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Conformal Anomalies – Recent Progress

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Abstract. We present a brief review of some recent results on conformal anomalies in four and more dimensions. The discussion is intended for relativists, so some background on the quantum origin of anomalies and of the ir simple properties in D=2 is also provided. Topics treated include a critical review of the effective gravitational action uniqueness problem and the derivation of beta functions, independent of ultraviolet behavior, from the type B anomaly.

1 Introduction

The subject of conformal (or Weyl) anomalies is almost precisely 20 years old, and has in its lifetime been connected with and influenced many important problems in relativity and particle physics, from Hawking radiation to conformal field theory and strings, as well as mathematics. The associated literature is correspondingly enormous and in this brief review I will concentrate only on the aspects of the problem that A. Schwimmer and I [1] as well as others, *e.g.*, [2, 3, 4] have been studying recently. Some of the details skipped over here may be found in these references; for some history see [5].

Since the quantum field theoretical background is not familiar to many relativists, I will begin with a (very) rapid introduction to anomalies as the result of a clash between classical symmetries and the quantum requirement of regularization. This will be illustrated in the simplest, but as usual, very special case of 2D where everything is unique and explicitly presentable in closed form, before going on to explain the generalization to four dimensions and higher. Here we will discuss both positive results as well as "what we know that isn't so," namely some widespread misunderstandings about the structure of effective gravitational actions, and what we still don't know. The emphasis throughout is on "classical" aspects that may particularly interest relativists.

2 Anomalies in General

Classical matter actions can be endowed with various formal invariances. The classic example here is that of chiral anomalies: a "charged" spinor field is invariant both under internal, "gauge", rotations and (up to terms in m) under chiral ones involving conjugation with γ_5. The corresponding Noether currents are the usual $j_\mu \sim \bar{\psi}\gamma_\mu\psi$ and the chiral current $j_{\mu 5} \sim \bar{\psi}\gamma_5\gamma_\mu\psi$. At the quantum level however, the regularization required to define and compute current correlation functions involving closed loops (even for the free field!) cannot simultaneously preserve both of these invariances – for example, a massive regulator clearly alters the chiral current's divergence, while other prescriptions would even destroy "charge" conservation. All this has no particular importance for the free field (since its currents are not the sources of anything) but as soon as there are even non-dynamical, external, fields present that couple to the currents, the consequences become very important indeed. In particular if one considers the closed loop triangle diagram represented by the time ordered correlator $< T(j_{\mu 5}(x)j_\alpha(y)j_\beta(z)) >$, there is a very physical effect: each of the j_a is coupled to an external A_α, while (the divergence of) $j_{\mu 5}$ represents a neutral pseudoscalar field (the π^0). Thus, the observed $\pi^0 \to 2\gamma$ decay is directly traceable to the – quantum – breakdown of chiral invariance through the single loop diagram with accompanying nonvanishing of the divergence of the above 3-point function (of course choosing to break gauge invariance instead would be catastrophic!). Furthermore, although we have here a closed loop, and had to regularize to obtain a well-defined answer, there are no infinities – this is a finite calculable process. The anomaly itself is proportional to the topological density $F_{\mu\nu}{}^*F^{\mu\nu}$, *i.e.*, the chiral current fails to be conserved, by $\partial_\mu j^{\mu 5} \sim \alpha F^*F$, and there is a corresponding effective action expressible in terms of the external fields that encodes the "backreaction" of the quantized matter (fermions) or the "external" pions and photons.

There is an even more obvious arena in which regularization (through introduction of a mass or of a cutoff or by formally continuing away from the physical dimension) destroys an invariance, namely that of conformally invariant systems involving only dimensionless parameters. Standard free field examples include the Maxwell action (but only at D=4) or a massless spinor or scalar in any D, all of whose dilation currents, $D^\mu = x^\nu T^\mu_\nu$, are conserved since for these systems both $\partial_\mu T^{\mu\nu} = 0$ and $T^\mu_\mu = 0$. [For the scalar field in $D > 2$, the usual stress tensor must be suitably "improved" in order to become traceless.] Now one may simply follow the same lines as for the chiral anomaly: Any regularization introduces a mass or alters the dimension, so that the closed loop contributions involving stress-tensor vertices lead to vacuum correlation functions involving (suitable numbers of) the $T_{\mu\nu}$ whose quantum invariances are diminished – either conservation or tracelessness is lost. Furthermore, one may introduce an external gravitational field coupled to the $T_{\mu\nu}$ vertices in such a way that the anomaly's properties may be expressed in purely geometric terms. The anomaly is again <u>finite</u> and cutoff-independent and the corresponding effective gravitational action that generates it represents the back-reaction of matter on the geometry – hence the connection with Hawking radiation. It is also related to the beta-function for the matter system in question (see appendix B), hence the special relevance to conformal field and string theory of the 2D anomaly.

