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The Histories of Chaotic Quantum Systems

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Summary

The histories for chaotic quantum systems are such that for long times the logical links are preserved but the causal links are forgotten.

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1 Introduction

In the many histories interpretation of quantum mechanics [1,2,3,4] one assigns probabilities to histories $P_1(t_1)\dots P_n(t_n)$ where the propositions P_k can be pictured as gates through which the system has to pass at a time t_k. A consistent probability assignment should observe two kinds of links.

(i) Logical links: $\overline{P}_k > P_k$ means that the gate P_k is contained in $\overline{P}_k$ and thus if we know that the system has passed through the smaller gate P_k we are sure that it has passed through the larger gate $\overline{P}_k$. Consequently a history with $\overline{P}_k$ should have a higher probability as the one with P_k.

(ii) Causal links: Since a dynamics works between the gates it should determine a causal order in which they are passed. Roughly if the motion is from left to right it is more likely that the system first passes through the gates on the left and then through the ones to the right than the other way round.

Surprisingly for finite quantum systems the logical links are not always respected whereas some causal links still exist. What I want to point out in this note is that for chaotic quantum systems, namely K-systems, in the limit of long times it is just the other way round. The logical links are respected, however, all causal relations are forgotten. In this respect they show the same behaviour as their classical counterpart.

2 Histories

In quantum logic propositions P are represented by projections onto subspaces of a Hilbert space $\mathcal{H}$. The order relation $P_1 > P_2$ means inclusion and the lattice operations $\vee$ and $\wedge$ are just the (linear) unions and intersections of these subspaces. Both operations are associative and commutative and monotonic with respect to the order relation:

$$(P_1 \vee P_2) \vee P_3 = P_1 \vee (P_2 \vee P_3), \qquad (P_1 \wedge P_2) \wedge P_3 = P_1 \wedge (P_2 \wedge P_3)$$
$$P_1 \vee P_2 = P_2 \vee P_1, \qquad P_1 \wedge P_2 = P_2 \wedge P_1 \tag{1}$$
$$P_1 > P_2 \Rightarrow P_1 \vee P_3 > P_2 \vee P_3 \qquad \text{and} \qquad P_1 \wedge P_3 > P_2 \wedge P_3.$$

P^c, the negation of P projects onto the orthogonal subspace and relations as in set theory

$$(P^c)^c = P, \qquad P^c \wedge Q^c = (P \vee Q)^c, \qquad P_1 > P_2 \Rightarrow P_1^c < P_2^c \tag{2}$$

hold. Algebraically, if P is considered as operator in $\mathcal{H}$, we have $P^c = 1 - P$, $P \wedge Q = \lim_{n\to\infty} P(QP)^n$ and $P \vee Q = P + Q$ if $PQ = 0$. A density matrix ρ assigns a probability $W(P) = \text{tr}\,\rho P$ to the truth of the proposition P. As function it is monotonic with respect to the order and $P_1 > P_2$ means in particular $W(P_1) = 1$ whenever $W(P_2) = 1$

or P_2 implies P_1. The lattice operations have their meaning "or", "and" in the sense that $W(P \vee Q) \geq \max\{W(P), W(Q)\}$, $= W(P) + W(Q)$ if $PQ = 0$ and

$$W(P) = 1, \qquad W(Q) = 1 \Rightarrow W(P \wedge Q) = 1. \tag{3}$$

However, subspaces of a Hilbert space may be oblique to each other such that the one of P_1 can intersect only at zero the ones of P_2 and P_2^c. In this case the classical distributive laws

$$\begin{aligned} P_1 \wedge (P_2 \vee P_3) &= (P_1 \wedge P_2) \vee (P_1 \wedge P_3) \\ P_1 \vee (P_2 \wedge P_3) &= (P_1 \vee P_2) \wedge (P_1 \vee P_3) \end{aligned} \tag{4}$$

break down

$$\begin{aligned} P_1 &= P_1 \wedge (P_2 \vee P_2^c) \neq (P_1 \wedge P_2) \vee (P_1 \wedge P_2^c) = 0 \\ P_1^c &= P_1^c \vee (P_2 \wedge P_2^c) \neq (P_1^c \vee P_2) \wedge (P_1^c \vee P_2^c) = 1. \end{aligned} \tag{5}$$

