Uniqueness of stationary, electro–vacuum black holes revisited

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Abstract. In recent years there has been some progress in the understanding of the global structure of stationary black hole space–times. In this paper we review some new results concerning the structure of stationary black hole space–times. In particular we prove a corrected version of the "black hole rigidity theorem", and we prove a uniqueness theorem for static black holes with degenerate connected horizons.

1   Introduction

The theory of black hole space–times seems to be one of the most elegant chapters of classical general relativity, that makes use of a whole range of techniques: global causality theory, the theory of partial differential equations, various aspects of Riemannian and Lorentzian geometry, etc. The foundations of the theory have been laid by Israel [1], Carter [2], Hawking [3, 4], and our knowledge of those space–times has been considerably expanded through the work in [5, 6, 7, 8] and others. A recent treatise which covers in detail various aspects of the theory is [9]. In spite of the considerable amount of work involved, several questions still remain to be investigated (*cf.* [10] for a recent review). In the last few years some of the open problems raised in that last reference have been solved, and in this review we shall shortly discuss some of those developments. I will also take this opportunity to give a complete proof of a corrected version of the so–called "rigidity theorem" for stationary black holes, as well as some steps of the proof of a uniqueness theorem for degenerate Reissner–Nordström black holes. We shall mostly concentrate on electro–vacuum space–times; a discussion of

various results on some other models can be found in [9, 11, 12]. The reader is also referred to [13] for an interesting new result concerning spherically symmetric "hairy" black holes.

Let us start by recalling that a space–time[1] is called *stationary* if there exists a Killing vector field X which approaches $\partial/\partial t$ in the asymptotically flat region *and* generates a one parameter groups of isometries (for the purposes of this paper space–times will be assumed to be asymptotically flat.) A space–time is called *static* if it is stationary and if there exists an isometry which changes time orientation. A space–time is called *axisymmetric* if there exists a Killing vector field Y which behaves like a *rotation* in the asymptotically flat region *and* generates a one parameter group of isometries, in the following (standard) sense: in an asymptotically Minkowskian coordinate system the partial derivatives of the Killing vector field asymptote to those of a rotational Killing vector field of Minkowski space–time, and all orbits of Y are, say, 2π periodic. It can be shown that this implies that there exists an axis of symmetry, that is, a set on which the Killing vector vanishes (*cf. e.g.* [14, Prop. 2.4]). That last property is often imposed as a part of the definition of axisymmetry.

It is worthwhile emphasizing that the above notions include the property that the orbits of the relevant Killing vectors are *complete*. This is a non–trivial requirement which plays an important role in the theory, and several steps of the proofs are wrong without that hypothesis. Hence, the question of completeness of the orbits, which we will discuss shortly in Section 4 below, is not only a question of mathematical purity, but is critical in several considerations.

It is widely expected that the Kerr–Newman black holes, together perhaps with the Majumdar–Papapetrou ones, exhaust the set of appropriately regular stationary electro–vacuum black holes. (An important question is, what does one mean by "appropriately regular".) This expectation is based on the following facts, which have been proved under various restrictive hypotheses, some of which we shall present in detail below:

1. The "rigidity theorem": non–degenerate[2] stationary analytic electro–vacuum black holes are either *static*, or *axisymmetric (cf.* Theorem 5.1 below).

2. The Reissner–Nordström non–degenerate black holes exhaust the family of *static* non–degenerate electro–vacuum black holes.

3. The non–degenerate Kerr–Newman black holes exhaust the family of non–degenerate *stationary–axisymmetric* electro–vacuum black holes (*cf.* Theorem 8.1 below).

Clearly, the rigidity theorem plays a key role here, reducing the classification problem of stationary black holes to that of classification of the static and of the axisymmetric ones. (More precisely, that theorem would play a key role if the analyticity condition there were

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[1] Throughout this paper the term space–time denotes a smooth, paracompact, connected, orientable and time–orientable Lorentzian manifold.

