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On Commuting Transfer Matrices

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Abstract. We study the non-singular R-matrices of the 8 vertex model satisfying the free Fermion condition, with a generalization that we allow non symmetric off diagonals.

Using the result shown in another paper that the transfer matrix T constructed from each such R-matrix commutes with a class of XYh-type Hamiltonian (somewhat generalized, and not necessarily selfadjoint), we show that such T-matrices commuting with each fixed Hamiltonian commute with each other at least for generic values of parameters.

In terms of the (Fermion) Clifford algebra obtained by the Jordan-Wigner transformation, the transfer matrix T for a generic value of parameters is shown to coincide with a constant multiple of elements of the group Spin $(2N, C)$ when multiplied by even and odd particle number projection operator.

1 Introduction

We consider the transfer matrix T for spin $1/2$ system (for example, see [1]) constructed as the trace of a product of R-matrix R along lattice $\{1, 2, \cdots, N\}$ of a finite length N with the periodic condition (eq. (2.4)).

Our main results are as follows:

(1) For each R of the type specified below, we find a class of Hamiltonians commuting with T.

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(2) The transfer matrices T, commuting with a fixed such Hamiltonian, are shown to commute each other.

The result (1) is stronger than what has been proved in that the class of R-matrices R we are considering include non-symmetric matrices. The result (2) seems to be much stronger than what follow from the known Yang-Baxter equations.

The R-matrices we consider are 4×4 matrices with 4 entries at 4 corners, 4 entries in the central 2×2 square, and zero for all other 8 entries (eq. (2.1)). The 8 non-trivial entries are complex numbers satisfying (1) a homogeneous second degree equation called the "free Fermion condition" (eq. (2.2)) and (2) the non-singular condition $\det R \neq 0$ (eq. (2.3)). We allow non-symmetric matrices, as emphasized above.

The 4-dimensional complex vector space on which R is operating will be identified with the tensor product of two 2-dimensional spaces, on each of which Pauli spin matrices (2×2 matrices) $\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_z^{(k)}$ are operating, where the upper index $k = a, b$ distinguishes the two component spaces of the tensor product.

We will be considering the algebra $\mathfrak{A}$ generated by spin matrices $\sigma_\alpha^{(j)}$, ($\alpha = x, y, z$) on lattice sites $j = 1, \cdots, N$. The R-matrix based on spin matrices $\sigma_\alpha^{(j)}$ and $\sigma_\beta^{(j+1)}$ will be denoted by $R_{j,j+1}$ ($a = j$, $b = j+1$ in the above notation). The Hamiltonians which we find to be commuting with T are of the following form:

$$H = \sum_{j=1}^{N} H_{j,j+1} \quad (H_{N,N+1} \equiv H_{N,1}) \tag{1.1}$$

$$H_{ab} = J_{11}\sigma_x^{(a)}\sigma_x^{(b)} + J_{12}\sigma_x^{(a)}\sigma_y^{(b)} + J_{21}\sigma_y^{(a)}\sigma_x^{(b)} + J_{22}\sigma_y^{(a)}\sigma_y^{(b)} + \lambda(\sigma_z^{(a)} + \sigma_z^{(b)}). \tag{1.2}$$

The commutativity of H and T is proved by generalizing the proof for symmetric R by Krinsky [2] to non-symmetric case. Naturally, we have to use somewhat more general Hamiltonians given above.

We prove the commutativity of $T's$ which commute with a fixed H by a method taken from [3]. It requires three properties of $T's$, namely the commutativity with a nontrivial H of the above form, the translation invariance and T being a multiple of an element of the group $\mathrm{Spin}(2N, \mathbb{C})$.

For the last property, we use the Jordan Wigner transformation to introduce Fermion creation and annihilation operators c_j^*, c_j ($j = 1, \cdots, N$) (eq. (4.1)). Their linear combinations denoted by $B(h)$ (eq. (4.7)) form a complex Clifford algebra of $2N$ dimension and define the group $\mathrm{Spin}(2N, \mathbb{C})$ which is denoted by $\widetilde{G}_c$ below.

In this connection, an important role is played by the modified R-matrix $\hat{R} = RP$ where P is the exchange operator of the two component spaces of the tensor product. R-matrices we are using are exactly characterized among all 4×4 matrices by its property that $\hat{R}$ is a constant multiple of an element in $\widetilde{G}_c$ (Proposition 5.2).

Returning to the transfer matrix T, we modify it by the shift T_0 of lattice sites (to the left): $\hat{T} = TT_0^{-1}$, where T and T_0 and hence $\hat{T}$ and T_0 commute. Note that T_0 does not depend on parameters, i.e. common for all parameter values. Therefore, the commutativity of T for two sets of parameter values is equivalent to the commutativity of the corresponding $\hat{T}$.

We then make a crucial observation (Lemma 2.1) that $\hat{T}$ is a cyclic product of $\hat{R}_{j,j+1}$, $j = 1, \cdots, N$ ($\hat{R}_{N,N+1} = \hat{R}_{N,1}$). This is a great simplification because the trace operation in the original definition of T (eq. (2.4)) is gone in this formula, which does not seem to have been noticed before. Because of this observation, we would obtain the conclusion that for a generic values of parameters $\hat{T}$ is a constant multiple of an element of $\tilde{G}_c$ if all $R_{j,j+1}$, $j = 1, \cdots, N$, were constant multiples of elements of $\tilde{G}_c$. The latter statement holds for $j = 1, \cdots, N-1$ but fails unfortunately for $j = N$ (i.e. for $R_{N,1}$) because of the presence of an extra operator $S = \prod_{j=1}^{N} \sigma_z^{(j)}$.

Because S takes eigenvalues ± 1 (due to $S = S^*, S^2 = 1$) and commute with elements of $\tilde{G}_c$, with Hamiltonians, and more generally with any even elements of $\mathfrak{A}$, we introduce the spectral projections $E_\pm$ of S for its eigenvalues ± 1 (eq. (5.5)). We then make analysis separately on two subspaces corresponding to $E_\pm$.

On the range of $E_\pm, S$ can be replaced by ± 1 and $R_{N,1}$ coincide with a constant multiple of an element of $\tilde{G}_c$. However, due to the cyclic product instead of the ordinary product, we have to make a further computation to reach the conclusion that for a generic values of parameters $\hat{T}$ coincides with a constant multiple $\hat{T}^\pm$ of elements of $\tilde{G}_c$, namely $\hat{T}E_\pm = \hat{T}^\pm E_\pm$.

We also find out that there exist quadratic expressions $H^\pm$ in creation and annihilation operators satisfying $HE_\pm = H^\pm E_\pm$ for the Hamiltonian H of the form (1.1). It then easily follows that $[H, T] = 0$ implies $[H^\pm, \hat{T}^\pm]E_\pm = 0$. However, it requires a nontrivial argument to obtain $[H^\pm, \hat{T}^\pm] = 0$.

Similarly, there exist shift automorphisms $U^\pm$ of the Clifford algebra such that

$$U^\pm \hat{T}^\pm (U^\pm)^{-1} E_\pm = \hat{T}^\pm E_\pm$$

follows easily from $[T_0, T] = 0$. However, it is non-trivial to obtain the translation invariance $U^\pm \hat{T}^\pm (U^\pm)^{-1} = \hat{T}^\pm$. Essential parts of the argument for the proof uses some explicit structure of Fock space and will be given in the Appendix.

As a consequence of all these arguments, we obtain three properties for $\hat{T}^\pm$ and hence mutual commutativity of $\hat{T}^\pm$ for different R-matrices commuting with the same Hamiltonian. This then imply the final conclusion about the commutativity of transfer matrices.

2 Transfer Matrix

(i) Boltzmann Weight.

In the 8-vertex model, which we will be discussing in this paper, the energy function $h(\lambda, \alpha, \lambda', \alpha')$ at a vertex depends on the configuration λ, α, λ', α' of 4 edges meeting at that vertex as in figure 1, where the configuration of each edge (i.e. each of λ, α, λ', α') takes 2 values, say ± 1.

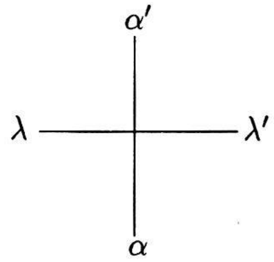

Figure 1. Configuration of edges at a vertex

We may consider the corresponding Boltzmann weight

$$R(\lambda, \alpha; \lambda', \alpha') = \exp -\beta h(\lambda, \alpha, \lambda', \alpha')$$

as 4×4 matrix in the following manner.

Let V be a two-dimensional complex vector space with an orthonormal basis $\{e_1, e_{-1}\}$. R is then interpreted as a 4×4 matrix acting on

$$v = \sum_{\lambda,\alpha=\pm 1} v_{\lambda\alpha}(e_\lambda \otimes e_\alpha) \in V \otimes V$$

by

$$Rv = \sum_{\lambda,\alpha=\pm 1} \Big(\sum_{\lambda',\alpha'=\pm 1} R(\lambda, \alpha; \lambda', \alpha') v_{\lambda'\alpha'} \Big) (e_\lambda \otimes e_\alpha).$$

With respect to the basis

$$\xi_1 = e_1 \otimes e_1, \quad \xi_2 = e_1 \otimes e_{-1}, \quad \xi_3 = e_{-1} \otimes e_1, \quad \xi_4 = e_{-1} \otimes e_{-1},$$

R takes the following form for our model.

$$R = \begin{bmatrix} a_+ & 0 & 0 & d \\ 0 & b_+ & c & 0 \\ 0 & c' & b_- & 0 \\ d' & 0 & 0 & a_- \end{bmatrix}. \tag{2.1}$$

We restrict ourselves to the case where the following "free Fermion condition" is satisfied

$$a_+ a_- + b_+ b_- - cc' - dd' = 0 \tag{2.2}$$

Furthermore, we impose a further condition

$$a_+ a_- - dd' \neq 0, \tag{2.3}$$

which is the condition (under (2.2)) for the existence of R^{-1}. (The significance of the condition (2.2) will become clear in the next section.)

We will be dealing with the general case of complex parameters without any other conditions than (2.2) and (2.3). In fact, for our main conclusion of the commutativity, the condition (2.3) is also not required, just as the limiting case of those satisfying (2.3).

In the following discussion, we have to deal with many 2-dimensional spaces V_j. In such a situation, the same matrix R acting on $V_j \otimes V_k$ will be denoted by R_{jk}.

(ii) Transfer Matrix.

We will be considering the 8-vertex model (2.2) on a two-dimensional $M \times N$ lattice with the periodic condition. The partition function will be of the form

$$Z = \sum e^{-\beta \sum h_n} = \sum \prod_n R_n(\cdot)$$

where the total energy $\sum h_n$ is the sum of interaction energy at all vertices, $R_n(\cdot)$ is the Boltzmann weight $R(\lambda, \alpha : \lambda', \alpha')$ (introduced in (i)) at the vertex n and the summation is over all configuration of edges. The summation will now be interpreted as product and traces of matrices in the following way.

If we look at a specific edge on m th row between j th and $(j+1)$ th columns, the only $R_n(\cdot)$ which depend on the configuration λ' of this edge are for two vertices $n = (m, j)$ and $n' = (m, j+1)$. We may consider R_n to be acting on $V_0 \otimes V_j$ and $R_{n'}$ acting on $V_0 \otimes V_{j+1}$, where V_j and V_{j+1} refers to configuration of edges on j th and $(j+1)$ th columns, respectively. Then the summation over λ' yields

$$\begin{aligned}&\sum_{\lambda'} R_n(\lambda, \alpha_j, \lambda', \alpha_j') R_{n'}(\lambda', \alpha_{j+1}, \lambda'', \alpha_{j+1}') \\ &= (R_n R_{n'})(\lambda, \alpha_j, \alpha_{j+1}; \lambda'', \alpha_j', \alpha_{j+1}')\end{aligned}$$

where R_n is acting on $V_0 \otimes V_j$, $R_{n'}$ acting on $V_0 \otimes V_{j+1}$, the product in $R_n R_{n'}$ is with respect to the action on the space V_0 with the resulting product matrix $R_n R_{n'}$ acting on $V_0 \otimes V_j \otimes V_{j+1}$. In this situation, we denote R_n as R_{0j}, $R_{n'}$ as $R_{0,j+1}$ and their product as $R_{0j} R_{0,j+1}$ with the understanding that R_{0j} is the matrix $R_n \otimes \mathbf{1}_{j+1}$ acting on $V_0 \otimes V_j \otimes V_{j+1}$ ($\mathbf{1}_{j+1}$ denotes the unit matrix E on V_{j+1}), R_{j+1} is a similar matrix for $R_{n'}$ and their product is then the usual product of matrices R_{0j} and $R_{0,j+1}$ acting on the same space $V_0 \otimes V_j \otimes V_{j+1}$.