This situation is realized already for one spin for

$$P_1 = \frac{1+\sigma_x}{2}, \qquad P_2 = \frac{1+\sigma_z}{2}, \qquad P_2^c = \frac{1-\sigma_z}{2}.$$

In this case the first line of (5) says it never happens that $\sigma_z = 1$ and $\sigma_x = 1$ or that $\sigma_z = -1$ and $\sigma_x = 1$ though $\sigma_z = \pm 1$ are the only possibilities. In physics this break down of classical logic is explained by saying that σ_x and σ_z cannot be measured simultaneously. However, one can measure them successively and according to the standard interpretation of quantum mechanics first measuring P_1 reduces ρ to $P_1 \rho P_1/\mathrm{tr}\ P_1 \rho P_1$. If one then measures P_2 one finds for the conditional probability $\mathrm{tr}\ P_2 P_1 \rho P_1 P_2/\mathrm{tr}\ P_1 \rho P_1$. Since the denominator is the probability to find P_1 the joint probability for finding first P_1 and then P_2 is

$$W(P_1, P_2) = \mathrm{tr}\ P_2 P_1 \rho P_1 P_2 = \mathrm{tr}\ \sqrt{\rho}\, P_1 P_2 P_1 \sqrt{\rho}. \tag{6}$$

Remarks

1. In the classical (commutative) situation we have:

   (i) $W(P_1, P_2) = W(P_1 \wedge P_2)$, hence
   (ii) $W(P_1, P_2) = W(P_2, P_1)$ and
   (iii) $W(P_1, P_2) \leq \min\{W(P_1), W(P_2)\}$
   (iv) $W(P_1, P_2) + W(P_1^c, P_2) = W(P_2)$.

   All these properties are lost in the quantum case. To see this take

$$P_1 = \frac{1+\sigma_x}{2}, \qquad P_2 = \frac{1-\sigma_z}{2}, \qquad \rho = \frac{1+\sigma_z}{2}, \qquad P_1 \wedge P_2 = 0,$$

$$P_1P_2P_1 = \frac{1}{2}P_1, \qquad P_2P_1P_2 = \frac{1}{2}P_2,$$

and thus

$$W(P_1,P_2) = \frac{1}{4} = W(P_1^c,P_2), \qquad W(P_2,P_1) = 0, \qquad W(P_2) = 0.$$

The orthodox quantum physicist would say the following to these failures.

ad (i) $P_1 \wedge P_2$ is the proposition that the spin has $\sigma_x = 1$, $\sigma_z = -1$ which never happens and thus has zero probability. Since ρ represents the state spin "up" there is a 50% chance to find $\sigma_x = 1$. If this happens there is another 50% chance to find subsequently $\sigma_z = -1$ which gives $W(P_1,P_2) = \frac{1}{4}$.

ad (ii) In the state ρ I have zero probability to find $\sigma_z = -1$ thus $W(P_2,P_1) = 0 \neq W(P_1,P_2)$. This expresses the noncommutativity of the influences of measuring σ_x and σ_z.

ad (iii) $W(P_1,P_2) = \frac{1}{4} > W(P_2) = 0$ is a little harder to swallow because it means that now the logical order relation is also lost as $W(P_2) = W(1,P_2)$ and $1 > P_1$. One can argue that the proposition represented by 1: "the spin points somewhere" is always true and does not require any measurement. Thus the implication $P_1 \Rightarrow \overline{P_1}$ does not necessarily imply $W(\overline{P_1},P_2) \geq W(P_1,P_2)$ since the measurements of $\overline{P_1}$ and P_1 may effect P_2 differently.

ad (iv) It says "something must have happened in the first place" and its failure has the same origin as the one of (iii). Ironically the non-distributivity of quantum logic changes a classical equality into an inequality $P_2 = P_2 \wedge (P_1 \vee P_1^c) > (P_2 \wedge P_1) \vee (P_2 \wedge P_1^c) = 0$ which goes in the other direction to what we have now

$$0 = W(P_2) < W(P_1^c,P_2) + W(P_1,P_2) = \frac{1}{2}.$$

2. One classical relation remains true, namely if P_2 implies P_1, that is $P_1 > P_2$ then $W(P_1,P_2) = W(P_2,P_1) = W(P_2)$. This means that the conditional probability to find P_1 given P_2 is equal to one.