[2] The condition of non–degeneracy is only needed here to be able to infer staticity, when the Killing vector field that asymptotes to $\partial/\partial t$ is tangent to the event horizon.

removed.) It turns out that this theorem is wrong as stated in [4]: in [15] a space–time has been constructed which satisfies all the hypotheses spelled–out in [4] and which is *not* axisymmetric. The construction consists in gluing together, in an appropriate way, two copies of the Kerr space–time, so that the isometry group of the resulting space–time is only ℝ: while there are still two linearly independent Killing vector fields globally defined, only one of them has complete orbits. This can be considered as a rather mild counter–example, but its existence shows that there is a potentially dangerous error in the proof of the theorem. We shall show in Section 5 below that some form of rigidity, weaker than that claimed in [4], can be obtained. Let us start by presenting some recent results which will be used in the proof.

2 The topology of black hole space–times

One of the steps of the proof of the rigidity theorem (both in [4] and below) consists in proving that connected components of black–hole event horizons must have $\mathbb{R} \times S^2$ topology. Moreover one also needs a simply–connected domain of outer communications. Both claims are insufficiently justified in [4]. The first complete proof of that result has been given in [16]. The results of that reference have been generalized by Galloway, as follows: Consider a space–time which is asymptotically flat at null infinity in the sense of the definition given on p. 282 of [17]. We shall say that $\mathscr{J}$ satisfies *the regularity condition* if

1. There exists a neighborhood $\widehat{\mathcal{O}}$ of $\mathscr{J}^+ \cup \mathscr{J}^-$ which is simply connected.

2. There exists a neighborhood $\mathcal{O} \subset \widehat{\mathcal{O}}$ of $\mathscr{J}^-$ such that for every compact set $K \subset \mathcal{O} \backslash \mathscr{J}^-$, the set $\partial J^+(K; \bar{M}) \cap \mathscr{J}^+$ is not empty.

We have the following result of Galloway, that does *not* assume stationarity[18]:

Theorem 2.1 (G. Galloway, 96) *Consider a space–time which is asymptotically flat at null infinity and which has a $\mathscr{J}$ which satisfies the regularity[3] condition. Suppose moreover that the Ricci tensor satisfies the null convergence condition:*

$$R_{\mu\nu}X^\mu X^\nu \geq 0 \quad \text{for all null vectors } X^\mu. \tag{2.1}$$

Then every globally hyperbolic domain of outer communications is simply connected.

The hypothesis of global hyperbolicity of the d.o.c. above is rather reasonable; it should be pointed out that it necessarily holds if the space–time itself is globally hyperbolic. All

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[3]The regularity condition of $\mathscr{J}$ can be replaced by other conditions, *cf. e.g.* [19]. While the regularity condition given here is not spelled out in detail in [20] and in [19], some form of causal regularity of $\mathscr{J}$ is needed for the arguments given there to go through. I am grateful to G. Galloway for discussions concerning this point.

the above mentioned results are based on the *topological censorship theorem* of Friedman, Schleich and Witt [20]. Further related results can be found in [21, 22, 23, 19].

To be able to apply this result to stationary black hole space–times we need to verify that the regularity condition of $\mathscr{J}$ is satisfied. Recall, first, that there are two ways of defining the domain of outer communications for a stationary space–time, the one given in [24], and the standard one using $\mathscr{J}$ (*cf. e.g.* [4, 17]). (Those definitions are equivalent for, say, electro–vacuum stationary space–times, but the former seems to be more convenient for many purposes.) The approach of [24] proceeds as follows: A space–like hypersurface Σ_{ext} diffeomorphic to $\mathbb{R}^3$ minus a ball will be said asymptotically flat if the fields (g_{ij}, K_{ij}) induced on Σ_{ext} by the space–time metric satisfy the fall–off conditions

$$|g_{ij} - \delta_{ij}| + r|\partial_\ell g_{ij}| + \cdots + r^k|\partial_{\ell_1 \cdots \ell_k} g_{ij}| + r|K_{ij}| + \cdots + r^k|\partial_{\ell_1 \cdots \ell_{k-1}} K_{ij}| \leq Cr^{-\alpha} , \quad (2.2)$$

for some constants C, $\alpha > 0$. An asymptotically flat hypersurface Σ (with or without boundary) will be said to have compact interior if Σ is of the form $\Sigma_{int} \cup \Sigma_{\text{ext}}$, with Σ_{int} compact. Let X be a Killing vector field which asymptotically approaches the unit normal to Σ_{ext}. Passing to a subset of Σ_{ext} we can without loss of generality assume that X is timelike on Σ_{ext}. If ϕ_t denotes the one parameter group of isometries generated by X, then an exterior four–dimensional asymptotically flat region can be obtained by moving Σ_{ext} around with the isometries;

$$M_{\text{ext}} = \cup_{t \in \mathbb{R}} \phi_t(\Sigma_{\text{ext}}) . \quad (2.3)$$

