vec{b}$ – which cannot be stated as an identity between functions on X.) Curiously (with respect to a statistical approach to M-branes), vector-bracket identities ('Basis Exchange Properties' [9]) also play an important role in combinatorical geometry.

From an aesthetic point of view, the most natural quadratic identity for (4) is

$$\sum_{\sigma \in S_{2M-1}} (\text{sign } \sigma) \{ \{ f_{\sigma 1}, \dots, f_{\sigma M} \} | f_{\sigma M+1}, \dots, f_{\sigma 2M-1} \} = 0.$$
 (57)

For M=3, e.g., one could see this to be a consequence of $(50)_3$ and (56) by grouping the 10 distinct terms in (57) according to whether $\{f_{\sigma 1}, f_{\sigma 2}, f_{\sigma 3}\}$ contains both f_4 and f_5 (3 terms, 'type A'), only one of them (3 'B-terms' and 3 'C-terms') or none of them (1 term, 'type D'); for the B (resp. C)-terms one can use (56) while (50) for the A-terms, to get $\pm \{f_4, f_5, \{f_1 f_2 f_3\}\}$ for each of the 4 types, and for the B and C-terms with a sign opposite to the one obtained from the D (and A) term(s). (57) (taken without the derivation-requirement) is a beautiful generalisation of Lie-algebras (M=2), and has recently started to attract the attention of mathematicians – mostly under the name of (M-1)-ary Lie algebras [10-13]. *

Unfortunately, all identities (50), (55)–(57), can be shown to be rigid, in the following sense: assuming that

$$[T_{\vec{m}_1}, \cdots, T_{\vec{m}_M}]_{\lambda} = g_{\lambda} ((\vec{m}_1, \cdots, \vec{m}_M)) T_{\vec{m}_1 + \cdots + \vec{m}_M}$$
(58)

with $g_{\lambda}(x)$ a smooth odd function proportional to $x + \lambda^n c x^n$ as $\lambda \to 0$ (n > 1) any of the above identities will require the constant c to be equal to zero (I have proved this

^{*}I would like to thank W. Soergel for mentioning refs. [10]/[11] to me and J.L. Loday for sending me a copy of [10] and [12]; also, I would like to express my gratitude to R. Chatterjee and L. Takhtajan for sending me their papers on Nambu Mechanics (cp. [5]).

only for M=3, and in the case of (57) – the a priori most promising case – for general M>2).

Concerning

$$g_{\lambda}\left((\vec{a}, \vec{b}, \vec{c}_{1})\right) g_{\lambda}\left((\vec{a} + \vec{b} + \vec{c}_{1}, \vec{c}_{2}, \vec{c}_{3})\right)$$

$$+ g_{\lambda}\left((\vec{a}, \vec{b}, \vec{c}_{2})\right) g_{\lambda}\left((\vec{a} + \vec{b} + \vec{c}_{2}, \vec{c}_{3}, \vec{c}_{1})\right)$$

$$+ g_{\lambda}\left((\vec{a}, \vec{b}, \vec{c}_{3})\right) g_{\lambda}\left((\vec{a} + \vec{b} + \vec{c}_{3}, \vec{c}_{1}, \vec{c}_{2})\right)$$

$$\stackrel{!}{=} g_{\lambda}\left((\vec{c}_{1}, \vec{c}_{2}, \vec{c}_{3})\right) g_{\lambda}\left((\vec{c}_{1} + \vec{c}_{2} + \vec{c}_{3}, \vec{a}, \vec{b})\right) ,$$

$$(59)$$

i.e. the deformation of $(50)_{M=3}$, one could assume $z:=(\vec{c}_1,\vec{c}_2,\vec{c}_3)\neq 0$, $\vec{a}=\sum_1^3\alpha_r\vec{c}_r$, $\vec{b}=\sum_1^3\beta_r\vec{c}_r$, such that $g(y):=\bar{g}_\lambda(y):=g_\lambda(zy)$ must satisfy

$$g(\alpha_2 \beta_3 - \alpha_3 \beta_2) \cdot g(1 + \alpha_1 + \beta_1)$$
+ cyclic permutations
$$= g(1) \cdot g(\alpha_2 \beta_3 - \alpha_3 \beta_2 + \text{cycl.})$$
(60)

for all α_r, β_r ; which is clearly impossible for any nonlinear g of the required form. (e.g., as in next to lowest order in λ the terms $\alpha_1(\alpha_2 \beta_3)^{n>1}$ appear only once).