3. That quantum mechanically the implication $P_1 \Rightarrow \overline{P_1}$ does not yield $W(\overline{P_1},P_2) \geq W(P_1,P_2)$ is perhaps more paradoxical than the failure of Bell's inequality since no locality assumptions but only logical implications are involved.

This preceding procedure can be generalized in two ways. One can measure an arbitrary number of projections P_α, $\alpha = 1,\ldots,r$ and one may let a time evolution $P \to P(t)$ intervene between the measurements. In this way one can assign to a sequence of "events" $P_{\alpha_1}(t_1), P_{\alpha_2}(t_2), \ldots, P_{\alpha_n}(t_n)$ (a "history" briefly written $\underline{\alpha}$ for the index set or the corresponding vector) a probability $W(\underline{\alpha}) = \mathrm{Tr}\ P_{\alpha_n}(t_n)\ldots P_{\alpha_1}(t_1)\rho P_{\alpha_1}(t_1)\ldots P_{\alpha_n}(t_n)$. For a complete set of projections, $P_\alpha P_{\alpha'} = \delta_{\alpha\alpha'}P_\alpha$, $\sum_\alpha P_\alpha = 1$, this gives a probability distribution over the set $\{\underline{\alpha}\}$ of histories:

$$W(\underline{\alpha}) \geq 0, \qquad \sum_{\underline{\alpha}} W(\underline{\alpha}) = 1. \tag{7}$$

Remark: Now even classically the commutativity $W(\alpha_1, \alpha_2) = W(\alpha_2, \alpha_1)$ does not hold anymore. As trivial example take for $P_{1,2}$ the characteristic functions $\chi_{(p_1,p'_1)}$ and $\chi_{(p_2,p'_2)}$ on the circle and as dynamics the shift $P(t) = \chi_{(p+t,p'+t)}$ and for W the Lebesgue measure μ. Then

$$W(1,2) = \mu((p_1, p'_1) \wedge (p_2 + t, p'_2 + t)) \neq \mu((p_2, p'_2) \wedge (p_1 + t, p'_1 + t)) = W(2,1).$$

If $(p_2, p'_2) = (p_1 - t, p'_1 - t)$ then $P_2(t) \Leftrightarrow P_1(0)$ but $P_2(0) \not\Leftrightarrow P_1(t)$. The absence of symmetry reflects the causal order of events, P_1 is a precondition for P_2 to happen at t but not vice versa.

To give a consistent description of the different histories some further relations hold in the classical case which are absent in quantum mechanics. They are all guaranteed if the system has a property called "decoherence".

Proposition I. Let P_α, $\sum_{\alpha=1}^r P_\alpha = 1$ be some orthogonal projectors. Between the properties which hold classically but not generally,

(i) $D(\underline{\alpha}', \underline{\alpha}) := \text{Tr } P_{\alpha'_1}(t_1) \dots P_{\alpha'_n}(t_n) \rho P_{\alpha_n}(t_n) \dots P_{\alpha_1}(t_1) = \delta_{\underline{\alpha},\underline{\alpha}'} W(\underline{\alpha})$ ("decoherence").

(ii) If $P_{\alpha_i} \neq 0 \; \forall \, i$ then $W(\underline{\alpha} + \underline{\alpha}') = W(\underline{\alpha}) + W(\underline{\alpha}')$ (using the same definition for W even if $P_\alpha + P_{\alpha'}$ need not be a projector).

(iii) If $\overline{P}_\alpha$ is another set of orthogonal projectors such that $\forall \; \alpha'_i \exists \alpha_i$, $i = 1 \dots n$, with $\overline{P}_{\alpha_i} > P_{\alpha'_i}$ then $\overline{W}(\underline{\alpha}) \geq W(\underline{\alpha}')$.