3 Conformal Anomaly in 2D

Two dimensions are always very special in physics, but there is nevertheless a lot to be learned from this simplest context. We will see that this is the one case where everything can be done explicitly to describe and use the anomaly, and also begin to see how higher dimensions will differ in fundamental ways; in particular it will become clear what the open problems are, and what apparently natural extensions beyond D=2 are in fact incorrect.

Here and throughout we will use dimensional regularization, in which the spacetime dimension is moved by ϵ from its integer value, *e.g.*, $D = 2 + \epsilon$, to have entirely finite unambiguous correlators before we face the delicate question of taking the $\epsilon = 0$ limit. Near D=2, we consider in flat space to begin with, the vacuum 2-point correlator,

$$K(q)_{\mu\nu\alpha\beta} = \langle T_{\mu\nu}(q)\, T_{\alpha\beta}(-q)\rangle \,, \tag{3.1}$$

where q is the external momentum of this 2-point closed loop (and time ordering is understood throughout). Here $T_{\mu\nu}$ represents the stress tensor operator of one of our massless systems, say a scalar, for which T^{μ}_{μ} vanishes at any dimension, integer or not, by "improvement". Specifically, let

$$T_{\mu\nu} = (\phi,_{\mu}\phi,_{\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\,\phi,_{\alpha}\phi^{,\alpha}) + \tfrac{(D-2)}{4(1-D)}\,(\partial_{\mu}\partial_{\nu} - \eta_{\mu\nu}\Box)\phi^2 \equiv T^0_{\mu\nu} + \Delta_{\mu\nu}\,. \tag{3.2}$$

This is the usual "minimal" stress tensor $T^0_{\mu\nu}$ supplemented by an identically conserved "superpotential" term $\Delta_{\mu\nu}$ that does not affect the Poincaré generators, and so is allowed. On shell ($\Box\phi = 0$), it is easy to see that $T_{\mu\nu}$ is traceless and conserved, $q^{\nu}\,T^{\mu}_{\nu}(q) = 0$. Then the regulated function $K_{\mu\nu\alpha\beta}$ must be proportional to projectors $P_{\mu\nu}(q) = (-q_{\mu}q_{\nu} + q^2\eta_{\mu\nu})$ on each of its indices as well as symmetric under interchange of the pairs $(\mu\nu)$ and $(\alpha\beta)$ and of course traceless in each pair. The unique such form, as also obtained by explicit integration over the internal loop momentum is

$$K_{\mu\nu\alpha\beta}(q) = f(D)/\epsilon\left\{(P_{\mu\alpha}P_{\nu\beta} + P_{\mu\beta}P_{\nu\alpha}) - \frac{2}{D-1}\,P_{\mu\nu}P_{\alpha\beta}\right\} q^{-2(1+\epsilon)} \tag{3.3}$$

where $f(D)$ is a finite constant depending on the field species. Since $P_{\mu\alpha}P_{\alpha\nu} = q^2\,P_{\mu\nu}$ and $P^{\mu}_{\mu} = (D-1)q^2$, it follow that K obeys all the above requirements. Now by power counting (in the loop integration) alone it follows *a priori* that K must be finite at D=2. We have mentioned that finiteness is a hallmark of anomalies, but it can be a very subtle one, as we shall see: There must be some hidden factor in the numerator to cancel the ϵ denominators at D=2. This is indeed the case for, but only for, the $q_{\mu}q_{\nu}q_{\alpha}q_{\beta}$ term in (3.1) which is manifestly proportional to ϵ. For the rest, the mechanism in question is what we have called "0/0" in

[1]: exactly at $\epsilon = 0$ the whole numerator vanishes identically, as is most easily seen by noting that exactly at D=2, $P_{\mu\nu} = \tilde{q}_\mu \tilde{q}_\nu$, $\tilde{q}_\mu \equiv \epsilon_{\mu\nu} q^\nu$. So each term in (3.3) is simply quartic in the $\tilde{q}$'s and their sum vanishes at D=2. A deeper, and more geometric statement of this comes about if we now introduce an external metric and couple each $T_{\mu\nu}$ to this metric in the usual way. Indeed, it is sufficient to use linearized coupling to the deviation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ from flat space, and then invoke covariance to obtain the full answer. The corresponding functional

$$W[h_{\mu\nu}] = \int\int h_{\mu\nu}(x)\langle T_{\mu\nu}(x) T_{\alpha\beta}(y)\rangle h_{\alpha\beta}(y) d^2x\, d^2y \tag{3.4}$$

is of course the (finite) effective gravitational action due to matter back-reaction (to one-loop order), which incorporates the anomaly in the limit $\epsilon \to 0$. Using the linearized identity (with the sign convention $R_{\mu\nu} \sim +\partial_\alpha \Gamma^\alpha_{\mu\nu}$)