Repeating the same procedure for summation over configurations on successive edges on m th row, we obtain the following matrix acting on $V_1 \otimes V_2 \otimes \cdots \otimes V_N$, called the transfer matrix.

$$T = tr_{V_0} R_{01} R_{02} \cdots R_{0N} \tag{2.4}$$

Here products and the trace are taken on the space V_0. The matrix elements of T will be of the form

$$T(\alpha_1, \alpha_2, \ldots, \alpha_N; \alpha_1', \alpha_2', \ldots, \alpha_N')$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ refers to the configuration of the edges on the first, the second, ..., the N th column between $(m-1)$ th and m th row and $\alpha_1', \alpha_2', \ldots, \alpha_N'$ refers to those between m th and $(m+1)$ th rows. The partition function Z is obtained as

$$Z = tr\, T^M$$

where products and the trace are now taken on the space $V_1 \otimes \cdots \otimes V_N$.

The quantity we want to discuss in this paper is the transfer matrix T itself as a matrix acting on

$$\mathbf{V} \equiv V_1 \otimes \cdots \otimes V_N \tag{2.5}$$

(iii) Translation invariance.

We introduce the translation operator T_0 on $\mathbf{V}$ by its action

$$T_0(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N}) = e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_N} \otimes e_{\alpha_1} \tag{2.6}$$

on an orthonormal basis $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N}$ of $\mathbf{V}$. It is the translation to the left. The inverse of T_0 is the translation to the right:

$$T_0^{-1}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N}) = e_{\alpha_N} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{N-1}} \tag{2.7}$$

Then we obtain

$$(\mathbf{1}_{V_0} \otimes T_0) R_{0j} (\mathbf{1}_{V_0} \otimes T_0)^{-1} = R_{0,j-1} \tag{2.8}$$

on $V_0 \otimes \mathbf{V}$. ($j-1$ for $j=1$ is to be understood as N.) Hence we obtain from (2.4) and the trace property $tr(AB) = tr(BA)$, the following translation invariance

$$T_0 T T_0^{-1} = T. \tag{2.9}$$

In the following, we will be discussing the property of

$$\hat{T} = T_0^{-1} T = T T_0^{-1} \tag{2.10}$$

which also commutes with T_0.

(iv) Basic formula for $\hat{T}$.

We introduce the following exchange operator acting on $V \otimes V$.

$$P(e_\alpha \otimes e_\beta) = e_\beta \otimes e_\alpha. \tag{2.11}$$

Namely

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

We then define

$$\hat{R} = RP = \begin{pmatrix} a_+ & 0 & 0 & d \\ 0 & c & b_+ & 0 \\ 0 & b_- & c' & 0 \\ d' & 0 & 0 & a_- \end{pmatrix}. \tag{2.13}$$

We now introduce the notion of circular product. Let $\hat{R}_{j,k}$ be a matrix acting on $V_j \otimes V_k$, namely $\hat{R}_{j,k} \in \mathfrak{B}(V_j) \otimes \mathfrak{B}(V_k)$ where $\mathfrak{B}(V)$ denotes the algebra of all 2×2 matrices acting on V. Then for $\hat{R}_{N,N+1} = \sum r_{kl} A_k \otimes B_l$ with $V_{N+1} = V_1$, $A_k \in \mathfrak{B}(V_N)$, $B_l \in \mathfrak{B}(V_1)$, we define the following circular product:

$$\prod_{j:1\to N}^{\text{circ.}} \hat{R}_{j,j+1} = \sum_{k,l} r_{kl}\, B_l\, \hat{R}_{1,2}\hat{R}_{2,3}\cdots \hat{R}_{N-1,N}\, A_k \tag{2.14}$$

where B_l is identified with $B_l \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_N$ and A_k with $\mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_{N-1} \otimes A_k$.

The circular product is almost the same as the ordinary product $\hat{R}_{1,2}\hat{R}_{2,3} \cdots \hat{R}_{N-1,N}\hat{R}_{N,1}$ except that the matrix of $\mathfrak{B}(V_1)$ contained in $\hat{R}_{N,1}$ should multiply the matrix of $\mathfrak{B}(V_1)$ contained in $\hat{R}_{1,2}$ from the left instead of from the right, namely their order of product is inverted. We also write it as

$$:\hat{R}_{1,2}\hat{R}_{2,3}\cdots \hat{R}_{N-1,N}\hat{R}_{N,1}:_1 \tag{2.15}$$

where $:\cdots:_1$ denote the inversion of the order of product of matrices belonging to $\mathfrak{B}(V_1)$ which are contained in each factor $\hat{R}_{j,j+1}$ (in fact those in $\hat{R}_{1,2}$ and in $\hat{R}_{N,1}$). We have the following formulas:

Formula 1 : $\hat{R}_{1,2}\hat{R}_{2,3}\cdots\hat{R}_{N,1}:_1 = {}^{t^1}\left\{({}^{t^1}\hat{R}_{1,2})\hat{R}_{2,3}\cdots({}^{t^1}\hat{R}_{N,1})\right\}$ (2.16)

where t^1 on the left shoulder indicate the transposition of matrices belonging to $B(V_1)$.

Formula 2 : $\hat{R}_{1,2}\hat{R}_{2,3}\cdots\hat{R}_{N,1}:_1 = :\hat{R}_{N,1}\hat{R}_{1,2}\cdots\hat{R}_{N-1,N}:_N$ (2.17)

Roughly speaking the circular product is invariant under circular permutation.

The following Lemma gives the basic formula for $\hat{T}$, which will be the basis of our computation in this paper.

Lemma 2.1

$$\hat{T} = \prod_{j:1\to N}^{\text{circ.}} \hat{R}_{j,j+1} \tag{2.18}$$

where $V_{N+1} = V_1$ and $\hat{R}_{j,j+1}$ is $\hat{R}$ of (2.13) acting on $V_j \otimes V_{j+1}$.

Proof. By definition (2.13)

$$\hat{R}_{j,j+1}(\alpha_j, \alpha_{j+1}; \alpha'_j, \alpha'_{j+1}) = R(\alpha_j, \alpha_{j+1}; \alpha'_{j+1}, \alpha'_j)$$

where α_j, α'_j and α_{j+1}, α'_{j+1} are labels for orthonormal basis vectors of V_j and V_{j+1}, respectively. Then

$$\begin{aligned}
&(\hat{R}_{j-1,j}\hat{R}_{j,j+1})(\alpha_{j-1}, \alpha_j, \alpha_{j+1}; \alpha'_{j-1}, \alpha'_j, \alpha'_{j+1}) \\
&= \sum_{\alpha''_j} R(\alpha_{j-1}, \alpha_j; \alpha''_j, \alpha'_{j-1})\, R(\alpha''_j, \alpha_{j+1}; \alpha'_{j+1}\alpha'_j) \\
&= \sum_{\lambda_j} R(\alpha_{j-1}, \alpha_j; \lambda_j, \alpha'_{j-1})\, R(\lambda_j, \alpha_{j+1}; \alpha'_{j+1}, \alpha'_j)
\end{aligned}$$

where the second equality is simply a relabelling of the summation index from α''_j to λ_j.

Repeating this computation, we obtain

$$\begin{aligned}
&\Big(\prod_{j:1\to N}^{\text{circ.}} \hat{R}_{j,j+1}\Big)(\alpha_1,\dots,\alpha_N;\alpha'_1,\dots,\alpha'_N)\\
&= \sum_{\lambda_1\cdots\lambda_N}\prod_{j=1}^{N} R(\lambda_j,\alpha_{j+1};\lambda_{j+1},\alpha'_j)
\end{aligned}$$

where the index $N+1$ is to be replaced by 1. Therefore

$$\begin{aligned}
&\Big\{T_0\Big(\prod_{j:1\to N}^{\text{circ.}} \hat{R}_{j,j+1}\Big)\Big\}(\alpha_1,\dots,\alpha_N;\alpha'_1,\dots,\alpha'_N)\\
&= \Big(\prod_{j:1\to N}^{\text{circ.}} \hat{R}_{j,j+1}\Big)(\alpha_N,\alpha_1,\dots,\alpha_{N-1};\alpha'_1,\dots,\alpha'_N)\\
&= \sum_{\lambda_1\cdots\lambda_N}\prod_{j=1}^{N} R(\lambda_j,\alpha_j;\lambda_{j+1},\alpha'_j)\\
&= tr_0(R_{01}\cdots R_{0N}) = T.
\end{aligned}$$

Q.E.D.

We note that the space V_0 in the definition of T is completely eliminated on the right hand side of the formula (2.18). This is the merit of this formula.

3 Free Fermion Condition

In this section, we clarify the implication of Free Fermion Condition (2.2) for the purpose of its application to our main theorem.

We use the following linear basis of 2×2 matrices in $\mathfrak{B}(V)$.

$$E=\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad i\sigma_y=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Pauli matrices σ_x, σ_y, σ_z in $\mathfrak{B}(V_j)$ will be denoted by $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\sigma_z^{(j)}$. Due to their orthogonality

$$tr(\sigma_\alpha\sigma_\beta)=2\delta_{\alpha\beta},$$

the coefficients of the expansion

$$A=\sum_\alpha a_\alpha\sigma_\alpha$$

for an arbitrary 2×2 matrix A can be obtained by

$$a_\alpha=\frac{1}{2}tr(\sigma_\alpha A).$$

(The coefficient of $i\sigma_y$ will be $-ia_y$.)

For expansion of $\hat{R}_{j,j+1} \in \mathfrak{B}(V_j \otimes V_{j+1})$, we may use the product of above bases for V_j and V_{j+1}. $\hat{R}$ is of a special form so that it is a direct sum of 2×2 matrices

$$\hat{R}^e = \begin{pmatrix} a_+ & d \\ d' & a_- \end{pmatrix} \quad \text{on} \quad (V \otimes V)_e \equiv \mathbb{C}e_1 \otimes e_1 + \mathbb{C}e_{-1} \otimes e_{-1} \tag{3.1}$$

and

$$\hat{R}^0 = \begin{pmatrix} c & b_+ \\ b_- & c' \end{pmatrix} \quad \text{on} \quad (V \otimes V)_0 \equiv \mathbb{C}e_1 \otimes e_{-1} + \mathbb{C}e_{-1} \otimes e_1, \tag{3.2}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore only the following 8 terms out of 16 possible terms appear in the expansion of $\hat{R}$:

$$\sigma_0 \otimes \sigma_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \sigma_x \otimes \sigma_x = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix},$$

$$\sigma_0 \otimes \sigma_z = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \qquad \sigma_x \otimes (i\sigma_y) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix},$$

$$\sigma_z \otimes \sigma_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \qquad (i\sigma_y) \otimes \sigma_x = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

$$\sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \qquad (i\sigma_y) \otimes (i\sigma_y) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

We now focus on the coefficient of $\sigma_z \otimes \sigma_z$, which is given by

$$\begin{aligned} (1/4)tr_4\{D(\sigma_z \otimes \sigma_z)\} &= (1/4)(d_{11} - d_{22} - d_{33} + d_{44}) \\ &= (1/4)\{tr_2 D_e - tr_2 D_0\} \end{aligned} \tag{3.3}$$

for any 4×4 matrix $D = (d_{ij}) \in \mathfrak{B}(V \otimes V)$ which is a direct sum of $D_e \in \mathfrak{B}((V \otimes V)_e)$ and $D_0 \in \mathfrak{B}((V \otimes V)_0)$. In this formula, tr_n denotes the trace of an $n \times n$ matrix ($n = 4, 2$).

Lemma 3.1 *$\hat{R}$ can be written in the form*

$$\hat{R} = e^D \tag{3.4a}$$

$$D = D_e \oplus D_0, \quad tr_2 D_e = tr_2 D_0 \tag{3.4b}$$

(namely, the vanishing of (3.3) for D) if and only if the following 3 conditions are satisfied by parameters:

(a) The free Fermion condition (2.2) holds.

(b) $\hat{R}$ is non-singular, i.e. the condition (2.3) holds.

(c) $\hat{R}$ is not of the following form:

$$\hat{R}^e = \lambda \mathbf{1} + N^e, \quad \hat{R}^0 = -\lambda \mathbf{1} + N^0 \tag{3.5}$$

for $\lambda \neq 0$ and non-zero nilpotent matrices N^e and N^0.

Remark If $\hat{R}$ has the form excluded in the condition (c), then $\hat{R}$ can be written as the product

$$\hat{R} = R_a R_b$$

of the two matrices

$$R_a = \lambda \mathbf{1} \oplus -\lambda \mathbf{1}, \quad R_b = 1 + N$$

where

$$N = \lambda^{-1} N^e \oplus -\lambda^{-1} N^0$$

is nilpotent. Both R_a and R_b satisfy conditions (a), (b), (c) and hence there exist, by this Lemma, D_a and D_b of the form (3.4b) such that

$$R_a = e^{D_a}, \quad R_b = e^{D_b},$$
$$\hat{R} = e^{D_a} e^{D_b}.$$

Thus $\hat{R}$ can be written as a product of 2 matrices of the form (3.4) but the product $\hat{R}$ can not be written in such a form.