(iv) $\sum_{\alpha_k=1}^r W(\alpha_1 \dots \alpha_n) = W(\alpha_1, \dots \alpha_{k-1}, \alpha_{k+1} \dots \alpha_n)$

there are the implications

$$\text{(iv)} \Leftarrow \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

Proof:

(i) ⇒ (ii) We assumed $P_{\alpha_i} \neq 0 \; \forall \, i$ such that $P_{\alpha'_i} \cdot P_{\alpha_i} = 0$ implies $P_{\alpha'_i} \neq P_{\alpha_i}$. Then (i) implies the vanishing of the cross terms which yields the multilinearity of $W(\underline{\alpha})$.

(ii) ⇒ (iii) $\overline{P}_{\alpha_i} > P_{\alpha'_i}$ implies $\overline{P}_{\alpha_i} = P_{\alpha'_i} + P_{\alpha''_i}$ with $P_{\alpha'_i} \cdot P_{\alpha''_i} = 0$ and thus $\overline{W}(\underline{\alpha}) = W(\underline{\alpha}') + W(\underline{\alpha}'') \geq W(\underline{\alpha}')$.

(i) ⇒ (iv) $W(\alpha_1, \dots \alpha_{k-1}, \alpha_{k+1} \dots \alpha_n) = W(\alpha_1, \dots \alpha_{k-1}, 1, \alpha_{k+1} \dots \alpha_n) =$ $= W(\alpha_1, \dots \alpha_{k-1}, \sum_{\alpha_i} P_{\alpha_i}, \alpha_{k+1}, \dots, \alpha_k) = \sum_{\alpha_k} W(\alpha_1, \dots \alpha_n)$ since the terms with $i \neq k$ are zero according to (i).

One might think that if the $t_i - t_{i-1}$ are macroscopic times that the system behaves classically and decoheres. That this is not so is shown in the appendix for one free particle in one dimension.

In this note I want to study the effect of chaoticity of quantum dynamics on the behaviour of $W(\underline{\alpha})$. As prototype of chaotic systems we shall take K-systems [5,6,7] whose properties can readily be generalized to quantum dynamics. For long time intervals $t_i - t_{i-1}$ their histories show the following simple features:

(i) They decohere.

(ii) $W(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a symmetric function.

Remarks

ad (i) This was to be expected since these systems, in contradistinction to finite quantum systems, are asymptotically abelian. Thus for a wide time mesh all the classical properties should hold. The only point is which degree of asymptotic abelianness insures that the properties of Prop. I hold. In general K-systems are only weakly but not strongly asymptotic abelian and this does not immediately imply I.

ad (ii) This means for long times the system forgets all causal links. There is a numerical measure of the forgetfulness of a dynamical system, the memory loss [9]. Classically K-systems have maximal memory loss but quantum mechanically it is only a subclass of K-systems, the entropic K-systems which share this property.

3 K-Systems

Definition (3.1) An algebraic K-system consists of an algebra $\mathcal{A}$ with an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ and a family $\mathcal{A}_n$, $n \in Z$, of subalgebras such that $\sigma(\mathcal{A}_n) = \mathcal{A}_{n+1}$ and

(i) $\mathcal{A}_{n+1} \supset \mathcal{A}_n$,

(ii) $\bigcup_n \mathcal{A}_n = \mathcal{A}$,

(iii) $\bigcap_n \mathcal{A}_n = C \cdot \mathbf{1}$.

Remarks (3.2)

1. We shall assume all $\mathcal{A}_n$ and $\mathcal{A}$ to be von Neumann algebras. General C^* dynamical systems will have several invariant states and they will not exhibit the necessary cluster properties unless they are extremal invariant. Correspondingly $\bigcup_n$ means algebraic union together with strong closure.

2. $\bigcap_n$ is the set theoretic intersection. Thus (iii) means that the isomorphism σ : $\mathcal{A}_n \leftrightarrow \mathcal{A}_{n+1}$ has no non-trivial invariant subalgebras. They would remain in the "tail" $\bigcap_n \mathcal{A}_n$ and conversely such a tail would be an invariant subalgebra.

If ω is a σ-invariant faithful state over $\mathcal{A}$ then it was shown in [7] that one has the following cluster properties.