The *domain of outer communications* (d.o.c.) associated with Σ_{ext} or with M_{ext} is then defined as

$$\langle\langle M_{\text{ext}} \rangle\rangle = J^+(M_{\text{ext}}) \cap J^-(M_{\text{ext}}) . \quad (2.4)$$

The black–hole event horizon $\mathcal{B}$ associated with the asymptotic end Σ_{ext} or with M_{ext} is defined as

$$\mathcal{B} = M \setminus J^-(M_{\text{ext}}) . \quad (2.5)$$

For electro–vacuum space–times one can now attach to M_{ext} a null conformal boundary using the prescription given in [25, Appendix] — the conditions needed for that construction are satisfied by [26]. It is straightforward to check that the regularity condition is satisfied by this completion (with $\mathcal{O} = M_{\text{ext}} \cup \mathscr{J}^-$, $\widehat{\mathcal{O}} = \mathcal{O} \cup \mathscr{J}^+$).

The spherical topology theorem is essentially a Corollary of Theorem 2.1. More precisely, let us consider a space–like hypersurface Σ which has a boundary on the event horizon. We wish to show that this boundary is a finite collection of spheres. Clearly some further hypotheses are needed for this result to hold. Let us start with some terminology. Consider a point p in the asymptotically flat region M_{ext} so that, in particular, X^a is timelike at p. Define $\mathcal{C} = \partial I^+(p; \langle\langle M_{\text{ext}} \rangle\rangle)$. Then $\mathcal{C}$ automatically is an achronal, Lipschitz hypersurface. We write $\mathcal{C}_{\text{ext}} = \mathcal{C} \cap M_{\text{ext}}$. The following has been proved in [16]:

Theorem 2.2 (P.C. & R. Wald, 94) *Let (M, g_{ab}) be a stationary spacetime containing a single asymptotically flat region, whose domain of outer communications, $\langle\langle M_{\text{ext}} \rangle\rangle$, is globally hyperbolic. Suppose that the null energy condition (2.1) holds. Assume that there exists an*

achronal, asymptotically flat slice, $\mathcal{S}$, of $\langle\langle M_{\text{ext}}\rangle\rangle$, whose boundary in M intersects the event horizon, $\mathcal{H}$, of any black holes in M in a compact cross–section, K. If $C \setminus C_{\text{ext}}$ has compact closure in M (where C and C_{ext} were defined above), then each connected component of K is homeomorphic to a sphere.

Theorem 2.2 above assumes that (M, g_{ab}) contains only one asymptotically flat region. This has been done purely for notational convenience, as we have the following consequence of [20] (*cf.* [16, Prop. 1]):

Proposition 2.3 *Let (M, g_{ab}) be a globally hyperbolic space–time with Cauchy surface Σ and with a one parameter group of isometries, ϕ_t, generated by a Killing vector field X^a. Suppose that Σ contains a (possibly infinite) number of asymptotic regions Σ_i, in which X^a is timelike and tends asymptotically to a non–zero multiple of the unit normal to Σ as the distance away from some fixed point $p \in \Sigma$ tends to infinity. Consider an asymptotically flat three–end Σ_i, and let $\mathcal{B}_i$ and $\mathcal{W}_i$ be the black– and white–hole regions with respect to Σ_i as defined above. Consider the domain of outer communications $\langle\langle M_i\rangle\rangle$ (defined as in (2.3)-(2.4), using Σ_i) If the null energy condition (2.1) holds, then*

$$M_i \cap J^{\pm}(\langle\langle M_j\rangle\rangle) = \emptyset \quad for \quad i \neq j \,.$$

This result shows, in particular, that in the Lichnerowicz–type Theorem 2.7 of [10] the condition (2.11) there is unnecessary,

There is of course an equivalent result in the context of Galloway's Theorem 2.1, under appropriate conditions on $\mathscr{J}$.