Proof of Lemma 3.1

The necessity follows from the following computation. From (3.4), it follows that

$$\hat{R} = e^D = e^{D_e} \oplus e^{D_0}.$$

Namely

$$\hat{R}^e = e^{D_e}, \quad \hat{R}^0 = e^{D_0}$$

The vanishing of (3.3) for D implies

$$\det \hat{R}^e = e^{trD_e} = e^{trD_0} = \det \hat{R}^0$$

where $\det A$ denotes the determinant of A, and we have used the formula $\det e^D = e^{trD}$ for a general matrix D. Due to

$$\det \hat{R}^e = a_+ a_- - dd', \quad \det \hat{R}^0 = cc' - b_+ b_-,$$

the free Fermion condition (2.2) follows.

For any matrix D, e^D has the inverse e^{-D} and hence it is non-singular. Finally, consider $\hat{R}$ of the form (3.5). Suppose

$$\hat{R}^e = e^{D_e},\ \hat{R}^0 = e^{D_0}. \tag{3.6}$$

If the Jordan form of D_e were diagonal, then e^{D_e} would be diagonal, in contradiction with the assumed form (3.5). The same argument holds for D_0. Hence

$$D_e = \mu_e \mathbf{1} + M_e,\ D_0 = \mu_0 \mathbf{1} + M_0$$

for some nilpotent M_e and M_0. By the requirement (3.6), we have

$$\begin{aligned} &e^{\mu_e} = \lambda,\ e^{\mu_0} = -\lambda, \\ &e^{M_e} = 1 + M_e = 1 + \lambda^{-1} N_e,\ e^{M_0} = 1 + M_0 = 1 - \lambda^{-1} N_0. \end{aligned}$$

Thus $e^{\mu_e - \mu_0} = -1$ and hence

$$\mu_0 = \mu_e + (2\ell + 1)\pi i$$

for some integer ℓ. Hence

$$Tr\,\mu_0 \mathbf{1} - Tr\,\mu_e \mathbf{1} = 2(2\ell + 1)\pi i \neq 0.$$

So (3.4b) is not satisfied. This completes the necessity proof.

Next we show the sufficiency. First consider the case where the Jordan normal form of $\hat{R}^e$ is diagonal. By condition (2), both $\hat{R}^e$ and $\hat{R}^0$ are non-singular and hence they can be written as e^{D_e} and e^{D_0} for some matrices D_e and D_0. If the eigenvalues of $\hat{R}^e$ are λ_{e_1} and λ_{e_2}, then eigenvalues of D_e can be taken to be

$$\mathrm{Log}\lambda_{e_1} + 2\ell_1 \pi i,\ \mathrm{Log}\lambda_{e_2} + 2\ell_2 \pi i$$

for any integers ℓ_1 and ℓ_2 (Log is a fixed branch of the logarithmic function) so that for one choice of D_e there is another choice D'_e satisfying

$$\hat{R}^e = e^{D_e} = e^{D'_e},\ Tr D'_e = Tr D_e + 2\ell\pi i$$

for any integer ℓ (simply by adjusting ℓ_2 with ℓ_1 fixed, for example). On the other hand the free Fermion condition implies

$$e^{Tr D_e} = \det \hat{R}^e = \det \hat{R}^0 = e^{Tr D_0}$$

so that

$$Tr D_e = Tr D_0 + 2\ell\pi i$$

for some integer ℓ. Then we can choose D'_e which satisfies $\hat{R}^e = e^{D'_e}$ and (3.4) by

$$Tr D'_e = Tr D_e + 2\ell\pi i = Tr D_0.$$

The same argument holds if $\hat{R}^0$ has a diagonal Jordan normal form. The remaining cases are when both $\hat{R}^e$ and $\hat{R}^0$ have non-diagonal Jordan normal form, i.e.

$$\hat{R}^e = \lambda_e \mathbf{1} + N^e,\ \hat{R}^0 = \lambda_0 \mathbf{1} + N^0$$

with some nilpotent N^e and N^0. By the free Fermion condition,

$$\lambda_e^2 = \det \hat{R}^e = \det \hat{R}^0 = \lambda_0^2.$$

Hence $\lambda_0 = \pm\lambda_e$. The case $\lambda_0 = -\lambda_e$ gives (3.5) with $\lambda = \lambda_e$ and is excluded by condition (c). If $\lambda_0 = \lambda_e = \lambda$, then we can fix one μ satisfying $e^\mu = \lambda$ and define

$$D_e = \mu\mathbf{1} + \lambda_e^{-1}N^e,\ D_0 = \mu\mathbf{1} + \lambda_0^{-1}N^0.$$

Then (3.4) is satisfied.

Q.E.D.

4 Fermion creation and annihilation operators

We will be dealing with the algebra

$$\mathfrak{A} = \mathfrak{B}(V_1) \otimes \cdots \otimes \mathfrak{B}(V_N).$$

The notation $\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}$ for Pauli matrices in each $\mathfrak{B}(V_j)$ will also be used to denote the corresponding operators in $\mathfrak{A}$, i.e. $\sigma_\alpha^{(j)}$ denotes also

$$\mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_{j-1} \otimes \sigma_\alpha^{(j)} \otimes \mathbf{1}_{j+1} \otimes \cdots \otimes \mathbf{1}_N$$

$(\alpha = x, y, z)$ where $\mathbf{1}_k$ denotes the unit matrix in $\mathfrak{B}(V_\mathbf{k})$. We denote by $\mathfrak{A}_j$ the subalgebra of $\mathfrak{A}$ generated by them.

(i) Jordan Wigner transformation.

We introduce the following operators:

$$c_j = S_j(\sigma_x^{(j)} - i\sigma_y^{(j)})/2, \tag{4.1a}$$

$$c_j^* = S_j(\sigma_x^{(j)} + i\sigma_y^{(j)})/2, \tag{4.1b}$$

$$S_j = \sigma_z^{(1)} \cdots \sigma_z^{(j-1)} \quad (S_1 = 1). \tag{4.1c}$$

The inverse transformation is given by

$$\sigma_z^{(j)} = 2c_j^*c_j - 1, \tag{4.2a}$$

$$\sigma_x^{(j)} = S_j(c_j + c_j^*), \tag{4.2b}$$

$$\sigma_y^{(j)} = iS_j(c_j - c_j^*), \tag{4.2c}$$

The operator c_j^* is the adjoint of c_j as can be derived from (4.1a) and (4.1b). They satisfy the following canonical anticommutation relations (CAR):

$$[c_j, c_k]_+ = [c_j^*, c_k^*]_+ = 0, \tag{4.3a}$$

$$[c_j, c_k^*]_+ = \delta_{jk} \quad (0 \text{ for } j \neq k, \quad 1 \text{ for } j = k) \tag{4.3b}$$

where $[A, B]_+ = AB + BA$. By (4.2), the set of all c_j and c_j^* $(j = 1, \cdots, N)$ generates

$$\mathfrak{A} = \mathfrak{B}(V_1 \otimes \cdots \otimes V_N) = \mathfrak{B}(V_1) \otimes \cdots \otimes \mathfrak{B}(V_N). \tag{4.4}$$

So $\mathfrak{A}$ is reinterpreted as a CAR algebra or Fermion algebra.

(ii) Self-dual CAR description [4].

For the sake of compact notation, we introduce the following notation. For each

$$f = (f_1, \cdots, f_N) \in \ell^2(\{1, \cdots, N\}),$$

we define

$$c(f) = \sum f_j c_j, \quad c^*(f) = \sum f_j c_j^*. \tag{4.5}$$

For each

$$h = \begin{pmatrix} f \\ g \end{pmatrix} \in L \equiv \ell^2(\{1, \cdots, N\}) \oplus \ell^2(\{1, \cdots, N\}), \tag{4.6}$$

we define

$$B(h) = c^*(f) + c(g) \in \mathfrak{A}. \tag{4.7}$$

It satisfies (and characterized by) the following self-dual CAR relations.

(1) $h \in L \to B(h) \in \mathfrak{A}$ is (complex) linear.

(2) $$[B(h_1)^*, B(h_2)]_+ = (h_1, h_2) \tag{4.8}$$

where the ℓ^2 inner product on L is defined by

$$(h_1, h_2) = (f_1, f_2) + (g_1, g_2)$$

for

$$h_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \in L, \quad h_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in L$$

and

$$(f, g) = \bar{f}_1, g_1 + \cdots + \bar{f}_n g_n$$

for $f, g \in \ell^2(\{1, \cdots, n\})$.

(3) $$B(h)^* \equiv B(\Gamma h) \tag{4.9}$$

where

$$\Gamma \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}. \tag{4.10}$$

The conjugate linear operator Γ on L is involutive ($\Gamma^2 = 1$) and antiunitary ($(\Gamma h_1, \Gamma h_2) = (h_2, h_1)$) and hence viewed as an abstract complex conjugation operator on L in the sense that there exists an orthonormal basis $e_1, \cdots, e_{2N}$ in L satisfying $\Gamma e_j = e_j$ for all $j = 1, \cdots, 2N$

and Γ is a (concrete) complex conjugation of components of vectors relative to such a Γ-invariant basis.

(iii) Quadratic combination

Let $A = (A_{ij})_{i,j=1,\cdots,2N}$ be $2N \times 2N$ matrix. It can be written in terms of $N \times N$ matrices $A^{k\ell}$ as

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}. \tag{4.11}$$

We define

$$\begin{aligned} (B, AB) &\equiv (c^*, A^{11}c) + (c^*, A^{12}c^*) + (c, A^{21}c) + (c, A^{22}c^*) \\ &= \sum_{i,j=1}^{N} (A_{ij}c_i^*c_j + A_{i,j+N}c_i^*c_j^* \\ &\quad + A_{i+N,j}c_ic_j + A_{i+N,j+N}c_ic_j^*), \end{aligned} \tag{4.12}$$

which covers all possible quadratic expressions of c_j and c_j^*, $j = 1, \cdots, N$. It has the following commutation relations

$$[(B, AB), B(h)] = B(\alpha(A)h), \tag{4.13}$$
$$[(B, A_1B), (B, A_2B)] = (B, [\alpha(A_1), \alpha(A_2)]B) \tag{4.14}$$
$$\alpha(A) \equiv A - \Gamma A^* \Gamma \tag{4.15}$$

which easily follows from the formula:

$$[AB, C] = A[B, C]_+ - [A, C]_+ B.$$

The set of all $\alpha(A)$ can be characterized as

$$\mathfrak{o}(2N, C) \equiv \{A \in B(L); \Gamma A^* \Gamma = -A\}. \tag{4.16}$$

Namely $\alpha(A)$ for any A belongs to (4.16), and $\alpha(A) = 2A$ holds for any A in the set (4.16). It is a complex linear set closed under commutator and hence is a Lie algebra.

The operator Γ may be viewed as a complexification and $\mathfrak{o}(2N, C)$ is the set of all anti-symmetric matrices (under transposition) with respect to a Γ-invariant orthonormal basis.

(iv) Bogoliubov automorphisms

From (4.13), we obtain for $A \in \mathfrak{o}(2N, C)$

$$e^{(B,AB)} B(h) e^{-(B,AB)} = B(e^{2A}h), \tag{4.17}$$

where

$$e^{2A} \in O(2N, C) = \{S \in \mathcal{B}(L); \Gamma S^* \Gamma = S^{-1}\}. \tag{4.18}$$

and $O(2N, \mathbb{C})$ is the complex orthogonal group.

For any $S \in O\ (2N, C)$, there exists a unique algebraic automorphism α_S of $\mathfrak{A}$ (not necessarily preserving $*$) satisfying

$$\alpha_S(B(h)) = B(Sh). \tag{4.19}$$

called a Bogoliubov automorphism. It becomes a $*$-automorphism if and only if U is unitary i.e.

$$U \in O(2N) = \{U \in B(L) : \ [\Gamma, U] = 0,\ UU^* = U^*U = 1\} \tag{4.20}$$

where $O(2N)$ is the real orthogonal group.

The equation (4.17) says that the similarity transformation by $e^{t(B,AB)}$ gives rise to a Bogoliubov automorphism:

$$Ad\ e^{t(B,AB)} = \alpha_{e^{2tA}}. \tag{4.21}$$

It obviously belongs to the connected component of the Lie group:

$$e^{2tA} \in SO(2N, C).$$

We shall denote $G = SO(2N)$ and $G_C = SO(2N, C)$. Their Lie algebras are

$$\mathfrak{g} = \{A \in B(L);\ [\Gamma, A] = 0,\ A^* = -A\},$$
$$\mathfrak{g}_c = \mathfrak{o}(2N, C).$$

The latter is the complexification of $\mathfrak{g}$.