Theorem (3.3) Let $(\mathcal{A}_n, \sigma, \omega)$ be a von Neumann K-system then $\forall\, b \in \mathcal{A}$, $n \in Z$, $\varepsilon > 0\ \exists\, M(b, n, \varepsilon)$ such that

$$|\omega(b\sigma^{-k}a) - \omega(b)\omega(a)| < \varepsilon\|a\| \qquad \forall\, a \in \mathcal{A}_n, \quad k \geq M.$$

Remarks (3.4)

1. The proof of (3.3) uses the modular automorphism of ω. This is why we need von Neumann algebras and ω to be faithful.
2. (3.3) expresses a uniformity over all $\mathcal{A}_n$ of the weak convergence of $\sigma^k(a)$ to $\omega(a)$. Though the set $\{\mathcal{A}_n\}$ is strongly dense in $\mathcal{A}$ uniformity $\forall\, a \in \mathcal{A}$ is impossible (take $\sigma^M b$ for a).

If σ^{-1} represents the time evolution, $\sigma^{-t}(P) = P(t)$ then the K-clustering (3.3) implies the features of the histories in this system which we stated in Section 2.

Theorem (3.5) Let $(\mathcal{A}_n, \sigma, \omega)$ be a von Neumann K-system with σ^{-1} the time evolution. Given a set of propositions $P_1 \ldots P_r \in \mathcal{A}$ and $\varepsilon > 0$, $n \in N$, then there exists T such that for each history $W(\underline{\alpha}) = \omega(P_{\alpha_1}(t_1) \ldots P_{\alpha_n}(t_n) \ldots P_{\alpha_1}(t_1))$ we have

$$\left| W(\underline{\alpha}) - \prod_{i=1}^{n} \omega(P_{\alpha_i}) \right| < \varepsilon$$

whenever $t_{i+1} - t_i > T\ \forall\, i$.

Proof: First we note that the strong density of $\{\mathcal{A}_n\}$ in $\mathcal{A}$ implies that given $P_i, = 1 \ldots r$ $\forall\, \varepsilon > \exists\, N$ with $(\|P_i - \tilde{P}_i)|\Omega\rangle\| < \varepsilon\ \forall\, i = 1 \ldots r$ where $\tilde{P}_i \in \mathcal{A}_N$, $\|\tilde{P}_i\| = 1$. $|\Omega\rangle$ is the cyclic vector in the GNS-Hilbert space corresponding to ω. This extends to histories because of the

Lemma

$$\|(P_{\alpha_1}(t_1)P_{\alpha_2}(t_2) \ldots P_{\alpha_n}(t_n) - \tilde{P}_{\alpha_1}(t_1) \ldots \tilde{P}_{\alpha_n}(t_n))|\Omega\rangle\| \leq 3\varepsilon(1 + c')^n$$

where $c' > 0$ is determined as follows. Since ω is faithful $|\Omega\rangle$ it is cyclic for the commutant $\mathcal{A}'$ and thus $\exists\, P_i' \in \mathcal{A}'$ such that $\|(\tilde{P}_i - P_i')|\Omega\rangle\| < \varepsilon\ \forall\, i$ and $c' = \max_i \|P_i'\|$.

Proof of the Lemma:

$$\prod_{i=1}^{n} P_{\alpha_i}(t_i) - \prod_{i=1}^{n} \tilde{P}_{\alpha_i}(t_i) = \prod_{i=1}^{n-1} P_{\alpha_1}(t_i)(P_{\alpha_n}(t_n) - \tilde{P}_{\alpha_n}(t_n)) - \left(\prod_{i=1}^{n-1} \tilde{P}_{\alpha_i}(t_i) - \prod_{i=1}^{n-1} \tilde{P}_{\alpha_i}(t_i)\right) \tilde{P}_{\alpha_n}(t_n)$$

and we can proceed by induction in n. We want to show $\|(\prod_{i=1}^n P_{\alpha_i} - \prod_{i=1}^n \tilde{P}_{\alpha_i})|\Omega\rangle\| \leq 3\varepsilon\ C(n)$ and we know already $C(1) < 1$. Since all P and $\tilde{P}$ have norm 1 the above decomposition says upon replacing $\tilde{P}_{\alpha_n}$ by $P'_{\alpha n}$

$$3\varepsilon C(n) \leq \varepsilon + 3\varepsilon c' C(n-1) + 2\varepsilon \Rightarrow C(n) \leq 1 + c' C(n-1) \Rightarrow C(n) \leq (1+c')^n.$$