Similarly, the deformation of (56) is impossible due to the analogous requirement

$$g(\alpha_3) g(\beta_2 - \beta_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1)) + \text{cycl.}$$

$$\stackrel{!}{=} -g(1) g((\alpha_1 \beta_2 - \alpha_2 \beta_1) + \text{cycl.}) . \tag{61}$$

Finally, concerning possible deformations of (57), let $(\vec{a}_1, \dots, \vec{a}_M) \neq 0$, and

$$\vec{a}_{M+\bar{r}} = \sum_{s=1}^{M} \alpha_s^{(\bar{r})} \vec{a}_s \ (\bar{r} = 1, \dots, M-1);$$
then $g \ (1 + \alpha_1^{(1)} + \dots + \alpha_1^{(M-1)}) \cdot g$

$$\underbrace{\begin{pmatrix} 1 \\ 0 & \vec{\alpha}^{(1)} \dots \vec{\alpha}^{(M-1)} \\ \vdots \\ 0 \end{pmatrix}}_{-\dots \ [1]},$$

e.g., contains (in next to lowest order in λ) a term $\alpha_1^{(1)} \cdot \alpha_1^{(2)} \cdot [1]$ (of total degree (M+1) in the $\alpha_s^{(\bar{r})}$), which cannot appear anywhere else (in the same order in λ), – in contradiction to the assumption that (57) should hold for $[\cdots]_{\lambda}$ (cp. (58)) replacing the curly bracket (4).

6. A Remark on Generalized Schild Actions

Consider

$$S := -\int d\varphi^0 \ d^M \varphi \ f(G) \ , \tag{62}$$

where $G := (-)^M$ det $(G_{\alpha\beta})$, $G_{\alpha\beta} := \frac{\partial x^{\mu}}{\partial \varphi^{\alpha}} \frac{\partial x^{\nu}}{\partial \varphi^{\beta}} \eta_{\mu\nu}$, $\eta_{\mu\nu} = \text{diag } (1, -1, \dots, -1)$, $\alpha, \beta = 0, \dots, M$ and f some smooth monotonic function like G^{γ} ($\gamma = 1$ resp. $\frac{1}{2}$ corresponding to a generalized Schild-, resp. Nambu-Goto, action for M-branes). Apart from a few subtleties (like $\gamma = 1$ allowing for vanishing G, while $\gamma = \frac{1}{2}$ does not) actions with different f are equivalent, in the sense that the equations of motion,

$$\partial_{\alpha} \left(f'(G)GG^{\alpha\beta} \partial_{\beta} x^{\mu} \right) = 0 \qquad \mu = 0 \cdots D - 1$$
 (63)

are easily seen to imply

$$\partial_{\alpha} G = 0 \qquad \alpha = 0, \cdots, M \tag{64}$$

(just multiply (63) by $\partial_{\epsilon}x_{\mu}$ and sum) – unless f(G) = const. \sqrt{G} , in which case (62) is fully reparametrisation invariant and a parametrisation may be assumed in which G = const. (such that (63) becomes proportional to $\partial_{\alpha}(G^{\alpha\beta} \partial_{\beta}x^{\mu})$ also in this case). Due to

$$G = \sum_{\mu_1 < \dots < \mu_{M+1}} \left\{ x^{\mu_1}, \dots, x^{\mu_{M+1}} \right\} \left\{ x_{\mu_1}, \dots, x_{\mu_{M+1}} \right\}$$
 (65)