Let $\tilde{G}_c$ be the subgroup of the group $\mathfrak{A}_u$ of all unitaries in $\mathfrak{A}$ generated by $e^{(B,AB)}$, $A \in \mathfrak{g}_c$. Its Lie algebra is the Lie algebra of all (B, AB) which is isomorphic to the Lie algebra $\mathfrak{g}_c$. Both $\tilde{G}_c$ and G_c are connected Lie groups with isomorphic Lie algebras. It is known that the universal covering group of G_c for $N \geqq 2$ covers G_c twice. On the other hand

$$-1 = e_z^{\pi i \sigma^{(j)}} = e^{\pi i (c_j^* c_j - c_j c_j^*)}$$

induce the trivial automorphism $id.$ of $\mathfrak{A}$ and $(c_j^* c_j - c_j c_j^*) = (B, AB)$ for an $A \in \mathfrak{g}_0$, so that the homomorphism $\tilde{G}_c \to G_c$ induced by the isomorphism of their Lie algebra is not one-to-one. Hence $\tilde{G}_c$ is isomorphic to the universal covering group of G_c and the map

$$V \in \tilde{G}_c \to AdV \in G_c \tag{4.22}$$

is exactly two-to-one, namely, for each $\bar{V} \in G_c$, there are two elements $\pm V \in \tilde{G}_c$ such that $Ad(\pm V) = \bar{V}$.

5 The operator $\hat{T}$ as an element of $\tilde{G}_c$

(i) By the Jordan-Wigner transformation, the eight 4×4 matrices in Section 3 as elements of $B(V_j \otimes V_{j+1}) = \mathfrak{A}_j \otimes \mathfrak{A}_{j+1}$ can now be written in terms of quadratic expressions of c's and

c^*'s by substitution of (4.2):

$$\begin{aligned}
\sigma_0 \otimes \sigma_0 &= \mathbf{1}, \quad \sigma_0 \otimes \sigma_z = c_{j+1}^* c_{j+1} - c_{j+1} c_{j+1}^*, \\
\sigma_z \otimes \sigma_0 &= c_j^* c_j - c_j c_j^*, \quad \sigma_x \otimes \sigma_x = (c_j - c_j^*)(c_{j+1} + c_{j+1}^*) \\
\sigma_x \otimes (i\sigma_y) &= -(c_j - c_j^*)(c_{j+1} - c_{j+1}^*), \quad (i\sigma_y) \otimes \sigma_x = -(c_j + c_j^*)(c_{j+1} + c_{j+1}^*) \\
(i\sigma_y) \otimes (i\sigma_y) &= (c_j + c_j^*)(c_{j+1} - c_{j+1^*}), \\
\sigma_z \otimes \sigma_z &= (c_j^* c_j - c_j c_j^*)(c_{j+1}^* c_{j+1} - c_{j+1} c_{j+1}^*).
\end{aligned}$$

By Lemma 3.1 and the subsequent Remark, we obtain the following result.

Lemma 5.1 *$\hat{R}_{j,j+1} \in \mathfrak{A}_j \otimes \mathfrak{A}_{j+1}$ $(j = 1, 2, \cdots, N-1)$ is in $\tilde{G}_c$ up to a constant multiple (i.e. $\hat{R}_{j,j+1} \in \mathbb{C}\tilde{G}_c$) if the following two conditions are satisfied.*

(1) Free Fermion Condition. (Condition (2.2))

(2) R is not singular. (Condition (2.3))

(The constant multiple comes from the term $\sigma_0 \otimes \sigma_0$ in D.) Note that $\tilde{G}_c$ is a group and hence

$$\hat{R}_{1,2}\hat{R}_{2,3} \cdots \hat{R}_{N-1,N}$$

is also in $\tilde{G}_c$ up to a constant multiple.

The R-matrix R we are considering is of the form (2.1) with parameters satisfying the free Fermion condition (2.2) and the non-singular condition (2.3). The matrix $\hat{R}$ which is the R-matrix R modified by the exchange operator P as in (2.13) is playing the central role in our discussion of the transfer matrix T or $\hat{T}$. It can be characterized among all 4×4 matrices by the following condition.

Proposition 5.2 *For a 4 by 4 matrix $\hat{R}$ to belong to $\tilde{G}_c$ up to a constant multiple, it is necessary and sufficient that $R = \hat{R}P$ is of the form* (2.1) *and the parameters satisfy* (2.2) *and* (2.3).

Proof. We have already seen the "if" part in Lemma 5.1. We now prove the necessity. We consider the even-oddness automorphism Θ of $\mathfrak{A}$ which is uniquely defined by $\Theta(c_j) = -c_j$, $\Theta(c_j^*) = -\Theta(c_j^*)$, $j = 1, \ldots, N$. Then (B, AB), being a quadratic expression in c's, is Θ-invariant, hence so is any element $R \in \tilde{G}_c$. This implies that $\hat{R}$ and R are of the form (2.1). Also, any element of $\tilde{G}_c$ is invertible, have to be non-singular and hence (2.3) is satisfied.

Finally we note that the free Fermion condition for R is the equality of determinants of two 2×2 matrices in $\hat{R}$, one in the central square and the other on the four corners. We already know by Lemma 3.1 that it is satisfied by $e^{(B,AB)}$ and hence by their products. In particular any element of $\tilde{G}_c$ (for the case $N = 2$) satisfies this property for $\hat{R}$ and hence the free Fermion condition for $R = \hat{R}P$.

Q.E.D.

Since $\hat{R}_{N,1}$ has a different form, we now want to discuss $\hat{R}_{N,1}$ along with other matters associated with the edge of our lattice.

(ii) Edge of the lattice

At the edge of the lattice, we have the following relations:

$$\left.\begin{array}{ll} \sigma_x^{(N)} & = S_N(c_N + c_N^*) = (c_N + c_N^*)\sigma_z^{(N)} S = (c_N - c_N^*)S \\ \sigma_y^{(N)} & = iS_N(c_N - c_N^*) = i(c_N - c_N^*)\sigma_z^{(N)} S = i(c_N + c_N^*)S \\ \sigma_z^{(N)} & = 2c_N^* c_N - 1 \end{array}\right\} \tag{5.1}$$

where

$$S = \sigma_z^{(1)} \cdots \sigma_z^{(N)} = \prod_{j=1}^{N} (c_j^* c_j - c_j c_j^*). \tag{5.2}$$

The operator S anticommutes with every $B(h)$, $h \in L$ and hence commutes with quadratic expressions of $c's$ and $c^{*\prime}s$. Since $S^* = S$ and $S^2 = 1$, any $Q \in SO(2N, \mathbb{C})$ and $A \in \mathfrak{o}(2N, \mathbb{C})$ split as follows:

$$Q = Q_+ + Q_-, \quad Q_\pm = QE_\pm, \tag{5.3}$$
$$A = A_+ + A_-, \quad A_\pm = AE_\pm, \tag{5.4}$$
$$E_\pm = (1 \pm S)/2. \tag{5.5}$$

We now investigate $\hat{R}_{N,1}$ and its role in the cyclic product more closely. In terms of Pauli spin matrices, we have

$$\begin{aligned} \hat{R}_{N,1} = \; & \alpha_{00}\mathbf{1} + \alpha_{z0}\sigma_z^{(1)} + \alpha_{0z}\sigma_z^{(N)} + \alpha_{zz}\sigma_z^{(1)}\sigma_z^{(N)} \\ & + \alpha_{xx}\sigma_x^{(1)}\sigma_x^{(N)} + \alpha_{yx}\sigma_y^{(1)}\sigma_x^{(N)} + \alpha_{xy}\sigma_x^{(1)}\sigma_y^{(N)} + \alpha_{yy}\sigma_y^{(1)}\sigma_y^{(N)}, \end{aligned}$$

with parameters α_{ij} determined by

$$\begin{aligned} a_+ &= \alpha_{00} + \alpha_{z0} + \alpha_{0z} + \alpha_{zz}, \quad a_- = \alpha_{00} - \alpha_{z0} - \alpha_{0z} + \alpha_{zz}, \\ c &= \alpha_{00} - \alpha_{z0} + \alpha_{0z} - \alpha_{zz}, \quad c' = \alpha_{00} + \alpha_{z0} - \alpha_{0z} - \alpha_{zz}, \\ d &= \alpha_{xx} - i\alpha_{yx} - i\alpha_{xy} - \alpha_{yy}, \quad d' = \alpha_{xx} + i\alpha_{yx} + i\alpha_{xy} - \alpha_{yy}, \\ b_+ &= \alpha_{xx} + i\alpha_{yx} - i\alpha_{xy} + \alpha_{yy}, \quad b_- = \alpha_{xx} - i\alpha_{yx} + i\alpha_{xy} + \alpha_{yy}. \end{aligned}$$

First we discuss the free Fermion condition, which is equivalent to

$$\alpha_{00}\alpha_{zz} - \alpha_{0z}\alpha_{z0} = \alpha_{xy}\alpha_{yx} - \alpha_{xx}\alpha_{yy}. \tag{5.6}$$

in terms of the new parameters.

With the abbreviation

$$\tilde{g} = \hat{R}_{1,2} \cdots \hat{R}_{N-1,N} \in \mathbb{C}\tilde{G}_c,$$

we have

$$\hat{T} = \sum_{(\mu,\nu)} \alpha_{\mu\nu} \sigma_\mu^{(1)} \tilde{g} \sigma_\nu^{(N)}$$

where $\sigma_\mu^{(1)}$ is in front of g by the definition of the cyclic product, $\sigma_0^{(1)} = \sigma_0^{(N)} = \mathbf{1}$, and the pair of indices (μ, ν) run over 8 possibilities for the index of α.

By substituting (4.2) with $j = 1$ and (5.1), we obtain

$$\begin{aligned} \hat{T} &= A_{00}\tilde{g} + A_{30}c_1^* c_1 \tilde{g} + A_{03}\tilde{g} c_N^* c_N + A_{33} c_1^* c_1 \tilde{g} c_N^* c_N \\ &\quad + (A_{11} c_1^* \tilde{g} c_N + A_{12} c_1^* \tilde{g} c_N^* + A_{21} c_1 \tilde{g} c_N + A_{22} c_1 \tilde{g} c_N^*) S, \end{aligned}$$

with

$$\begin{aligned} A_{00} &= \alpha_{00} - \alpha_{0z} - \alpha_{z0} + \alpha_{zz}, \quad A_{33} = 4\alpha_{zz}, \\ A_{30} &= 2(\alpha_{z0} - \alpha_{zz}), \quad A_{03} = 2(\alpha_{0z} - \alpha_{zz}), \\ A_{11} &= \alpha_{xx} - i\alpha_{yx} + i\alpha_{xy} + \alpha_{yy} = b_2, \\ A_{12} &= -\alpha_{xx} + i\alpha_{yx} + i\alpha_{xy} + \alpha_{yy} = -d, \\ A_{21} &= \alpha_{xx} + i\alpha_{yx} + i\alpha_{xy} - \alpha_{yy} = d', \\ A_{22} &= -\alpha_{xx} - i\alpha_{yx} + i\alpha_{xy} - \alpha_{yy} = -b_1. \end{aligned}$$

The condition (5.6) is equivalent to

$$A_{00}A_{33} - A_{30}A_{03} = A_{11}A_{22} - A_{21}A_{12}. \tag{5.7}$$

When multiplied by $E_\pm$, S can be replaced by ± 1.

Hence we have $\hat{T}E_\pm = \hat{R}_\pm \tilde{g} E_\pm$ with

$$\begin{aligned} \hat{R}_\pm &= A_{00} + A_{30} c_1^* c_1 + A_{03} \mathfrak{c}' \mathfrak{c} + A_{33} c_1^* c_1 \mathfrak{c}' \mathfrak{c} \\ &\quad \pm (A_{11} c_1^* \mathfrak{c} + A_{12} c_1^* \mathfrak{c}' + A_{21} c_1 \mathfrak{c} + A_{22} c_1 \mathfrak{c}') \end{aligned}$$

where

$$\mathfrak{c} = \tilde{g} c_N \tilde{g}^{-1} = \alpha_g(c_N), \quad \mathfrak{c}' = \tilde{g} c_N^* \tilde{g}^{-1} = \alpha_g(c_N^*)$$

are $B(gh_N)$ and $B(gh'_N)$ for the test function h_N and h'_N satisfying $c_N = B(h_N)$, $c_N^* = B(h'_N)$ and g is the element of G_c, $\alpha_g = Ad\,\tilde{g}$ being the Bogoliubov automorphism due to $\tilde{g} \in \mathbb{C}\tilde{G}_c$. We note that CAR relations are preserved:

$$\mathfrak{c}^2 = (\mathfrak{c}')^2 = 0, \quad [\mathfrak{c}, \mathfrak{c}']_+ = 1.$$

We now show for a generic value of parameters that $\hat{R}_\pm \in \mathbb{C}\,\tilde{G}_c$. This will imply $\hat{R}_\pm \tilde{g} \in \mathbb{C}\,\tilde{G}_c$. We project out the first component of the test functuions gh_N and gh'_N as

$$\begin{aligned} \mathfrak{c} &= B(gh_N) = \beta_1 c_1 + \beta_2 c_1^* + \bar{\mathfrak{c}} \\ \mathfrak{c}' &= B(gh'_N) = \beta'_1 c_1 + \beta'_2 c_1^* + \bar{\mathfrak{c}}' \end{aligned}$$

where $\bar{c}$ and $\bar{c}'$ are linear combinations of c_j, c_j^*, $j = 2, \ldots, N$ and satisfying

$$\begin{aligned}
&\bar{c}^2 + \beta_1\beta_2 = \mathfrak{c}^2 = 0, \quad (\bar{c}')^2 + \beta_1'\beta_2' = (\mathfrak{c}')^2 = 0, \\
&[\bar{c}, \bar{c}']_+ + \beta_1\beta_2' + \beta_1'\beta_2 = [\mathfrak{c}, \mathfrak{c}']_+ = 1.
\end{aligned}$$