Since $\sigma(P) = U^{-1}PU$ and $U|\Omega\rangle = |\Omega\rangle$ all the estimates are uniform in the t_i. Thus we have shown $|W(P_{\underline{\alpha}}) - W(\tilde{P}_{\underline{\alpha}})| < 6\varepsilon(1+c')^n$ and since n and the P_i and therefore c' are fixed it means that effectively we may assume all P_i to be in some $\mathcal{A}_N$. To apply (3.3) we still have to bring together the factor referring to the same time and again we shall proceed inductively. Let α_t denote the modular automorphism of ω such that we have the KMS-condition satisfied: $\omega(ab) = \omega((\alpha_{-i}b)a)$. The elements for which α_t can be continued analytically such that $\|\alpha_{-i}(b)\| < \infty$ are strongly dense in $\mathcal{A}$ such that $\forall\, \varepsilon > 0$ $\exists\, \hat{P}_k$ such that $(\|\tilde{P}_k - \hat{P}_k)|\Omega\rangle\| < \varepsilon$ and $\|\alpha_{-i}\hat{P}_k\| < c\ \forall\ k = 1\ldots r$. Then

$$\begin{aligned}&|\langle\Omega|\tilde{P}_{\alpha_1}(t_1)\ldots\tilde{P}_{\alpha_n}(t_n)\ldots\tilde{P}_{\alpha_1}(t_1)|\Omega\rangle \\ &\quad - \langle\Omega|(\alpha_{-i}\hat{P}_{\alpha_1})\tilde{P}_{\alpha_1}\tilde{P}_{\alpha_2}(t_2-t_1)\ldots\tilde{P}_{\alpha_n}(t_n-t_1)\ldots\tilde{P}_2(t_2-t_1)|\Omega\rangle| < \varepsilon.\end{aligned}$$

Since all $\tilde{P}_{\alpha_i} \in \mathcal{A}_n$ and $(\alpha_{-i}\hat{P}_{\alpha_1})\tilde{P}_{\alpha_1} \in \mathcal{A}$ we can appeal to (3.3) to show that if $t_2 - t_1 \geq M_1$ and thus all $t_k - t_1 > M_1 = M(\alpha_{-i}\hat{P}_{\alpha_1})\tilde{P}_{\alpha_1}, N, \varepsilon)$, $k = 2\ldots n$ we get

$$|\langle\Omega|\tilde{P}_{\alpha_1}(t_1)\ldots\tilde{P}_{\alpha_n}(t_n)\ldots\tilde{P}_{\alpha_1}(t_1)|\Omega\rangle - \omega(\tilde{P}_{\alpha_1})\langle\Omega|\tilde{P}_{\alpha_2}(t_2)\ldots\tilde{P}_{\alpha_n}(t_n)\ldots\tilde{P}_{\alpha_2}(t_2)\rangle| < 5\varepsilon$$

$\forall\ t_2 - t_1 > M_1$. To collect the ε's we used $\omega((\alpha_{-i}\hat{P}_{\alpha_1}\tilde{P}_{\alpha_1}) = \omega(\tilde{P}_{\alpha_1}(\tilde{P}_{\alpha_1} + \hat{P}_{\alpha_1} - \tilde{P}_{\alpha_1}))$ and

$$\begin{aligned}\omega(\tilde{P}^2_{\alpha_i}) &= \omega(P^2_{\alpha_i} + P_{\alpha_i}(\tilde{P}_{\alpha_i} - P_{\alpha_i}) + (\tilde{P}_{\alpha_i} - P_{\alpha_i})P_{\alpha_i} + (P_{\alpha_i} - \tilde{P}_{\alpha_i})^2) \\ &= \omega(\tilde{P}_{\alpha_i} + (P_{\alpha_i} - \tilde{P}_{\alpha_i})(1 + P_{\alpha_i}) + P_{\alpha_i}(\tilde{P}_{\alpha_i} - P_{\alpha_i}) + (P_{\alpha_i} - \tilde{P}_{\alpha_i})^2).\end{aligned}$$

Thus by a finite number of induction steps we come to (3.5) with $T = \max_i M_i$ and a suitable redefinition of ε.