$$G^L_{\mu\nu} = \frac{\Box^{-1}}{2}\left(P_{\mu\nu} P_{\alpha\beta} - P_{\mu\alpha} P_{\nu\beta}\right) h_{\alpha\beta}\ , \quad R^L = P^{\mu\nu} h_{\mu\nu}\ , \quad P_{\mu\nu} \equiv (-\eta_{\mu\nu}\Box + \partial^2_{\mu\nu}) \tag{3.5}$$

we see that

$$W^L[h] \sim \frac{1}{\epsilon}\int d^Dx\, d^Dy \left[4 G^L_{\mu\nu}\, \Box^{-1}\, G^L_{\mu\nu} + \frac{2\epsilon}{(D-1)} R^L\, \Box^{-1}\, R^L\right] \tag{3.6}$$

where we have dropped the (irrelevant) $\Box^\epsilon$ part. Now <u>at</u> D=2, the identity $P_{\mu\nu} = \tilde{q}_\mu \tilde{q}_\nu$ we found earlier precisely implies that $G^L_{\mu\nu} \equiv 0$, an identity well known to be valid to all orders in $h_{\mu\nu}$, *i.e.*, for the full Einstein tensor. Thus, we obtain a meaningful prescription for W^L by <u>defining</u> the numerator to be taken <u>at</u> D=2, where $G^L_{\mu\nu} = 0$, which leaves the unique finite form

$$W^L[h] \sim \tfrac{1}{2}\textstyle\int d^2x\, d^2y\, R^L(x)\Box^{-1}(x,y) R^L(y)\ , \qquad \Box[\Box^{-1}(x,y)] \equiv \delta(x-y)\ , \tag{3.7}$$

for our effective action to lowest order. Note also the single pole structure $\Box^{-1}$, defined of course as the flat space scalar propagator (with some choice of boundary condition). This is traceable back to the hard-core one-loop Feynman diagram origin of our fancy effective action, a fact it will be essential to remember also in higher D. Now we can easily improve (3.7) to a fully covariant form, namely

$$W[g] \sim \int\int d^2x\, d^2y (\sqrt{-g}\, R)(x) \langle x|(\sigma qrtg\, \Box)^{-1}|y\rangle (\sqrt{-g}\, R)(y) \tag{3.8}$$

in terms of the full curved space propagator indicated. This is the celebrated Polyakov action, up to an overall (equally celebrated!) coefficient. Let me make two further important remarks about W. The first is to remind us where the anomalies are: Even though the formal operator matter action is Weyl invariant,[1] the resulting W is not. That is, if we vary $g_{\mu\nu}$ in W, using the fact that in 2D,

$$\delta(\sqrt{-g}\,\Box) \equiv \delta(\partial_\mu \sqrt{-g}\, g^{\mu\nu}\partial_\nu) = 0\ , \qquad \delta(\sqrt{-g}\, R) = -2\sqrt{-g}\,\Box\sigma \tag{3.9}$$

[1]Recall that, in a general geometry, flat space conformal invariance is promoted to Weyl invariance, where Weyl transformation are given by $\delta g_{\mu\nu} = 2\sigma(x) g_{\mu\nu}(x)$, $\delta\phi(x) = \alpha\sigma(x)\phi(x)$ and α indicates the space-time dimension of the matter field in question; in 2D a scalar field has $\alpha = 0$, while a spinor has $\alpha = -1/2$, etc.

we find that

$$\delta W[g]/\delta\sigma(x) \sim \sqrt{-g}\,R(x) \equiv \mathcal{A}(x)\ , \tag{3.10}$$

does not vanish. A corollary is that the anomaly $\mathcal{A}(x)$, being the variational derivative of an action, must obey the reciprocity relation

$$\delta\mathcal{A}(x)/\delta\sigma(x') = \delta\mathcal{A}(x')/\delta\sigma(x)\ , \tag{3.11}$$

which it does, since $\delta\mathcal{A}/\delta\sigma' = \Box\delta^2(x-x') = \Box'\delta^2(x-x) = \delta\mathcal{A}'/\delta\sigma$.

The second observation about this effective action is that it contains a single pole; this means in our context an excitation of the (hereby induced) gravitational field. We can see this by Polyakov's observation that in the conformal gauge, $g_{\mu\nu} = e^{2\phi(x)}\eta_{\mu\nu}$ (always locally reachable in 2D) $W[g]$ reduces to $\int d^2x\,\phi\Box\phi$, *i.e.*, that its (single) Euler–Lagrange equation is $2\Box\phi = -R = 0$. This 2D characteristic has general validity (see Appendix A). We can also notice that the self-interacting W form comes from integrating out the scalar field ϕ in

$$W = \int d^2x\,\sqrt{-g}\,R\phi + \int d^2x\sqrt{-g}\,\phi\Box\phi\ . \tag{3.12}$$