3 The structure of isometry groups of asymptotically flat space–times.

A prerequisite for studying stationary space–times is the understanding of the structure of the isometry groups which can arise, together with their actions. A reasonable restriction which one may wish to impose, in addition to asymptotic flatness, is that of timelikeness of the ADM momentum of the space–times under consideration. As is well known, this property will hold in all space–times which are sufficiently regular and satisfy an energy condition (*cf.* [27, 28] for a list of references). The first complete proof, in the context of black hole space–times, has been recently given by [29]. For the theorem that follows we do not assume anything about the nature of the Killing vectors or of the matter present; it is therefore convenient to use a notion of asymptotic flatness which uses at the outset four–dimensional coordinates. A metric on Ω will be said to be *asymptotically flat* if there exist $\alpha > 0$ and $k \geq 0$ such that

$$|g_{\mu\nu} - \eta_{\mu\nu}| + r|\partial_\alpha g_{\mu\nu}| + \cdots + r^k|\partial_{\alpha_1} \cdots \partial_{\alpha_k} g_{\mu\nu}| \leq Cr^{-\alpha} \tag{3.1}$$

for some constant C ($\eta_{\mu\nu}$ is the Minkowski metric). Ω will be called *a boost–type domain*, if

$$\Omega = \{(t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^3 : |\vec{x}| \geq R, |t| \leq \theta r + C\}, \tag{3.2}$$

for some constants $\theta > 0$ and $C \in \mathbb{R}$. Let ϕ_t denote the flow of a Killing vector field X. $(M, g_{\mu\nu})$ will be said to be *stationary–rotating* if the matrix of partial derivatives of X^μ asymptotically approaches a rotation matrix in Σ_{ext}, and if ϕ_t moreover satisfies

$$\phi_{2\pi}(x^\mu) = x^\mu + A^\mu + O(r^{-\epsilon}), \qquad \epsilon > 0$$

in the asymptotically flat end, where A^μ is a timelike vector of Minkowski space–time (in particular $A^\mu \neq 0$). One can think of $\partial/\partial\phi + a\partial/\partial t$, $a \neq 0$ as a model for the behavior involved. The interest of that definition stems from the following result, proved in [14]:

Theorem 3.1 (P.C. & R. Beig, 96) *Let $(M, g_{\mu\nu})$ be a space–time containing an asymptotically flat boost–type domain Ω, with time–like (non–vanishing) ADM four momentum p^μ, with fall–off exponent $\alpha > 1/2$ and differentiability index $k \geq 3$ (see eq. (3.1) below). We shall also assume that the hypersurface $\{t = 0\} \subset \Omega$ can be Lorentz transformed to a hypersurface in Ω which is asymptotically orthogonal to p^μ. Suppose moreover that the Einstein tensor $G_{\mu\nu}$ of $g_{\mu\nu}$ satisfies in Ω the fall–off condition*

$$G_{\mu\nu} = O(r^{-3-\epsilon}), \qquad \epsilon > 0. \tag{3.3}$$

Let G_0 denote the connected component of the group of all isometries of $(M, g_{\mu\nu})$. If G_0 is non–trivial, then one of the following holds:

1. $G_0 = \mathbb{R}$, *and $(M, g_{\mu\nu})$ is either stationary, or stationary–rotating.*

2. $G_0 = U(1)$, *and $(M, g_{\mu\nu})$ is axisymmetric.*

3. $G_0 = \mathbb{R} \times U(1)$, *and $(M, g_{\mu\nu})$ is stationary–axisymmetric.*

4. $G_0 = SO(3)$, *and $(M, g_{\mu\nu})$ is spherically symmetric.*

5. $G_0 = \mathbb{R} \times SO(3)$, *and $(M, g_{\mu\nu})$ is stationary–spherically symmetric.*

The reader should notice that Theorem 3.1 excludes boost–type Killing vectors (as well as various other behavior). This feature is specific to asymptotic flatness at spatial infinity; see [30] for a large class of vacuum space–times with boost symmetries which are asymptotically flat in light–like directions. The theorem is sharp, in the sense that the result is not true if p^μ is allowed to vanish or to be non–time–like.