This proves the asymptotic symmetry of $W(\underline{\alpha})$ in $\alpha_1\ldots\alpha_n$. To see the decoherence one notes that if we have to the right $P_{\alpha'}$ we get an expression

$$\begin{aligned}|\langle\Omega|(\alpha_{-i}\hat{P}_{\alpha'_k})\tilde{P}_{\alpha_k}|\Omega\rangle| &= |\langle\Omega|\tilde{P}_{\alpha_k}\hat{P}_{\alpha'_k}|\Omega\rangle < \varepsilon + |\langle\Omega|\tilde{P}_{\alpha_k}\tilde{P}_{\alpha'_k}|\Omega\rangle| \\ &\leq 2\varepsilon + |\langle\Omega|P_{\alpha_k}\tilde{P}_{\alpha'_k}|\Omega\rangle \leq 3\varepsilon\end{aligned}$$

for $\alpha'_k \neq \alpha_k$ since the P's are orthogonal.

Remarks

1. Schrödinger put it succinctly: "All we have in quantum theory is a succession of events but we cannot fill the gaps between them". In the language used here this comes to the following question. If we start with a state ρ and refine it by measuring $P_{\alpha_1}, P_{\alpha_2}\ldots$ can it be sharpened to the degree that we can make a safe prediction what happens between P_{α_k} and $P_{\alpha_{k+1}}$? Is there a P_{α_ℓ}, $t_k \leq t_\ell \leq t_{k+1}$

such that $W(P_{\alpha_1}, P_{\alpha_2}, \ldots P_{\alpha_k} P_{\alpha_\ell} P_{\alpha+1} \ldots P_{\alpha_n}) = W(P_{\alpha_1}, P_{\alpha_2}, \ldots P_{\alpha_k}, P_{\alpha_{k+1}}, \ldots P_{\alpha_n})$ and $W(P_{\alpha_1}, P_{\alpha_2} \ldots P_{\alpha_k} P^c_{\alpha_\ell} P_{\alpha_{k+1}} \ldots P_{\alpha_n}) = 0$. Keeping the P_{α_i} and the t_i fixed this can always be achieved by taking $P_{\alpha_\ell} = P_{\alpha_k}(t_k - t_\ell)$ or $P_{\alpha_\ell} = P_{\alpha_{k+1}}(t_{k+1} - t_\ell)$. This reflects only the deterministic nature of the time evolution. What Schrödinger probably ment was that this does not hold for all P_ℓ or P^c_ℓ which certainly is true in the noncommutative case where the decomposition of unity in minimal projections is not unique. In our case when we let the $t_{i+1} - t_i$ go to infinity in fact it holds for no $P_{\alpha_\ell} \neq 1$ whatsoever. The reason is that the above probabilities get an extra factor $\omega(P_{\alpha_\ell})$ or $\omega(P^c_{\alpha_\ell})$ and $\omega(P_{\alpha_\ell}) = 0$ or $\omega(P^c_{\alpha_\ell}) = 0$ are excluded by the faithfulness of ω.

2. In general quantum dynamical systems the consistency conditions Prop. I hold if the $P_1 \ldots P_r$ are taken from an abelian subalgebra of $\mathcal{A}$ which is invariant under the time evolution. If $\mathcal{A}$ has a nontrivial center it would be a candidate for such a subalgebra but if ω is KMS the center is elementwise invariant and the dynamics becomes trivial. If $\mathcal{A}$ is simple there may be no invariant abelian subalgebras and the consistency conditions may never be satisfied. For K-systems the situation is much better since in the long time limit (I,(i)) holds for any set of projections.