This is also the Wess–Zumino form of the action: suppose we introduce a "Weyl-compensator" field ϕ which varies as $\delta\phi = \sigma(x)$. Then the first term gives the desired anomaly $\sqrt{-g}\,R$ when we vary ϕ. However there is also the extra contribution from varying $\sqrt{-g}\,R$ which yields $-2\sqrt{-g}\,\Box\phi$; the second term in (3.12) precisely cancels this unwanted piece (recall that $\delta\sqrt{-g}\,\Box = 0$!). The form (3.12) is also obtainable by taking the action (3.8) and subtracting from it its Weyl-invariantized version $W[g_{\mu\nu}e^{-2\phi}]$. The expansion in ϕ is just (3.12).

Two final geometric remarks that will be relevant later: the first is that there is but one anomaly term possible because the integrability condition (3.11) has only one solution with the desired dimensionality – *i.e.*, with a local scalar density $\mathcal{A}(x)$ that is itself scale invariant. The second is that we would have obtained the correct W, *i.e.*, the correct "numerator" in W by using the fact that in 2D any quantity antisymmetric in more than 2 indices vanishes, *e.g.*, any $A^{\alpha\nu}_{[\alpha\beta}A^{\alpha\beta}_{\mu\nu]} \equiv 0$ where $\mathcal{A}_{[\mu\nu][\alpha\beta]}$ has the algebraic symmetries of the Riemann tensor, antisymmetric in each pair and symmetric under their interchange; that is in fact the useful way [1] to understand higher-dimensional W's.

This pedagogical survey has been intended to illuminate the more complicated D=4 and higher situations below; consequently we skip entirely the subjects of conformal field theory and strings which have the conformal anomaly as a base (even any reasonable list of references would swamp our text).

4 Four Dimensions

Let us summarize the lessons from D=2: any matter system will lead to an effective gravitational action through its coupling to external geometry at the one-loop level (for free fields, there are of course no higher loops!). If, in particular the matter is classically Weyl-invariant, then the process of regularization necessarily leads to an effective action that is not both diffeo- and Weyl- invariant, the anomalous part, $\mathcal{A}(x) = \delta W[ge^{2\sigma(x)}]/\delta\sigma(x)|_{\sigma=0}$, being finite and local, although the effective action is non-local, with the characteristic single denominator $\Box^{-1}$ inherited from the loop integral. In addition it was possible to give the full nonlinear form (3.8) of this action, representing the single possible anomaly, the Euler density $E_2 \equiv \sqrt{-g}\,R$. This action was furthermore unique; no other conformal-invariant functional exists at D=2. Now we turn to 4D and higher even dimensions (anomalies can only occur at even dimensions, as can be already understood from the simple fact that no local scalar density can exist in odd dimensions that is even constant scale invariant).

Let us begin backwards, and ask for a list of candidate anomalies, that is scalar densities $\mathcal{A}(x)$ that are local, expandable in $h_{\mu\nu}$ (since they can be obtained perturbatively), scale invariant and obey the integrability condition (3.11) that permits an effective action (whose existence is also perturbatively guaranteed). The list here consists of the three independent ways to square a curvature, most usefully the combinations

$$E_4 = \sqrt{-g}\,(R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2)\ , \quad ; \sqrt{-g}\,C^2\ , \quad \sqrt{-g}\,R^2 \tag{4.1}$$

where C is the Weyl tensor and E_4 is the Gauss–Bonnet topological density that generalizes the Euler density E_2. In addition, there is what we shall see is a trivial candidate

$$a = \sqrt{-g}\,\Box R \tag{4.2}$$

and also the Hirzebruch density $R_{\mu\nu\alpha\beta}{}^{*}\!R^{\mu\nu\alpha\beta}$, a parity-odd topological quantity that we will not discuss further here except to mention that it is itself Weyl invariant (by the cyclic identity $R_{[abc]d} = 0$), just like $\sqrt{-g}\,C^2$. The Weyl variation of E_4 embodies reciprocity since

$$\delta\, E_4(x)/\delta\sigma(x') = G^{\mu\nu}(x) D_\mu D_\nu \delta(x-x') = G^{\mu\nu}(x') D'_\mu D'_\nu \delta(x-x') = \delta E_4(x')/\delta\sigma(x) \tag{4.3}$$

owing to the identical conservation of $G^{\mu\nu}$. In any dimension $D = 2n$, E_{2n} being a total divergence, will behave like E_4 with $G^{\mu\nu}$ replaced by a higher order identically conserved tensor (which, like $G^{\mu\nu}$, vanishes identically in all lower n!), so that the Euler density is always a legal candidate. [The proof is simple: in components