As shown in the Appendix (Lemma A.5), $\bar{c}$ and $\bar{c}'$ are linear combinations

$$\bar{c} = \xi c_0 + \eta c_0', \quad \bar{c}' = \xi' c_0 + \eta' c_0'$$

of c_0 and c_0' which are linear combinations of c_j, c_j^*, $j = 2, \ldots, N$ and satisfy

$$c_0^2 = (c_0')^2 = 0, \quad [c_0, c_0']_+ = 1 \ ,$$

under the condition

$$[\bar{c}, \bar{c}']_+^2 - 4\bar{c}^2(\bar{c}')^2 = 1 - 2(\beta_1\beta_2' + \beta_1'\beta_2) + (\beta_1\beta_2' - \beta_1'\beta_2)^2 \neq 0. \tag{$*$}$$

The fact that they are linear combinations of c_j, c_j^*, $j = 2, \ldots, N$ implies

$$[c_0, c_1]_+ = [c_0, c_1^*]_+ = [c_0', c_1]_+ = [c_0', c_1^*]_+ = 0.$$

By using these relations, we can compute $\bar{c}^2$, $(\bar{c}')^2$ and $[\bar{c}, \bar{c}']_+$. Substituting the result into earlier relations, we obtain

$$\begin{aligned}
&\xi\eta + \beta_1\beta_2 = \xi'\eta' + \beta_1'\beta_2' = 0, \\
&\xi\eta' + \xi'\eta + \beta_1\beta_2' + \beta_1'\beta_2 = 1.
\end{aligned}$$

By substitution, we have

$$\begin{aligned}
\hat{R}_\pm &= A_{00}' + A_{30}'c_1^*c_1 + A_{03}'c_0'c_0 + A_{33}'c_1^*c_1c_0'c_0 \\
&\quad \pm A_{11}'c_1^*c_0 \pm A_{12}'c_1^*c_0' \pm A_{21}'c_1c_0 \pm A_{22}'c_1c_0',
\end{aligned}$$

with

$$\begin{aligned}
A_{00}' &= A_{00} + A_{03}(\beta_1'\beta_2 + \eta\xi') \pm A_{21}\beta_2 \pm A_{22}\beta_2', \\
A_{30}' &= A_{30} + A_{03}(\beta_1\beta_2' - \beta_2\beta_1') + A_{33}(\eta\xi' + \beta_1\beta_2') \\
&\quad \pm A_{11}\beta_1 \mp A_{21}\beta_2 \pm A_{12}\beta_1' \mp A_{22}\beta_2', \\
A_{03}' &= A_{03}(\xi\eta' - \xi'\eta), \quad A_{33}' = A_{33}(\xi\eta' - \xi'\eta), \\
A_{11}' &= A_{11}\xi + A_{12}\xi' \pm A_{03}(\xi\beta_2' - \xi'\beta_2) \pm A_{33}(\xi\beta_2' - \xi'\beta_2), \\
A_{12}' &= A_{12}\eta' + A_{11}\eta \pm A_{03}(\eta\beta_2' - \eta'\beta_2) \pm A_{33}(\eta\beta_2' - \eta'\beta_2), \\
A_{21}' &= A_{21}\xi + A_{22}\xi' \pm A_{03}(\xi\beta_1' - \xi'\beta_1), \\
A_{22}' &= A_{22}\eta' + A_{21}\eta \pm A_{03}(\eta\beta_1' - \eta'\beta_1).
\end{aligned}$$

We now compute expressions corresponding to two sides of (5.7).

$$\begin{aligned}
A'_{00}A'_{33} - A'_{03}A'_{30} &= (\xi\eta' - \xi'\eta)\{(A_{00}A_{33} - A_{03}A_{30}) \\
&\quad +(\beta_2\beta'_1 - \beta'_2\beta_1)A_{03}(A_{03} + A_{33}) \pm \beta_2 A_{21}(A_{03} + A_{33}) \\
&\quad \pm\beta'_2 A_{22}(A_{03} + A_{33}) \mp \beta'_1 A_{12}A_{03} \mp \beta_1 A_{11}A_{03}\}, \\
A'_{11}A'_{22} - A'_{12}A'_{21} &= (\xi\eta' - \xi'\eta)\{(A_{11}A_{22} - A_{12}A_{21}) \\
&\quad +(\beta_2\beta'_1 - \beta'_2\beta_1)A_{03}(A_{03} + A_{33}) \pm \beta_2 A_{21}(A_{03} + A_{33}) \\
&\quad \pm\beta'_2 A_{22}(A_{03} + A_{33}) \mp \beta'_1 A_{12}A_{03} \mp \beta_1 A_{11}A_{03}\}.
\end{aligned}$$

Thus we have

$$A'_{00}A'_{33} - A'_{03}A'_{30} = A'_{11}A'_{22} - A'_{12}A'_{21}. \tag{5.8}$$

We can now go through the same computation from σ's to c's backwards to obtain the formula

$$\begin{aligned}
\hat{R}_{\pm} &= \alpha'_{00}\mathbf{1} + \alpha'_{z0}\sigma_z^{(1)} + \alpha'_{0z}\tilde{\sigma}_z + \alpha'_{zz}\sigma_z^{(1)}\tilde{\sigma}_z \\
&\quad +\alpha'_{xx}\sigma_x^{(1)}\tilde{\sigma}_x + \alpha'_{yx}\sigma_y^{(1)}\tilde{\sigma}_x + \alpha'_{xy}\sigma_x^{(1)}\tilde{\sigma}_y + \alpha'_{yy}\sigma_y^{(1)}\tilde{\sigma}_y
\end{aligned}$$

where

$$\alpha'_{00}\alpha'_{zz} - \alpha'_{0z}\alpha'_{z0} = \alpha'_{xy}\alpha'_{yx} - \alpha'_{xx}\alpha'_{yy}, \tag{5.9}$$

and

$$\tilde{\sigma}_z = 2\bar{c}'\bar{c} - 1, \quad \tilde{\sigma}_x = \sigma_z^{(1)}(\bar{c} + \bar{c}'), \quad \tilde{\sigma}_y = i\sigma_z^{(1)}(\bar{c} - \bar{c}').$$

Then $\tilde{\sigma}_x$, $\tilde{\sigma}_y$ and $\tilde{\sigma}_z$ commute with $\sigma_\alpha^{(1)}$ and satisfy the algebraic properties of Pauli spin matrices

$$\begin{aligned}
&\tilde{\sigma}_\alpha^2 = 1, \\
&\tilde{\sigma}_\alpha\tilde{\sigma}_\beta = -\tilde{\sigma}_\beta\tilde{\sigma}_\alpha = i\tilde{\sigma}_\gamma
\end{aligned}$$

for $(\alpha, \beta, \gamma) = (x, y, z)$, (y, z, x) and (z, x, y), except their hermiticity.

For the application of Lemma 3.1, we need the non-singular condition

$$\det \hat{R}_{\pm} \neq 0. \tag{5.10}$$

For a specific set of values of the parameters $a_1 = a_2 = c = c' = 1, b_1 = b_2 = d = d' = 0$, all $\hat{R}_{j,j+1}, j = 1, \cdots, N-1$ are the identity matrix and hence $\hat{R}_{\pm}$ is also an identity matrix. Therefore (5.10) holds. By continuity, (5.10) holds in an open neighbourhood of this specific set of values of parameters. Thus for a generic values of the parameters (more precisely, possibly excluding the algebraic manifolds defined by $\det \hat{R}_{\pm} = 0$), (5.10) holds.

Similarly, in the above proof, we need the condition $(*)$. For the above specific set of values of the parameters, the left hand side of $(*)$ is 1 and hence $(*)$ is satisfied in an open neighbourhood of the specific set of values. Thus for a generic values of parameters, $(*)$ also holds.

We can now apply Lemma 3.1 to $\hat{R}_{\pm}$. The proof of Lemma 3.1 does not use the hermiticity of Pauli spin matrices, and hence $\hat{R}_{\pm}$ is either of the form e^D or a product of two such matrices where D is a quadratic expression of $c_1, c_1^*, \bar{c}$ and $\bar{c}'$. Hence we have $\hat{R}_{\pm} \in \mathbb{C}\,\tilde{G}_c$. (Note that $tr\sigma_\alpha = 0$ follows from the relations $2i\sigma_\gamma = [\sigma_\alpha, \sigma_\beta]$.)

The foregoing computation shows the following result.

Proposition 5.3 *For $\hat{T}$ of (2.10) (the ratio of the transfer matrix by translation operator), the following hold for a generic values of the parameters.*

$$\hat{T}E_{\pm} = k_{\pm}V_{\pm}E_{\pm}, \quad V_{\pm} \in \tilde{G}_c, \quad k_{\pm} \in \mathbb{C}. \tag{5.11}$$

We note that

$$\det\ \hat{R}_{j,j+1} = (a_+a_- - dd')^{1/2}$$

under the free Fermion condition and det $\hat{R}_{\pm}$ is also of the same form with deformed parameters. The parameter k is needed to make det $V_{\pm} = 1$. Thus k^2 is a polynomial of the parameters $a_+, a_-, b_+, b_-c, c', d$ and d'. The ambiguity $\pm$ for $V_{\pm}$ arising from the square root of k^2 disappears when mapped into G_c.

6 Shift Invariance

The operator S defined by (5.2) implements the automorphism Θ of the algebra $\mathfrak{A}$ ($\Theta(A) = SAS^{-1}, A \in \mathfrak{A}$) satisfying

$$\left.\begin{array}{ll} \Theta(c_j) = -c_j, & \Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}, \\ \Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)}, & \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)}, \end{array}\right\} \tag{6.1}$$

$j = 1, \ldots, N$. Hence S commutes with $\hat{R}_{j,j+1}$ and $\hat{T}$ in (2.18).

From the definition (6.1), it follows that S and $E_{\pm}$ are invariant under the shift discussed in Section 2 (iii). Namely

$$[T_0, \hat{T}E_{\pm}] = 0. \tag{6.2}$$

The subalgebra of $\mathfrak{A}$ consisting of all Θ-invariant elements A of $\mathfrak{A}$ (i.e. $\Theta(A) = A$) will be denoted $\mathfrak{A}_+$. It is generated as a C^*-algebra by quadratic expressions in c_j and c_j^*, i.e. by c_jc_k, $c_jc_k^*$, $c_j^*c_k$ and $c_j^*c_k^*$ $(j, k = 1, \ldots, N)$. T_0, $\hat{T}$, S, $E_{\pm}$ are all elements of $\mathfrak{A}_+$.

(i) Shift Automorphisms

The shift operator T_0 introduced in Section 2 (iii) induces the following shift automorphism of $\mathfrak{A}$.

$$\tau(A) = (Ad\,T_0^{-1})A = T_0^{-1}AT_0. \tag{6.3}$$

For example, for $\alpha = x, y, z$,

$$\tau(\sigma_\alpha^{(j)}) = \sigma_\alpha^{(j+1)} \quad \text{if } j \neq N.$$
$$\tau(\sigma_\alpha^{(N)}) = \sigma_\alpha^{(1)}.$$

If $j < k \neq N$, then

$$c_j c_k = \{(\sigma_x^{(j)} - i\sigma_y^{(j)})/2\}\sigma_z^{(j)} \cdots \sigma_z^{(k-1)}(\sigma_x^{(k)} - i\sigma_y^{(k)})/2$$

and hence

$$\tau(c_j c_k) = c_{j+1}c_{k+1}. \tag{6.4}$$

Similarly,

$$\tau(c_j c_k^*) = c_{j+1}c_{k+1}^*, \quad \tau(c_j^* c_k) = c_{j+1}^* c_{k+1}, \quad \tau(c_j^* c_k^*) = c_{j+1}^* c_{k+1}^* \tag{6.5}$$

By the canonical anticommutation relations (4.3), these equations imply the same equations for the case $k < j \neq N$.