We find it likely that there exist no electro–vacuum, asymptotically flat space–times which have no black hole region, which are stationary–rotating and for which $G_0 = \mathbb{R}$. A similar statement should be true for domains of outer communications of regular black hole space–times. It would be of interest to prove this result. Let us also point out that the Jacobi ellipsoids [31] provide a Newtonian example of solutions with a one dimensional group of symmetries with a "stationary–rotating" behavior.

Theorem 3.1 is used in the proof of Theorem 4.2 below.

4   Killing vectors vs. isometry groups

In general relativity there exist at least two ways for a solution to be symmetric: there might exist

1. a Killing vector field X on the space–time (M, g), or there might exist

2. an action of a (non–trivial) connected Lie group G on M by isometries.

Clearly 2 implies 1, *but 1 does not need to imply 2* (remove *e.g.* points from a space–time on which an action of G exists). In the uniqueness theory, as presented *e.g.* in [4, 32, 9, 33], one *always* assumes that an action of a group G on M exists. This is equivalent to the statement, that the orbits of all the (relevant) Killing vector fields are complete. In [34] and [10] completeness of orbits of Killing vectors was shown for vacuum and electro-vacuum space–times, under various conditions. The results obtained there are not completely satisfactory in the black hole context, as they do not cover degenerate black holes. Moreover, in the case of non–degenerate black holes, the theorems proved there assume that all the horizons contain their bifurcation surfaces, a condition which one may wish not to impose *a priori* in some situations (*cf.* the discussion in Section 6 below). The following result, which takes care of those problems and which does *not* assume any field equations, has been proved in [15]:

Theorem 4.1 *Consider a space–time (M, g_{ab}) with a Killing vector field X and suppose that M contains an asymptotically flat three–end Σ_{ext}, with X time–like in Σ_{ext}. (Here the metric is assumed to be twice differentiable, while asymptotic flatness is defined in the sense of eq. (3.1) with $\alpha > 0$ and $k \geq 0$.) Suppose that the orbits of X are complete through all points $p \in \Sigma_{\text{ext}}$. If $\langle\langle M_{\text{ext}} \rangle\rangle$ is globally hyperbolic, then the orbits of X through points $p \in \langle\langle M_{\text{ext}} \rangle\rangle$ are complete.*

In [15] a generalization of this result to stationary–rotating space–times has also been given. A theorem of Nomizu [35], together with Theorem 3.1 give the following result [15], which will be used in the proof of Theorem 5.1 below:

Theorem 4.2 *Consider an analytic space–time (M, g_{ab}) with a Killing vector field X with complete orbits. Suppose that M contains an asymptotically flat three–end Σ_{ext} with time-like ADM four–momentum, and with $X(p)$ time–like for $p \in \Sigma_{\text{ext}}$. (Here asymptotic flatness is defined in the sense of eq. (2.2) with $\alpha > 1/2$ and $k \geq 3$, together with eq. (3.3) .) Let $\langle\langle M_{ext} \rangle\rangle$ denote the domain of outer communications associated with Σ_{ext} as defined below; assume that $\langle\langle M_{\text{ext}} \rangle\rangle$ is globally hyperbolic and simply connected. If there exists a Killing vector field Y, which is not a constant multiple of X, defined on an open subset $\mathcal{O}$ of $\langle\langle M_{\text{ext}} \rangle\rangle$, then the isometry group of $\langle\langle M_{\text{ext}} \rangle\rangle$ (with the metric obtained from (M, g_{ab}) by restriction) contains $\mathbb{R} \times U(1)$.*

We emphasize that no field equations or energy inequalities are assumed above. Note that simple connectedness of the domain of outer communications necessarily holds when a positivity condition is imposed on the Einstein tensor of g_{ab}, as shown by Galloway's Theorem 2.1 above. Similarly the hypothesis of time–likeness of the ADM momentum will follow if one assumes existence of an appropriate space–like surface in (M, g_{ab}). It should be emphasized that no claims about isometries of $M \setminus \langle\langle M_{\text{ext}} \rangle\rangle$ (with the obvious metric) are made.