$$E_{2n} \sim \mathcal{E}^{\mu_1\ldots\mu_{2n}}\mathcal{E}^{\nu_1\ldots\nu_{2n}} R_{\mu_1\mu_2\nu_1\nu_2}\ldots R_{\mu_{2n-1}\mu_{2n}\nu_{2n-1}\nu_{2n}}$$

with Weyl variation of $R_{\mu\nu\alpha\beta} \sim g_{\mu\alpha}D_\nu D_\beta\sigma$ + cyclic, so $\delta E_{2n} \sim G^{\nu\beta}D_\nu D_\beta\,\sigma$. It is easy to see that $G^{\nu\beta}$ is the metric variation of $I = \int g_{\nu\beta}G^{\nu\beta}d^{2n}x$, so it is identically conserved, as is also checked directly using the Riemann tensor's Bianchi identities.] This unique term we have called type A. Likewise, at all higher dimensions, there will be appropriate generalizations $\sqrt{-g}\,C_1\ldots C_n$ of $\sqrt{-g}\,CC$ at n=2; these are called type B, and they clearly increase in number with dimension since there are (at the very least) more independent ways to contract indices among the greater number of Weyl tensors. Finally, the $\sqrt{-g}\,R^2$ term in (4.1) is forbidden: it fails the integrability test – obviously there is only one identically conserved tensor linear in curvature, namely the Einstein tensor, in 4D. The term $a(x)$ of (4.2) is integrable, but irrelevant because it stems from a purely <u>local</u> action such as $\int d^4x\sqrt{-g}\,R^2$, a form that is in any case needed as a counterterm to the well-known "two-point" infinity (rather than the "three-point" nonlocal anomaly). This pattern persists for all $D = 2n$:

there is either the single, type A, E_{2n} term corresponding to the single conserved "$G^{\mu\nu}$" tensor of rank (n-1) (that is the "Einstein tensor" of the action $\int d^{2n}x\sqrt{-g}\,E_{(n-2)}$) or the increasingly large type B set of Weyl invariants $\sqrt{-g}\,C^n$ just discussed, in addition to local anomalies. [There is very nice agreement, incidentally, between the present analysis and cohomology arguments such as those of [6] and references therein.] The specific coefficients of the various anomaly terms have been tabulated for all massless free fields (in D=2, 4 at least), in terms of their spin content, but there is more, for interacting systems, that involves their β-functions. Here I only have space to sketch the effective action problem and some attractive, but alas invalid, closed form solutions of it.

Let us first dispose of the action problem for type B: what $W[g]$ gives $\sqrt{-g}\,C^2$ upon Weyl variation? Clearly, since $\sqrt{-g}\,C^2$ is itself inert, we want some thing of the form $W_B \sim \int d^4x\sqrt{-g}\,C^2X$ where X is a scalar that varies as $\sigma(x)$, to linear order, say. The only such diffeo-invariant candidates (*i.e.*, scalars) are (to linearized order in $h_{\mu\nu}$) $R/\Box$ and $\ln\Box$; they have profoundly different origins in terms of scale dependence, and the (only) correct choice is [1] $\ln\Box$. Indeed, this was the first nonlocal anomaly to be discovered in D=4, and the correct W_B was already given there; nevertheless the wrong choice has often cropped up since. The definition of the nonlocal $\ln\Box$ (more correctly $\ln\Box/\mu^2$, where μ is a regularization scale) is straightforward and, to the operative cubic order in $h_{\mu\nu}$, its location in the integral may simply be taken "between" the two Weyl tensors. The closed form extension of this action is not known explicitly, but must exist.

In type A, we need a $W_A[g]$ with a single pole; to lowest (cubic) order in $h_{\mu\nu}$, it has the somewhat inelegant form [1]

$$W_{3,d=4} = \int d^4x\sqrt{-g}\,\Box^{-1}\left[\frac{1}{2}\,R^2_{\mu\nu\alpha\beta}R + 10R_{\mu\nu}R^{\nu\alpha}R^{\mu}_{\alpha} - 13R^2_{\mu\nu}\,R + \frac{41}{18}\,R^3 + 6R_{\mu\nu\alpha\beta}R^{\mu\alpha}R^{\nu\beta}\right] \tag{4.4}$$

which, however, derives in a direct way from the "0/0" ideas of D=2 by using the $\epsilon \to 0$ limit of the cubic form $C^{\mu\nu}_{[\mu\nu}C^{\alpha\beta}_{\alpha\beta}C^{\lambda\sigma}_{\lambda\sigma]}$ together with a "floating" $\Box^{-1}$ that for present purposes can be between any two of the factors. It is essential to a correct prescription that it be consistent with the known field-theoretic rules for anomalies, as well as with, of course, the Ward identities; this is fulfilled by (4.4). [Indeed, to get E_4 from varying (4.4) required frequent use of this apparatus!] To date, we have been unable to find a closed form, however. There does exist a very elegant closed form expression whose Weyl variation simply yields E_4; the only problem is that is is wrong, *i.e.*, it cannot arise from a loop integral. The form in question rests on a simple analogy with the Polyakov form (3.8) in 2D; there, $\delta_\sigma\,E_2 = (\sqrt{-g}\,\Box)\sigma$ while $\delta_\sigma(\sqrt{-g}\,\Box) = 0$, which immediately justifies (3.8). Can this be promoted to $D = 4$? First, it is clear that the analog of $\sqrt{-g}\,\Box$ must be something like $\sqrt{-g}\,\Box^2$ in order to be even constant scale invariant (let alone Weyl invariant). Indeed, the correct operator is