For the case $j < k = N$, we have

$$\begin{aligned} \tau(c_j c_N) &= \frac{1}{2}(\sigma_x^{(j+1)} - i\sigma_y^{(j+1)})\sigma_z^{(j+1)} \cdots \sigma_z^{(N)}\frac{1}{2}(\sigma_x^{(1)} - i\sigma_y^{(1)}) \\ &= c_{j+1}Sc_1 = -Sc_{j+1}c_1. \end{aligned} \tag{6.6}$$

Similarly, we obtain

$$\tau(c_j c_N^*) = -Sc_{j+1}c_1^*, \quad \tau(c_j^* c_N) = -Sc_{j+1}^* c_1, \quad \tau(c_j^* c_N^*) = -Sc_{j+1}^* c_1^* \cdots \tag{6.7}$$

Let $\tau_\pm^{\mathrm{CAR}}$ be the $*$-automorphisms of $\mathfrak{A} = \mathfrak{A}^{\mathrm{CAR}}$ uniquely determined by

$$\left.\begin{aligned} &\tau_\pm^{\mathrm{CAR}}(c_j) = c_{j+1} \quad \text{if } j \neq N, \\ &\tau_\pm^{\mathrm{CAR}}(c_N) = \pm c_1 \end{aligned}\right\} \tag{6.8}$$

Lemma 6.1 *For any $A \in \mathfrak{A}_+$,*

$$\tau(AE_\pm) = \tau_\mp^{\mathrm{CAR}}(A)E_\pm \tag{6.9}$$

Since $SE_\pm = \pm E_\pm$ and $\tau(E_\pm) = E_\pm$, (6.9) holds for $A = c_j c_k, c_j c_k^*, c_j^* c_k, c_j^* c_k^*$ $(j, k = 1, \cdots, N)$ by (6.4)–(6.7) and (6.8). Hence (6.9) holds for any A in $\mathfrak{A}_+$, which is generated by quadratic expressions in $c's$ and $c^*{}'s$.

(ii) Shift Invariance

The transfer matrix $\hat{T}$ (T modified by T_0) belongs to $\mathfrak{A}_+$ and coincides with a constant multiple of an element of $\tilde{G}_c$ when projected to E_+ and to E_-:

$$\hat{T}E_\pm = k_\pm V_\pm E_\pm, \quad V_\pm \in \tilde{G}_c, \ k_\pm \in \mathbb{C}. \tag{6.10}$$

Proposition 6.2 *Assume $N > 2$. Then*

$$\tau_\mp^{\mathrm{CAR}}(V_\pm) = V_\pm \tag{6.11}$$

The following technical Lemma, to be proved in the Appendix, is the basis for the proof.

Lemma 6.3 *Assume $N > 2$. If $V_\pm \in \tilde{G}_c$ satisfies $V_\pm E_\pm = E_\pm$, then*

$$V_+\Omega = \Omega, \quad Ad\ V_+ = \pm 1\ (\in G_c) \tag{6.12}$$
$$V_-\Omega = \pm\Omega, \quad Ad\ V_- = \pm 1\ (\in G_c) \tag{6.13}$$

where we have two possibilities for V_+ and for V_- given by the choice of the sign $\pm$. The $\pm$ in two equations in (6.13) *should be the same.*

Proof of Proposition 6.2 By (6.2) and (6.9), we have

$$\begin{aligned} \tau_\mp^{\mathrm{CAR}}(V_\pm)E_\pm &= \tau(V_\pm E_\pm) = k_\pm^{-1}\tau(\hat{T}E_\pm) \\ &= k_\pm^{-1}\hat{T}E_\pm = V_\pm E_\pm. \end{aligned}$$

Hence

$$V_\pm^{-1}\tau_\mp^{\mathrm{CAR}}(V_\pm)E_\pm = E_\pm, \quad V_\pm^{-1}\tau_\mp^{\mathrm{CAR}}(V_\pm) \in \tilde{G}_c.$$

We can now apply Lemma 6.3. We know that $V_\pm$ depends polynomially on 8 parameters of R except for the over-all factors $k_\pm$ and hence $V_\pm$ are continuous in these parameters. We also know that $\hat{R} = 1$ and hence $V_\pm = 1$ for a specific set of values of parameters:

$$a_+ = a_- = c = c' = 1, \quad b_+ = b_- = d = d' = 0.$$

Therefore the choice of $\pm$ in Lemma 6.3, which has to depend continuously on parameters of R and has to be $+$ for a specific value of parameters, has to be $+$ for all parameter values, as the set of allowed parameter values is connected (being generic points of the irreducible algebraic manifold defined by the free Fermion condition). Then Lemma 6.3 (i.e. eqs. (6.12) and (6.13) with $+$ sign) implies

$$V_\pm^{-1}\tau_\mp^{\mathrm{CAR}}(V_\pm) = 1.$$

Q.E.D.

(iii) Fourier analysis.

The automorphisms $\tau_{\pm}^{\mathrm{CAR}}$ introduced above are Bogoliubov $*$-automorphisms corresponding to the following transformations of the test function L of (4.6), which may be viewed as

$$\begin{aligned} L &= l^2(\{1,\dots,N\}) \otimes \mathbb{C}^2, \\ h &= \begin{pmatrix} f \\ g \end{pmatrix} = f \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in L, \\ & \qquad (f, g \in l_2(\{1,\dots,N\})). \end{aligned}$$

Defining

$$\begin{aligned} (U_{\pm} f)_j &= f_{j-1} \quad \text{if} \quad j \neq 1, \\ (U_{\pm} f)_1 &= \pm f_N \end{aligned}$$

for $f \in l^2(\{1,\dots,N\})$, we have

$$\tau_{\pm}^{\mathrm{CAR}}(B(h)) = B((U_{\pm} \otimes id)h)$$

for $h \in L$.

Eigenvectors of $U_{\pm}$ are given as follows.

$$\begin{aligned} (e_{+}^{(j)})_k &= e^{2\pi i\, jk/N}, \quad k = 1,\dots,N, \\ (e_{-}^{(j)})_k &= e^{\pi i(2j+1)k/N}, \quad k = 1,\dots,N, \end{aligned}$$

where $j = 0, 1, 2, \dots, N-1$. They satisfy

$$\begin{aligned} U_{+} e_{+}^{(j)} &= e^{-2\pi i j/N} e_{+}^{(j)}, \\ U_{-} e_{-}^{(j)} &= e^{-\pi i(2j+1)/N} e_{-}^{(j)}. \end{aligned}$$

Both set of vectors $\{e_{\pm}^{(j)}\}_j$ are orthonormal bases of $l^2(\{1,\dots,N\})$. If any linear operator V on L commutes with $U_{+} \otimes id$, then

$$V(e_{+}^{(j)} \otimes v) = e_{+}^{(j)} \otimes V^{(j)} v, \quad v \in \mathbb{C}^2$$

where $V^{(j)}$, $j = 0, \dots, N-1$ are 2×2 matrices acting on $\mathbb{C}^2$. Thus V is described by $\{V^{(j)}\}$. The same situation holds when U_{+} is replaced by U_{-}.

If $[H, V] = 0$ and if both H and V commute with U_{+}, then $[H^{(j)}, V^{(j)}] = 0$ for all j. Similarly for U_{-}.

For 2×2 matrices A and B with A not a multiple of the unit matrix $\mathbf{1}$, $[A, B] = 0$ implies that B is a linear combination of $\mathbf{1}$ and A. Thus we obtain the following result.

Proposition 6.4. *Assume that all H, V_1 and V_2 (acting on L) commute with U_{+} (or with U_{-}) and if all $H^{(j)}$ are not multiples of the unit matrix, then $[H, V_1] = 0$ and $[H, V_2] = 0$ implies $[V_1, V_2] = 0$.*

Proof is immediate as $[H, V_k]$ implies $[H^{(j)}, V_k^{(j)}] = 0$ for all j and, since $H^{(j)}$ is not a multiple of $\mathbf{1}$, we obtain $[V_1^{(j)}, V_2^{(j)}] = 0$, which implies $[V_1, V_2] = 0$.

7 Hamiltonian

(i) Fourier analysis of Hamiltonian

The Hamiltonian operator we will be considering is given by (1.1) and (1.2). In terms of creation and annihilation operators introduced in Section 4, we have

$$\begin{aligned} H_{j,j+1} &= \beta_{11}c_j^*c_{j+1} + \beta_{12}c_j^*c_{j+1}^* + \beta_{21}c_jc_{j+1} + \beta_{22}c_jc_{j+1}^* \\ &\quad +2\lambda(c_j^*c_j + c_{j+1}^*c_{j+1} - 1) \end{aligned} \tag{7.1}$$

for $j = 1, \ldots, N-1$ and

$$\begin{aligned} H_{N,1} &= -(\beta_{11}c_N^*c_1 + \beta_{12}c_N^*c_1^* + \beta_{21}c_Nc_1 + \beta_{22}c_Nc_1^*)S \\ &\quad +2\lambda(c_N^*c_N + c_1^*c_1 - 1), \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} \beta_{11} &= -J_{11} - iJ_{12} + iJ_{21} - J_{22}, \\ \beta_{12} &= -J_{11} + iJ_{12} + iJ_{21} + J_{22}, \\ \beta_{21} &= J_{11} + iJ_{12} + iJ_{21} - J_{22}, \\ \beta_{22} &= J_{11} - iJ_{12} + iJ_{21} + J_{22}. \end{aligned}$$

We now define bilinear Hamiltonians $H_\pm$ by setting $S = \pm 1$ in (7.2), so that

$$HE_\pm = H_\pm E_\pm. \tag{7.3}$$

Because $H_\pm$ are quadratic expressions of c's and c^*'s, they induce linear transformations $K_\pm$ on L through the following relations:

$$[H_\pm, B(h)] = B(K_\pm h), \quad h \in L. \tag{7.4}$$

We now compute $K_\pm$.

We have

$$\begin{aligned} [H_\pm, c_j^*] &= \beta_{11}c_{j-1}^* + \beta_{21}(c_{j-1} - c_{j+1}) - \beta_{22}c_{j+1}^* + 4\lambda c_j^*, \\ [H_\pm, c_j] &= -\beta_{11}c_{j+1} + \beta_{12}(c_{j-1}^* - c_{j+1}^*) + \beta_{22}c_{j-1} - 4\lambda c_j, \end{aligned}$$

for $j = 2, \ldots, N-1$ and

$$\begin{aligned} [H_\pm, c_N^*] &= \pm\beta_{21}c_1 \pm \beta_{22}c_1^* + \beta_{11}c_{N-1}^* + \beta_{21}c_{N-1} + 4\lambda c_N^*, \\ [H_\pm, c_N] &= \pm\beta_{11}c_1 \pm \beta_{12}c_1^* + \beta_{12}c_{N-1}^* + \beta_{22}c_{N-1} - 4\lambda c_N, \\ [H_\pm, c_1^*] &= \mp\beta_{11}c_N^* \mp \beta_{21}c_N - \beta_{21}c_2 - \beta_{22}c_2^* + 4\lambda c_1^*, \\ [H_\pm, c_1] &= \mp\beta_{12}c_N^* \mp \beta_{22}c_N - \beta_{11}c_2 - \beta_{12}c_2^* - 4\lambda c_1, \end{aligned}$$

where the latter 4 equations can be identified with the first two equations where $j = N$ and 1, respectively, and $c_{N+1} = \mp c_1$, $c^*_{N+1} = \mp c^*_1$, $c_0 = \mp c_N$, $c^*_0 = \mp c^*_N$.

Therefore

$$K_\pm = \begin{pmatrix} 4\lambda + \beta_{11}U^*_\mp - \beta_{22}U_\mp & \beta_{12}(U^*_\mp - U_\mp) \\ \beta_{21}(U^*_\mp - U_\mp) & -4\lambda - \beta_{11}U_\mp + \beta_{22}U^*_\mp \end{pmatrix}. \tag{7.5}$$

By Fourier transform of the preceding section, we obtain

$$K^{(j)}_+ = \begin{pmatrix} 4\lambda + \beta_{11}e^{\pi i(2j+1)/N} - \beta_{22}e^{-\pi i(2j+1)/N}, & 2i\beta_{12}\sin\frac{(2j+1)\pi}{N} \\ 2i\beta_{21}\sin\frac{(2j+1)\pi}{N}, & -4\lambda - \beta_{11}e^{-\pi i(2j+1)/N} + \beta_{22}e^{\pi i(2j+1)/N} \end{pmatrix}. \tag{7.6}$$

$$K^{(j)}_- = \begin{pmatrix} 4\lambda + \beta_{11}e^{\frac{2\pi ij}{N}} - \beta_{22}e^{-\frac{2\pi ij}{N}}, & 2i\beta_{12}\sin\frac{2\pi i}{N} \\ 2i\beta_{21}\sin\frac{2\pi j}{N}, & -4\lambda - \beta_{11}e^{-\frac{2\pi ij}{N}} + \beta_{22}e^{\frac{2\pi ij}{N}} \end{pmatrix}. \tag{7.7}$$

For a generic values of parameters $(J_{\alpha\beta}, \lambda)$, we see that $K^{(j)}_\pm$, $j = 0, \ldots, N-1$ are not multiples of the unit matrix and hence we can apply Proposition 6.4 to $K_\pm$.

(ii) Commutativity of Hamiltonians and transfer matrices.

The following result is obtained in [5] by using a generalization of Krinsky's method [2].