Appendix

We start with a particle at rest in an interval $(-L/2, L/2) \subset \mathbf{R} : \varphi_0(x) = 1/\sqrt{L}\ \forall\, |x| < L/2$, zero otherwise. The first proposition we measure is whether the particle is in the subinterval $\Delta = (-a, a)$, $a \leq L/2$. It corresponds to the projection operator

$$P_\Delta = \chi_\Delta(x) = \begin{matrix} 1 & \text{for } |x| < a \\ 0 & \text{otherwise.} \end{matrix}$$

The probability for this to be true is

$$\omega_{\varphi_0}(P_\Delta) = \langle \varphi_0 | \chi_\Delta | \varphi_0 \rangle = \frac{2a}{L}.$$

According to the reduction postulate this measurement changes the wave function to $\psi_0 = \frac{1}{\sqrt{2a}}\,\chi_\Delta(x)$. We assume a free time evolution $H = p^2$ and to calculate $\psi_t = e^{-ip^2 t}\psi_0$ we need the Fourier transform

$$\begin{aligned} \tilde{\psi}_0 &= \int_{-\infty}^{\infty} \frac{dx\, e^{ipx}}{\sqrt{2\pi}}\, \psi_0(x) = \frac{1}{\sqrt{4\pi a}} \int_{-a}^{a} dx\, e^{ipx} = \frac{1}{\sqrt{\pi a}}\, \frac{\sin ap}{p}, \\ \tilde{\psi}_t &= \frac{e^{-ip^2 t}}{\sqrt{\pi a}}\, \frac{\sin ap}{p}. \end{aligned}$$

Next we measure whether the momentum is in an interval $\Delta' = (b, b+\varepsilon)$. Thus the conditional probability given P_Δ to find $\tilde{P}_{\Delta'}$ after the time t is

$$\omega_{\psi_t}(\tilde{P}_{\Delta'}) = \frac{1}{\pi a} \int_b^{b+\varepsilon} \frac{dp}{p^2}\, \sin^2 ap = \frac{1}{\pi} \int_{ab}^{ab+a\varepsilon} d\alpha\, \frac{\sin^2 \alpha}{\alpha^2}.$$

Now for fixed b and ε the joint probability to find first P_Δ and after time t $\tilde{P}_{\Delta'}$ is

$$W(\Delta,\Delta') = \langle\varphi_0|P_\Delta\tilde{P}_{\Delta'}P_\Delta|\varphi_0\rangle = \omega_{\varphi_0}(P_\Delta)\cdot\omega_{\psi_t}(\tilde{P}_{\Delta'}) = \frac{2a}{L\pi}\int_{ab}^{ab+a\varepsilon} d\alpha\,\frac{\sin^2\alpha}{\alpha^2}. \qquad (A.1)$$

It should be monotonically increasing in a. This follows since $\bar{a} > a \Rightarrow P_{\bar{\Delta}} > P_\Delta$ or if one finds the particle in $(-a,a)$ one is sure that it is in $(-\bar{a},\bar{a})$. However (A.1) is not monotonic in a. To see this take a such that $ab = \pi/2$ and ε such that $a\varepsilon \ll 1$. Then

$$W(\Delta,\Delta') = \frac{1}{bL}\int_0^{a\varepsilon} d\gamma\,\frac{\cos^2\gamma}{(\pi/2+\gamma)^2} \simeq \frac{2\varepsilon}{\pi L b^2}.$$

Next consider $\bar{a} = \pi/b > a$ and again $\bar{a}\varepsilon \ll 1$. Then

$$W(\bar{\Delta},\Delta') = \frac{2}{bL}\int_0^{\bar{a}\varepsilon} d\gamma\,\frac{\sin^2\gamma}{(\pi+\gamma)^2} \simeq \frac{2\pi\varepsilon^3}{3Lb^4}.$$

Thus we have $W(\Delta,\Delta') > W(\bar{\Delta},\Delta)$ for $\pi^2\varepsilon^2/3 < b^2$ which we are free to choose. There is no escape to the conclusion that the reduction postulate does not lead to a classically consistent probabilistic interpretation of quantum mechanics.

Remarks

1. $H = p^2$ is of no importance, any $H = f(p)$ which conserves the momentum leads to the same conclusion.
2. In contradistinction to the chaotic quantum systems which we studied in Section 3 here even for macroscopic times some propositions do not decohere.
3. There are some projections of x and p which have a common eigenfunction [8] and for those this paradox would not appear.

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