$$\Delta = \Box^2 + 2D_\mu(R^{\mu\nu} - \tfrac{1}{3}\,g^{\mu\nu}R)D_\nu \tag{4.5}$$

a fairly simple (self-adjoint) generalization. [At 6D and beyond such a $\Delta \sim \Box^n + \ldots$ also exists but is no longer unique [2]]. Likewise, while E_4 does not quite vary correctly, the quantity

$$\bar{E}_4 \equiv (E_4 - \tfrac{2}{3}\,\Box R)\ , \quad \delta\bar{E}_4 = \sqrt{-g}\,\Delta\sigma \tag{4.6}$$

does, *i.e.*,

$$\delta\left(\int \bar{E}_4\Delta(\sqrt{-g}\,\Delta)^{-1}\bar{E}_4\right)/\delta\sigma(x) = \bar{E}_4(x)\,. \tag{4.7}$$

This differs from the desired E_4 by a local anomaly $\frac{2}{3}\,\Box R$, which means that the action is rather the one varied below:

$$\delta\left(\frac{1}{2}\int \bar{E}_4(\sqrt{-g}\,\Delta)^{-1}\bar{E}_4 - \frac{1}{18}\int R^2\right) = E_4\,. \tag{4.8}$$

The trouble with this form, however, is that it has a double pole, already to order h^3, and hence is not viable. Likewise, the form $\int \bar{E}_4(\sqrt{-g}\,\Delta)^{-1}\sqrt{-g}\,C^2$, whose variation gives $\sqrt{-g}\,, C^2$, has the wrong scale dependence. In fact its lowest (cubic) part is just $\sim \int C^2\,R/\Box$ which is incorrect. What is also interesting is that the "bad" type A form (4.8) is equivalent to the Wess–Zumino (WZ) expression that yields E_4, so that here too the 2D reasoning fails. Rather than give the mechanism behind the general WZ construction that "mechanically" yields the desired action, the result is sufficiently simple that we can reach it iteratively. We start by introducing the Weyl compensator field X, $\delta X = \sigma(x)$ and with the obvious zeroth ansatz

$$W_0 = \int d^4x\, E_4 X\,. \tag{4.9}$$

We must, however, compensate for the fact that the Weyl variation of E_4 gives the unwanted contribution $G^{\mu\nu}D_\mu D_\nu X$ to $\mathcal{A}(x)$ by adding

$$W_1 = \frac{1}{2}\int\int d^4x\, G^{\mu\nu}D_\mu X\, D_\nu X\,. \tag{4.10}$$

Now, however, we get an unwanted contribution from $\delta G^{\mu\nu} \sim (D^\mu D^\nu - g^{\mu\nu}\Box)(D_\mu X D_\nu X)$, requiring a cubic term $W_2 \sim (D_\mu X)^2\Box X$; then a quartic term $W_3 \sim \int d^4x\sqrt{g}\,(D_\mu X)^2(D_\nu X)^2$ is needed to cancel the contribution from $\delta(\Box)$ in W_2. The full closed form WZ action is then the appropriate sum,

$$W_{\rm WZ} = \int\int d^4x\left\{E_4 X + aG_{\mu\nu}D_\mu X\, D_\nu X + b\Box X(D_\mu X)^2 + c[(D_\mu X)^2]^2\right\}\,. \tag{4.11}$$

Unlike its 2D counterpart, however, this form is neither Gaussian, nor does it even have a kinetic term $\sim \int X\Box X$ at all, so we cannot go from it to a closed form, and setting $X \sim R/\Box$ as a lowest approximation introduces unacceptable $\Box^{-2}$ terms. This is not surprising, because one can show that this $W_{\rm WZ}$ (4.11) is closely related to the "Polyakov" expression (4.8) and its W. Indeed, one can show that (4.11) is just

$$W_{\rm WZ} = W[g] - W[ge^{-2X}]\,. \tag{4.12}$$

The last term on the right side being manifestly Weyl invariant, the two clearly yield the same anomaly.