Theorem 7.1. *The Hamiltonian H of (1.1) and the transfer matrix T of (2.4) for the R-matrix (2.1) commute if the following conditions hold in addition to the free Fermion condition (2.2).*

$$(K - iL) = \beta c'd, \quad (K + iL) = \beta cd', \tag{7.8}$$

$$2J = \beta(a_-b_+ + b_-a_+), \tag{7.9}$$

$$4\lambda = \beta(a^2_+ - b^2_+ + b^2_- - a^2_-), \tag{7.10}$$

where

$$J = J_{11} + J_{22}, \quad K = J_{11} - J_{22}, \quad L = J_{12} + J_{21}, \tag{7.11}$$

and $\beta \in \mathbb{C}$ is an additional parameter introduced for the description of the conditions.

The free Fermion condition defines an irreducible algebraic manifold in the space of parameters $a_\pm, b_\pm, c, c', d$ and d'. For each set of values of these parameters on this manifold and for $\beta \in \mathbb{C}$, the parameters K, L, J and λ are uniquely given by (7.8), (7.9) and (7.10). Therefore, the conditions (7.8), (7.9), (7.10) and (2.2) (which are the condition for the commutativity $[H, T] = 0$ in Theorem 7.1), defines an irreducible algebraic manifold in the space of parameters

$$a_\pm, b_\pm, c, c', d, d', J_{\alpha\beta}, \lambda, \beta. \tag{7.12}$$

Thus, for any given generic values of the parameters K, L, J and λ, the set of the other parameters satisfying (7.8), (7.9), (7.10) and (2.2) will again form an irreducible algebraic manifold.

We note that the commutativity has been obtained in [5] also for other cases, but in this paper, we concentrate on the parameters satisfying the above conditions.

(iii) Commutativity of $H_\pm$ and $\hat{T}_\pm$.

Since H is shift invariant, the commutativity of H and T by Theorem 7.1 implies

$$[H, \hat{T}] = 0, \quad \text{or} \quad \hat{T} H \hat{T}^{-1} = H, \tag{7.13}$$

under the same condition. By applying $E_\pm$, we obtain

$$(\hat{T}_\pm H_\pm \hat{T}_\pm^{-1} - H_\pm) E_\pm = 0. \tag{7.14}$$

We now use the following fact proved in Appendix.

Lemma 7.2. *Let H be a quadratic expression in c's and c^*'s. If $HE_+ = 0$ or $HE_- = 0$, then $H = 0$.*

Since $(\hat{T}_\pm H_\pm \hat{T}_\pm^{-1} - H_\pm)$ is a quadratic expression when $\hat{T}_\pm \in \mathbb{C}\hat{G}_c$, we have

$$\hat{T}_\pm H_\pm \hat{T}_\pm^{-1} = H_\pm \quad \text{or} \quad [\hat{T}_\pm, H_\pm] = 0 \tag{7.15}$$

for a generic values of the parameters $a_\pm, b_\pm, c, c', d, d'$.

8 Commutativity of transfer matrices

For a given values of parameters $J_{\alpha\beta}$ and λ in the Hamiltonian H of (1.1), the transfer matrix T and hence $\hat{T}$ for the parameter values $a_\pm, b_\pm, c, c', d$ and d' (for some value of β) satisfying (7.8), (7.9), (7.10) and (2.2) commute with H by Theorem 7.1. Furthermore, for a generic values of parameters $a_\pm, b_\pm, c, c', d$ and d', we have

$$\hat{T} E_\pm = k_\pm V_\pm E_\pm, \quad V_\pm \in \tilde{G}_c. \tag{8.1}$$

Then, by (7.14) and the commutativity $[\hat{T}, H] = 0$, we obtain

$$[V_\pm, H_\pm] = 0. \tag{8.2}$$

By proposition 6.2, we also have

$$\tau_\mp^{\mathrm{CAR}}(V_\pm) = V_\pm.$$

Also by explicit expressions (7.1) and (7.2), we have

$$\tau_\mp^{\mathrm{CAR}}(H_\pm) = H_\pm.$$

For a generic values of parameters $(J_{\alpha\beta}, \lambda)$, $K_\pm^{(j)}$, $j = 0, \ldots, N-1$ are not multiples of the unit matrix and hence, by applying Proposition 6.4, we obtain the commutativity

$$[\bar{V}_\pm^{(1)}, \bar{V}_\pm^{(2)}] = 0 \tag{8.3}$$

where $V_\pm^{(1)}$ and $V_\pm^{(2)}$ are $V_\pm$ for two sets of generic values of parameters $a_\pm, b_\pm, c, c', d$ and d', both sets of parameters satisfying (7.8), (7.9), (7.10) and (2.2) for the given generic values of parameters $(J_{\alpha\beta}, \lambda)$, and the corresponding $\bar{V}_\pm^{(1)}$, $\bar{V}_\pm^{(2)} \in G_c$ are defined by $V_\pm^{(l)} B(h) V_\pm^{(l)-1} = B(\bar{V}_\pm^{(l)} h)$, $l = 1, 2$. The commutativity (8.3) implies in general

$$V_\pm^{(2)} V_\pm^{(1)} (V_\pm^{(2)})^{-1} (V_\pm^{(1)})^{-1} = \mathbf{1} \quad \text{or} \quad -\mathbf{1} \tag{8.4}$$

due to double covering of G_c by $\tilde{G}_c$. However the generic points $(a_\pm, b_\pm, c, c', d, d')$ satisfying (7.8), (7.9), (7.10) and (2.2) for a given generic values of $(J_{\alpha\beta}, \lambda)$ are connected by the irreducibility of the manifold defined by (7.8), (7.9), (7.10) and (2.2). Hence we may continuously deform $V_\pm^{(1)}$ to $V_\pm^{(2)}$, keeping the relation (8.4), in which the choice of $+\mathbf{1}$ and $-\mathbf{1}$ has to be constant by continuity. When $V_\pm^{(1)}$ is replaced by $V_\pm^{(2)}$, the left hand side is $\mathbf{1}$ and hence the right hand side of (8.4) must be $\mathbf{1}$ also for a generic values of parameters for $V_\pm^{(1)}$ which are different from $V_\pm^{(2)}$. Therefore, we have the commutativity

$$[V_\pm^{(1)}, V_\pm^{(2)}] = 0.$$

This implies the commutativity of $\hat{T}^{(1)}$ and $\hat{T}^{(2)}$ ($\hat{T}$ for the two sets of parameter values). As pointed out in Section 1, T_0 is common for all parameter values and commutes with $\hat{T}$. Therefore, we obtain the commutativity of the transfer matrices

$$[T^{(1)}, T^{(2)}] = 0 \tag{8.5}$$

for two generic sets of parameters $(a_\pm, b_\pm, c, c', d, d')$, both set satisfying (7.8), (7.9), (7.10) and (2.2) for a given generic values of $(J_{\alpha\beta}, \lambda)$.

Once we obtain the commutativity (8.5) for a generic values of parameters, we may obtain the commutativity at general values of parameters $(J_{\alpha\beta}, \lambda)$ and $(a_\pm, b_\pm, c, c', d, d')$ by changing their values and taking limits, always keeping relations (7.8), (7.9), (7.10) and (2.2).

Appendix Lemmas about the CAR algebra.

In this appendix, we have the standing assumption $N > 2$.

(i) Proof of Lemma 6.3.

We divide the proof into two cases of E_+ and E_-.

Lemma A.1. *Assume $N > 2$, $V \in \tilde{G}_c$ and $VE_+ = E_+$. Then*

$$V\Omega = \Omega, \quad VB(h)V^{-1} = \pm B(h), \quad h \in L. \tag{A.1}$$

Proof. We shall be working on a Fock space with the vacuum vector Ω, annihilated by all c_j.

$$c_j\Omega = 0, \quad j = 1, \dots, N. \tag{A.2}$$

To distinguish c_j and c_j^* in the dual formalism of Section 4, we introduce a projection operator p acting on L defined by

$$p(f \oplus g) = f \oplus 0 \quad (f \oplus g \in L).$$

Then $B(h) = c(g)$ if and only if $ph = 0$. Similarly $B(h) = c^*(f)$ if $ph = h$. (In general, $B(h) = c^*(f) + c(g)$ for $h = f \oplus g$.)

The operator E_+ is a projection operator to vectors in the Fock space with even number of particles. In particular, $VE_+ = E_+$ implies

$$V\Omega = VE_+\Omega = E_+\Omega = \Omega. \tag{A.3}$$

Since $V \in \tilde{G}_c$, there is $\bar{V} \in G_c$ such that

$$(AdV)B(h) = VB(h)V^{-1} = B(\bar{V}h).$$

For any h satisfying $ph = 0$ (or $(1-p)h = h$),

$$B(h)\Omega = 0$$

by (A.2). We then obtain for such vector h in L

$$0 = VB(h)\Omega = VB(h)V^{-1}V\Omega = B(\bar{V}h)\Omega.$$

Hence $B(\bar{V}h)$ is an annihilation operator and

$$p\bar{V}h = 0.$$

Since $(1-p)h \equiv h_1$ satisfies $ph_1 = 0$ for any $h \in L$, we have

$$p\bar{V}(1-p) = 0. \tag{A.4}$$

Similarly, $VE_+ = E_+$ implies (by taking adjoint of both sides of this equation) V^*E_+ $(= E_+V^*) = E_+$ and hence

$$V^*\Omega = \Omega.$$

Using

$$\begin{aligned} &V^*B(h)(V^*)^{-1} = (V^{-1}B(h)^*V)^* = (V^{-1}B(\Gamma h)V)^* \\ &= B(\bar{V}^{-1}\Gamma h)^* = B(\Gamma\bar{V}^{-1}\Gamma h), \end{aligned} \tag{A.5}$$

we obtain

$$\begin{aligned} 0 &= V^*B((1-p)h)\Omega = V^*B((1-p)h)(V^*)^{-1}V^*\Omega \\ &= B(\Gamma\bar{V}^{-1}\Gamma(1-p)h)\Omega \end{aligned}$$

for any $h \in L$ and hence, due to $p\Gamma = \Gamma(1-p)$,

$$0 = p\Gamma\bar{V}^{-1}\Gamma(1-p) = \Gamma(1-p)\bar{V}^{-1}p\Gamma$$

which implies

$$(1-p)\bar{V}^{-1}p = 0 \tag{A.6}$$

By inserting $1 = p + (1-p)$ in the middle of $\bar{V}\bar{V}^{-1} = 1$ and using (A.4), we obtain

$$\begin{aligned}(p\bar{V}p)(p\bar{V}^{-1}p) &= p\bar{V}(p+(1-p))\bar{V}^{-1}p = p \\ &= (p\bar{V}^{-1}p)(p\bar{V}p)\end{aligned} \tag{A.7}$$

where the last equality is by the finite dimensionality of L.

By (A.6), we obtain

$$\begin{aligned}0 &= (1-p)(\bar{V}\bar{V}^{-1})p = (1-p)\bar{V}(p+(1-p))\bar{V}^{-1}p \\ &= (1-p)\bar{V}p(p\bar{V}^{-1}p).\end{aligned}$$

By multiplying $p\bar{V}p$ from the right and using (A.7), we obtain

$$0 = (1-p)\bar{V}p.$$

Combining with (A.4), we obtain

$$\bar{V} = p\bar{V}p + (1-p)\bar{V}(1-p). \tag{A.8}$$

Therefore we may write $\bar{V}(f \oplus 0) = \bar{V}_1 f \oplus 0$ and $\bar{V}(0 \oplus g) = 0 \oplus \bar{V}_2 g$. Since $VE_+ = E_+$ implies $V = 1$ on vectors with an even number of particles, we have

$$\begin{aligned}&Vc^*(f_1)\cdots c^*(f_{2n})\Omega = c^*(f_1)\cdots c^*(f_{2n})\Omega \\ &= (Vc^*(f_1)V^{-1})(Vc^*(f_2)V^{-1})\cdots\Omega = c^*(\bar{V}_1 f_1)\cdots c^*(\bar{V}_1 f_{2n})\Omega.\end{aligned}$$

This implies on the Fock space

$$f_1 \wedge f_2 \wedge \cdots \wedge f_{2n} = \bar{V}_1 f_1 \wedge \bar{V}_1 f_2 \wedge \cdots \wedge \bar{V}_1 f_{2n}. \tag{A.9}$$

Let us assume that $\bar{V}_1 f$ is not proportional to f and derive a contradiction. Since $N > 2$, there exists f_2 orthogonal to both f and $\bar{V}_1 f$. We then take $n = 2$, $f_1 = f$ and f_2 to be just that f_2. Then $\bar{V}_1 f$ can not be a linear combination of f and f_2 and hence $f \wedge f_2$ cannot be equal to $\bar{V}_1 f \wedge f_3$ with any $f_3 = \bar{V}_1 f_2$. ($\bar{V}_1 f \wedge (f \wedge f_2) \neq 0$ but $\bar{V}_1 f \wedge (\bar{V}_1 f \wedge f_3) = 0$.) Hence $\bar{V}_1 f = \lambda_f f$ for any $f \in L$. This is possible only if λ_f is independent of f, namely $\bar{V}_1 = \lambda\mathbf{1}$. By substituting this into (A.9), we obtain $\lambda^2 = 1$ and hence

$$\bar{V}_1 = \pm\mathbf{1}.$$

Since $V^* \in \tilde{G}_c$ and $V^* E_+ = E_+$ (as proved above), we obtain

$$(\overline{V^*})_1 = \pm\mathbf{1}.$$

By (A.5), this implies.

$$\bar{V}_2 = \pm\mathbf{1}.$$

Because of the CAR relations,

$$(f, g) = V[c^*(f), c(g)]_+ V^{-1} = (\bar{V}_1 f, \bar{V}_2 g),$$

the signs of $\bar{V}_1$ and $\bar{V}_2$ must be the same and we obtain

$$\bar{V} = \pm\mathbf{1}, \quad VB(h)V^{-1} = \pm B(h).$$

Q.E.D.