(63) may be written as (cp. [14] for strings, and [15] for membranes, in the case of $\gamma = 1$ resp. $\frac{1}{2}$)

$$\{f'(G)\{x^{\mu_1},\cdots,x^{\mu_{M+1}}\},\ x_{\mu_2},\cdots,x_{\mu_{M+1}}\}=0,$$
 (66)

whose deformed analogue (note the similarity between G = const. and condition (3.9) of [16])

$$[[x^{\mu_1}, \cdots, x^{\mu_{M+1}}], x_{\mu_2}, \cdots, x_{\mu_{M+1}}] = 0$$
(67)

looks very suggestive when thinking about space-time quantization in M-brane theories.

7. Multidimensional Integrable Systems from M-algebras

Several ideas used in the context of integrable systems are based on bilinear operations. Our problems to extend results about low (especially 1+1) dimensional integrable field theories to higher dimensions may well rest on precisely this fact. Already some time ago, attempts were made to overcome this difficulty by generalizing Lax-pairs to -triples ([3], p. 72) and Hirota's bilinear equations for ' τ -functions' [17] to multilinear equations ([3], p. 107-111).

At that time, good examples were lacking, and – not being an exception to the rule that generalisations involving the number of dimensions (of one sort or an other) are usually hindered by implicitely low dimensional point(s) of view – the proposed generalisation of

Hirota-operators may have still been too naive; while hoping to come back to the question of multidimensional τ -functions in the near future, I would now like to give an example (M > 3 will then be obvious) for an equation of the form

$$\dot{\mathcal{L}} = \frac{1}{\rho} \left\{ \mathcal{L}, \, \mathcal{M}_1, \, \mathcal{M}_2 \right\} \tag{68}$$

being equivalent to the equations of motion of a compact 3 dimensional manifold $\widehat{\sum} \subset \mathbb{R}^4$ (described by a time-dependent 4-vector $x^i(\varphi^1, \varphi^2, \varphi^3, t)$), moving in such a way that its normal velocity is always equal to the induced volume density \sqrt{g} (on $\widehat{\sum}$) devided by a fixed non-dynamical density $\rho(\varphi)$ ('the' volume density of the parameter manifold):

$$\dot{x}_{1} = \frac{1}{\rho} \{x_{2}, x_{3}, x_{4}\}$$

$$\dot{x}_{2} = -\frac{1}{\rho} \{x_{3}, x_{4}, x_{1}\}$$

$$\dot{x}_{3} = \frac{1}{\rho} \{x_{4}, x_{1}, x_{2}\}$$

$$\dot{x}_{4} = -\frac{1}{\rho} \{x_{1}, x_{2}, x_{3}\}.$$
(69)

With the curly bracket defined as before (cp. (4)), it will be an immediate consequence of (68) that

$$Q_n := \int_{\Sigma} d^3 \varphi \, \rho(\varphi) \, \mathcal{L}^n \tag{70}$$

is time-independent (for any n).

In [2] evolution-equations of the form (69) (in any number of dimensions) were shown to correspond to the diffeomorphism invariant part of an integrable Hamiltonian field theory (as well as to a gradient flow); one way to solve such equations is to note ([18], [2]) that the time at which the hypersurface will pass a point \vec{x} in space will simply be a harmonic function.

In any case, the (a) form of $(\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2)$ that will yield the equivalence of (69) with (68) is:

$$\mathcal{L} = (x_1 + ix_2)\frac{1}{\lambda} + (x_3 + ix_4)\frac{1}{\mu} + \mu(x_3 - ix_4) - \lambda(x_1 - ix_2)$$

$$\mathcal{M}_1 = \frac{\mu}{2}(x_3 - ix_4) - \frac{1}{2\mu}(x_3 + ix_4)$$

$$\mathcal{M}_2 = \frac{\lambda}{2}(x_1 - ix_2) + \frac{1}{2\lambda}(x_1 + ix_2)$$
(71)

(involving two spectral parameters, λ and μ). Surely, this observation will have much more elegant formulations, and conclusions.

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