At this point, then, we have two different D=4 actions, to leading (cubic) order about flat space for both type A and B anomalies, but only one correctly reproduces the underlying loop physics. In 2D, where there was only type A, this action was furthermore unique; no ΔW can be constructed that is Weyl invariant. Thus, knowledge of the anomaly determined the whole effective gravitational action there (see Appendix A). [Of course, a less impressive way to say this is that since the general 2D metric is conformally flat, only the $\delta W/\delta\sigma(x)$ is relevant anyway!] Is there similar uniqueness in $D \geq 4$, *i.e.*, do we expect that knowledge of the anomaly also determines the effective action here? From the above parenthetic remark, we should expect a negative answer. Indeed, let us show how to construct at least type A-like (with $\Box^{-1}$ behavior) ΔW's that are Weyl invariant to the same, lowest, order as that of our W of (4.4) itself. The idea is very simple. Consider in 4D the local cubic Weyl invariants, which are in fact the known type B anomalies in 6D. Although there are 2 such invariants in 6D (namely the apparently different ways of tracing the product of 3 Weyl tensors), they are equivalent in 4D owing to the identity $C^{\mu\nu}_{[\mu\nu}\, C^{\alpha\beta}_{\alpha\beta}\, C^{\lambda\sigma}_{\lambda\alpha]} \equiv 0$ here (6 indices are antisymmetrized). Thus the action $\Delta W = \int d^4x\, \mathrm{tr} C^3/\Box$ is clearly Weyl invariant to lowest order in $h_{\mu\nu}$. [The reason one cannot use the same idea in 2D, with C^2 as the invariant is of course that the Weyl tensor vanishes identically here.] Whether these leading order Weyl invariants really persist to all orders is not immediately clear, though there is no reason to doubt it (the overall scale invariance is formally preserved by $\int d^4x\sqrt{-g}\, C^3/\Box$ for example).[2] On the other hand, there seems to be no ambiguity in the type B actions, involving the $\ln \Box$ factors.

That there is room for ambiguity does not of course mean that it is always present; indeed a very recent interesting paper [7] on 4D conformal systems (of a very special type) derived the gravitational action uniquely from the anomaly. However, this uniqueness is probably related to the higher symmetry (Kähler structure) of the 4-manifold there. In any case what is really important is whether the coefficients of the type A effective action can be exploited as in 2D CFT to relate different conformal systems.

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References

[1] S. Deser and A. Schwimmer, Phys. Lett. **B309** (1993) 279.

[2] D.R. Karakhanyan, R.P. Manvelyan, and R.L. Mkrtchyan, Yerevan preprint (1996).

[3] H. Osborn and A. Petkou, Ann. Phys. **231** (1994) 311; J. Erdmenger and H. Osborn, hep-th 9605009 preprint.

[4] L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger, hep-th 9607110 preprint.

[2] Another possible set of ambiguities derive from a different set of 6D integral Weyl invariants that begins with terms like $C\Box C$ plus cubic curvatures. There are two such, given in [2]. [One of them is found in (25c) of [1], but two corrections must be made there: the relative sign of the $C\Box C$ and of the remaining terms is wrong and only the integral, not the density itself is Weyl invariant.] It is conceivable that the 4D integral of these quantities divided by the $\Box$ operator may also have an invariance to cubic order; the quadratic part from $C\Box C$ for example reduces to the (irrelevant)local invariant $\int d^4x\,\sqrt{-g}C^2$.

[5] M.J. Duff, Class. Quant. Grav. **11** (1994) 1387.

[6] L. Bonora, P. Pasti and M. Bregola, Class. Quant. Grav. **3** (1986) 635.

[7] A. Losev, G. Moore, N. Nekrasov, and S. Shatashvili, hep-th 9606082 preprint.

Appendix A

Varying 2D gravitational actions

Strictly speaking, one cannot first fix a gauge in an action, and then deduce the field equations by varying the remaining field components in a gauge theory; one would then in general miss the constraints, such as the Gauss law (fixing $A_0 = 0$) or the Hamiltonian constraints in Einstein theory (fixing $g_{0\mu} = \eta_{0\mu}$). However, it is intuitively clear that since there is only one independent metric component in 2D, it must be an exception and that the three field equations $\delta I/\delta g_{\mu\nu} = 0$ reduce to only one "real" one and two "Bianchi identities". For orientation, consider first the Polyakov action in linearized approximation. In terms of the variables in a 1+1 decomposition, $h \equiv h_{11}$, $N = h_{00}$, $L \equiv h_{01}$, the linearized curvature is

$$R^L \equiv \partial^2_{\alpha\beta} h^{\alpha\beta} - \Box h^\alpha_\alpha = (h'' - 2\dot{L}' + \ddot{N}) - (h-N)'' + (\ddot{h} - \ddot{N}) = \ddot{h} + N'' - 2\dot{L}'$$