Lemma A.2. *Assume $N > 2$, $V \in \tilde{G}_c$ and $VE_- = E_-$. Then*

$$V\Omega = \pm\Omega, \quad VB(h)V^{-1} = \pm B(h).$$

(The choice of + and − in the two places should be the same.)

Proof. As before, there is $\bar{V} \in G_c$ such that

$$VB(h)V^{-1} = B(\bar{V}h).$$

Since $B((1-p)h_1)B(h_2)\Omega = ((1-p)h_1, \Gamma h_2)\Omega$, we obtain

$$\begin{aligned}((1-p)h_1, \Gamma h_2)V\Omega &= VB((1-p)h_1)V^{-1}VB(h_2)\Omega \\ &= B(\bar{V}(1-p)h_1)B(h_2)\Omega,\end{aligned}$$

where we have used $VE_- = E_-$ (and hence $VB(h_2)\Omega = B(h_2)\Omega$). This holds for all h_2 only if $B(\bar{V}(1-p)h_1)$ is an annihilation operator, namely

$$p\bar{V}(1-p) = 0.$$

As before, we obtain by the use of $V^*E_- = E_-$

$$(1-p)\bar{V}^{-1}p = 0$$

and hence by the same argument as before

$$\bar{V} = p\bar{V}p + (1-p)\bar{V}(1-p).$$

Denoting $\bar{V}(f \oplus g) = (\bar{V}_1 f \oplus \bar{V}_2 g)$, we obtain

$$0 = c(g)\Omega = Vc(g)V^{-1}V\Omega = c(\bar{V}_2 g)V\Omega.$$

Since $\bar{V}_2$ is invertible (because $\bar{V} \in G_c$ is invertible), $c(\bar{V}_2 g)$ for all g exhausts all annihilation operators and hence

$$V\Omega = \lambda\Omega$$

for some λ. We then have from $VE_- = E_-$

$$\begin{aligned} c^*(f)\Omega &= Vc^*(f)\Omega = Vc^*(f)V^{-1}V\Omega \\ &= \lambda c^*(\bar{V}_1 f)\Omega. \end{aligned}$$

This implies $\lambda \neq 0$ and

$$\bar{V}_1 = \lambda^{-1}.$$

Since $N > 2$, we have linearly independent f_1, f_2, f_3, for which $VE_- = E_-$ implies

$$\begin{aligned} 0 &\neq c^*(f_1)c^*(f_2)c^*(f_3)\Omega = Vc^*(f_1)c^*(f_2)c^*(f_3)\Omega \\ &= \lambda c^*(\bar{V}f_1)c^*(\bar{V}f_2)c^*(\bar{V}f_3)\Omega = \lambda^{-2}c^*(f_1)c^*(f_2)c^*(f_3)\Omega. \end{aligned}$$

This implies $\lambda^2 = 1$ and $\lambda = \pm 1$. Hence

$$V\Omega = \pm\Omega \quad \text{and} \quad \bar{V}_1 = \pm\mathbf{1}$$

with the same sign.

Working with $V^* \in \tilde{G}_c$ which satisfies $V^*E_- = E_-$, we obtain

$$(\overline{V^*})_1 = \pm\mathbf{1}.$$

By using (A.5) again, this implies

$$\bar{V}_2 = \pm\mathbf{1}$$

and the CAR relations imply that $\bar{V}_1$ and $\bar{V}_2$ must have the same sign (of ± 1). Therefore

$$VB(h)V^{-1} = \pm B(h)$$

with the same sign as $V\Omega = \pm\Omega$. Q.E.D.

(ii) Proof of Lemma 7.2.

Again, we divide into two cases of E_+ and E_-

Lemma A.3. *Let H be a quadratic expression in c's and c^*'s. If $HE_+ = 0$, then $H = 0$.*

Proof. Let $[H, B(h)] = B(Kh)$. By $HE_+ = 0$, we obtain $H\Omega = 0$ and hence

$$0 = HB((1-p)h)\Omega = [H, B((1-p)h)]\Omega = B(K(1-p)h)\Omega.$$

Hence $B(K(1-p)h)$ is an annihilation operator and we have

$$pK(1-p) = 0. \tag{A.10}$$

We also note that $HE_+ = 0$ implies

$$H^* E_+ = (E_+ H)^* = (HE_+)^* = 0.$$

Furthermore,

$$[H^*, B(h)] = -[H, B(h)^*]^* = -B(K\Gamma h)^* = B(-\Gamma K \Gamma h).$$

By the same argument as above we obtain

$$p\Gamma K\Gamma(1-p) = 0,$$

which implies

$$(1-p)Kp = \Gamma(p\Gamma K\Gamma(1-p))\Gamma = 0$$

Together with (A.10) we obtain

$$K = pKp + (1-p)K(1-p).$$

In other words, $K(f \oplus g) = (K_1 f \oplus K_2 g)$.

From $HE_+ = 0$, we obtain

$$\begin{aligned} 0 &= HB(h_1)B(h_2)\Omega = [H, B(h_1)]B(h_2)\Omega + B(h_1)[H, B(h_2)]\Omega \\ &= B(Kh_1)B(h_2)\Omega + B(h_1)B(Kh_2)\Omega. \end{aligned}$$

If $h_1 = f_1 \oplus 0$ and $h_2 = f_2 \oplus 0$, this implies

$$K_1 f_1 \wedge f_2 = -f_1 \wedge K_1 f_2. \tag{A.11}$$

If $K_1 f_1$ is not proportional to f_1, we may take non-zero f_2 to be orthogonal to both f_1 and $K_1 f_1$ (due to $N > 2$). Then f_1 is not a linear combination of f_2 and $K_1 f_1$ and hence this equation can not hold (unless $K_1 f_1 = K_1 f_2 = 0$ which implies that $K f_1$ is a zero multiple of f_1). Therefore $K_1 = \lambda$ (by an exactly the same argument as in (i)). However, (A.11) then shows that $\lambda = 0$. Hence $K_1 = 0$.

By the same argument for H^*, we obtain $(\Gamma K \Gamma)_1 = 0$, which implies $K_2 = 0$ and hence $K = 0$. This implies $H = \mu\mathbf{1}$ and hence $HE_+ = 0$ implies $\mu = 0$. Namely $H = 0$. Q.E.D.

Lemma A.4. *Let H be a quadratic expression in c's and c^*'s. If $HE_- = 0$, then $H = 0$.*

Proof. There exists an operator K on L such that $[H, B(h)] = B(Kh)$ for $h \in L$. From $HE_- = 0$, we have for any $h \in L$

$$0 = HB(h)\Omega = [H, B(h)]\Omega + B(h)(H\Omega) = B(Kh)\Omega + B(h)(H\Omega).$$

Since H is quadratic in c's and c^*'s, $H\Omega$ is a linear combination of Ω and two particle vectors.

If it has two particle vector parts, then there exists $h \in L$ such that $B(h)(H\Omega)$ has 3 particle parts due to $N > 2$. Since $B(Kh)\Omega$ is a one-particle vector, this is not possible. Hence

$$H\Omega = \lambda\Omega, \quad B(Kh)\Omega = -\lambda B(h)\Omega. \tag{A.12}$$

This then implies $pK(1-p) = 0$ (due to $B((1-p)h)\Omega = 0$).

Working with H^* which also satisfies $H^*E_- = 0$, we obtain $p\Gamma K\Gamma(1-p) = 0$ (due to $[H^*, B(h)] = -B(\Gamma K \Gamma h)$) and hence $(1-p)Kp = 0$. Therefore

$$K = pKp + (1-p)K(1-p).$$

By substituting this into (A.12) when $ph = h$ and $B(h)$ is a creation operator, we obtain

$$pKp = -\lambda.$$

From the same computation for H^*, we obtain

$$H^*\Omega = \mu\Omega, \quad (1-p)K(1-p) = \mu.$$

From the CAR relations, we have

$$\begin{aligned} 0 &= [H, [B(h_1), B(h_2)]_+] \\ &= [B(Kh_1), B(h_2)]_+ + [B(h_1), B(Kh_2)] \\ &= (Kh_1, \Gamma h_2) + (h_1, \Gamma K h_2), \end{aligned}$$

we obtain $\Gamma K \Gamma = -K^*$ and hence $\mu = \bar{\lambda}$.

Again using $HE_- = 0$, we obtain

$$\begin{aligned} 0 &= Hc^*(f_1)c^*(f_2)c^*(f_3)\Omega \\ &= [H, c^*(f_1)]c^*(f_2)c^*(f_3)\Omega + c^*(f_1)[H, c^*(f_2)]c^*(f_3)\Omega \\ &\quad + c^*(f_1)c^*(f_2)[H, c^*(f_3)]\Omega + c^*(f_1)c^*(f_2)c^*(f_3)H\Omega \\ &= -2\lambda c^*(f_1)c^*(f_2)c^*(f_3)\Omega. \end{aligned}$$

By $N > 2$, we may choose mutually orthogonal non-zero f_1, f_2, f_3, for which $c^*(f_1)c^*(f_2)c^*(f_3)\Omega \neq 0$ and hence $\lambda = 0$. Therefore $K = 0$.

This implies that H is a multiple of identity, and $HE_- = 0$ then implies $H = 0$. Q.E.D.

(iii) Finally we prove the following Lemma needed in Section 5 after the equation (5.7).

Lemma A.5. *Let $\bar{c}$ and $\bar{c}'$ be linear combinations of c's and c^*'s. If $\alpha = \bar{c}^2$, $\beta = \bar{c}'^2$, $\gamma = [\bar{c}, \bar{c}']$ satisfy*

$$\gamma^2 - 4\alpha\beta \neq 0,$$

then there exist linear combinations c_0 and c_0' of $\bar{c}$ and $\bar{c}'$ such that $\bar{c}$ and $\bar{c}'$ are linear combinations of c_0 and c_0' and

$$c_0^2 = c_0'^2 = 0, \quad [c_0, c_0']_+ = \mathbf{1}.$$

Proof. We first look for solutions of

$$0 = (x\bar{c} + y\bar{c}')^2 = x^2\alpha + \gamma xy + \beta y^2$$

We have three cases.

(1) We consider the case $\gamma^2 \neq 4\alpha\beta$, $\alpha \neq 0$. Then we have two solutions

$$x_1 = -\gamma + \sqrt{\gamma^2 - 4\alpha\beta}, \qquad y_1 = 2\alpha$$
$$x_2 = -\gamma - \sqrt{\gamma^2 - 4\alpha\beta}, \qquad y_2 = 2\alpha$$

where $\sqrt{\gamma^2 - 4\alpha\beta}$ is one of the square root of $\gamma^2 - 4\alpha\beta$. Set

$$A = x_1\bar{c} + y_1\bar{c}', \quad B = x_2\bar{c} + y_2\bar{c}'.$$

We then have $A^2 = B^2 = 0$. Furthermore

$$\begin{aligned} (A - B)/(2\sqrt{\gamma^2 - 4\alpha\beta}) &= \bar{c}, \\ (2\alpha)^{-1}(A - x_1\bar{c}) &= \bar{c}', \end{aligned}$$

so that both $\bar{c}$ and $\bar{c}'$ are linear combinations of A and B. Finally,

$$\begin{aligned} [A, B]_+ &= 2\alpha x_1 x_2 + \gamma(x_1 y_2 + x_2 y_1) + 2\beta y_1 y_2 \\ &= 8\alpha^2\beta + 8\alpha^2\beta - 4\alpha\gamma^2 = 4\alpha(4\alpha\beta - \gamma^2) \neq 0. \end{aligned}$$

Now we can set, for example,

$$c_0 = A, \quad c_0' = (4\alpha(4\alpha\beta - \gamma^2))^{-1}B.$$

They satisfy

$$c_0^2 = c_0'^2 = 0, \quad [c_0, c_0']_+ = \mathbf{1}$$

Furthermore, $\bar{c}$ and $\bar{c}'$ are linear combinations of c_0, and c_0'.

(2) Case $\gamma^2 \neq 4\alpha\beta$, $\beta \neq 0$. By exchanging the role of $\bar{c}$ and $\bar{c}'$, we obtain the desired result.

(3) Case $\gamma \neq 0$, $\alpha = \beta = 0$. Already $c_0 = \bar{c}$, $c_0' = \gamma^{-1}\bar{c}'$ satisfy all the properties.

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