and consequently varying either the constraint variables (N, L) or the "dynamical" h will give the same $\Box^{-1} R^L = 0$ equation, which implies $R^L = 0$. The content of this equation is of course most obvious in conformal gauge, $h_{\mu\nu} = \phi\, \eta_{\mu\nu}$, where $R^L(\phi\, \eta_{\mu\nu}) \equiv -\Box\phi$. Because there is only one independent equation here, it would naturally also have been found by immediately fixing the gauge in $\int\int R^L \Box^{-1} R^L \to \int d^2x\, \phi \Box \phi$ and varying to find $\Box\phi = -R = 0$. In covariant form, varying (3.8) gives $(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)(R/\Box) = 0$, whose trace part is indeed $\sim \Box(R/\Box) = 0$. The remaining two components are automatically satisfied by $R = 0$; their separate content is that $\partial_t \partial_x (R/\Box) = 0 = (\partial_t^2 + \partial_x^2)(R/\Box)$, which is only formally (a bit) stronger than $\Box(R/\Box) = 0$. In the nonlinear case, this is less evident because the $(\sqrt{-g}\,\Box)^{-1}$ factor actually depends on different combinations of the metric, but the result nevertheless is valid: To justify it, let us vary the full Polyakov action (3.8) under all $\delta g_{\mu\nu}$. The variation of the $(\sqrt{g}\,\Box)^{-1}$ factor yields a term $\Delta_{\mu\nu}$ that is identically traceless, since $\sqrt{g}\,\Box$ depends only on the unimodular combination, $\sqrt{-g}\, g^{\mu\nu}$. More specifically, $\Delta_{\mu\nu}$ has the form of the usual scalar field's stress tensor with $\phi \sim R/\Box$. The variation of $\sqrt{-g}\, R$ yields the form $(D_\mu D_\nu - g_{\mu\nu}\Box)(R/\Box)$, whose trace is just R itself, as expected. Hence the trace of the full field equation already implies R=0, thereby automatically fulfilling the other two (traceless) components of the equations, modulo the formal point just made for the linear case.

The above result, that conformal variation gives all the information is less obvious for other 2D actions, even for the local $I = \int d^2x \sqrt{-g}\, R^2$. Its Euler equations, using $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$, are $\mathcal{G}^{\mu\nu} \equiv (-D_\mu D_\nu + g_{\mu\nu}\Box)R + \frac{1}{2} g_{\mu\nu} R^2 = 0$ which obey the Bianchi identities $D_\nu \mathcal{G}^{\mu\nu} \equiv 0$. The trace equation is $\Box R + R^2 = 0$, which seems to be weaker than the tensorial $\mathcal{G}_{\mu\nu} = 0$; but we know f rom our linearized discussion that they really are not. In conformal gauge, of course, $I = 4 \int d^2x e^{-2\phi} (\Box\phi)^2$, leading to the single equation for ϕ that represents the above trace.

Appendix B

Type B and β functions

As a novel example of the relation between the type B anomaly and β-functions, which also brings out the role of a scale in type B, and how the "invariance clash" is seen at a simple diagrammatic l evel, we take 4D self-interacting ϕ^4 theory, which is of course classically scale invariant. The relevant (Fourier transformed) correlators

$$L_{\mu\nu}(q;k,p) \equiv \langle T(T_{\mu\nu}(q)\phi^2(k_1)\phi^2(k_2))\rangle \qquad K(k) = \langle T(\phi^2(k)\phi^2(-k))\rangle$$

are not purely among stress tensors, but instead represent the triangle with one graviton and two ϕ^4 corners, and the pure scalar 2-point loop respectively. [There is also a contact term where a graviton emerges from one of the 2-point loop's ends, but that can be redefined into $L_{\mu\nu}$ by appropriate subtraction.] There are then two separate Ward identities representing (linearized) Weyl and coordinate invariance, and they cannot both be maintained – one signal is that K is logarithmically divergent and hence requires introduction of a scale: $K(k^2) = \ln k^2/\mu^2$. Decomposing $L_{\mu\nu}$ into invariant amplitudes after Fourier transforming and expanding in tensorial combinations of the two external momenta, one finds three relations among the four independent amplitudes, and that the UV divergences embodied by the cutoff in K cancel, because only $K(k_2^2) - K(k_1^2)$ enters. The invariance clash is best seen by going to a special point in momentum space where $(k_1 + k_2)^2 = 0$, $k_1^2 = k_2^2$. There, one discovers that the same structure function is simultaneously constrained by the respective Ward identities both to vanish and to be proportional to $k^2 dK/dk^2$. Explicit diagrammatic calculations confirm that K indeed has a δ-function discontinuity. Choosing to preserve conservation, then, has resulted in the conclusion that T^μ_μ is proportional to $\lambda\phi^4$, *i.e.*, to the beta function of the theory. Note that this whole calculation of the beta function has been entirely in the infrared domain and does not involve UV properties. Similar results have also been found very recently in the second paper of [3].