5   The rigidity theorem

Let us briefly recall the strategy used in [4] to prove the rigidity theorem: If the Killing vector field X is not tangent to the generators of the horizon, the authors of [4] argue that there exists another Killing vector Y defined in a neighborhood of the horizon. On p. 329 of [4] they next assert: "... One then extends the isometries by analytic continuation. ..." This last step does not work, the underlying reason being essentially that *maximal analytic extensions are not unique.* As we show below, Theorem 4.2 above takes care of this problem, at the price of a somewhat weaker conclusion.

It is conceivable that the set of hypotheses of [4] can be modified so that the desired conclusion, perhaps in the form of a statement concerning isometries of the d.o.c., can be achieved. It should, however, be stressed that the general setup of [4] does not allow Majumdar–Papapetrou black holes, and it is likely that no degenerate black holes can fit into this setup. On the other hand our formulation below is compatible with geometries of Majumdar–Papapetrou type. To be precise, we have the following theorem, the proof of which is a mixture of the arguments of [36] and [4], together with Theorem 4.2:

Theorem 5.1 *Consider an analytic stationary electro–vacuum space–time (M, g_{ab}) with Killing vector field X, and let $\langle\langle M_{\text{ext}} \rangle\rangle$ be an asymptotically flat globally hyperbolic domain of outer communications in M. (Here asymptotic flatness is understood in the sense of (2.2) with $\alpha > 0$ and $k \geq 1$, together with eq. (3.3).) Assume that there exists an achronal asymptotically flat slice S of $\langle\langle M_{\text{ext}} \rangle\rangle$, whose boundary in M intersects the event horizon $\mathcal{E}$ of the black hole region in a compact cross–section K. Suppose finally that there exists a connected component of $\mathcal{E} \cap J^{+}(M_{\text{ext}})$, say $\mathcal{E}_1$, which is an analytic submanifold of M. If X is not tangent to the generators of $\mathcal{E}_1$, then:*

1. *The isometry group of $\langle\langle M_{\text{ext}} \rangle\rangle$ contains $\mathbb{R} \times U(1)$.*

2. *The event horizon is a Killing horizon in the following sense: There exists a neighborhood of $\overline{\langle\langle M_{\text{ext}} \rangle\rangle}$ in $J^{+}(M_{\text{ext}})$ on which a Killing vector field Z, tangent to the generators of $\mathcal{E}_1$, is defined.*

Let us point out that the above result generalizes immediately to stationary–rotating space–times. The only supplementary hypothesis needed is that of asymptotic flatness with

$\alpha > 1/2$ and $k \geq 3$ in the sense of eq. (3.1) (in the stationary case that property follows from the fall–off hypotheses spelled–out above).

To prove Theorem 5.1, the following result will be needed:

Lemma 5.2 *Under the hypotheses of Theorem 5.1 there exists a neighborhood $\mathcal{O}$ of $\overline{\langle\langle M_{\mathrm{ext}} \rangle\rangle}$ in the space–time $(J^+(M_{\mathrm{ext}}), g|_{J^+(M_{\mathrm{ext}})})$ such that the following hold:*

1. *For all $p \in \mathcal{O}$ we have $X(p) \neq 0$.*

2. *There exists a a time function t on $\mathcal{O}$ such that the one parameter group of isometries ϕ_t generated by X acts on $\mathcal{O}$ by time translations:*

$$t(\phi_s(p)) = t(p) + s .$$

(In particular $\mathcal{O}$ is invariant under ϕ_t.)

Lemma 5.2 can be established by a straightforward (though somewhat lengthy) adaptation of the proofs in [24, Section 3] to the current situation. A simplifying strategy is to construct t in two steps, the first being the construction of t on $\overline{\langle\langle M_{\mathrm{ext}} \rangle\rangle} \cap J^+(M_{\mathrm{ext}})$.

We can now pass to the proof of Theorem 5.1. By Theorem 2.1 and Lemma 5.2 each connected component of K is diffeomorphic to a sphere (*cf.* the arguments in [16]). Let us choose one of those components, say K_1. There exists a neighborhood $\mathcal{U}$ of K_1 in $\mathcal{E}_1$ together with a coordinate system $(u, x^a = (\theta, \varphi))$ in which the tensor field $\hat{g}$ induced from g on $\mathcal{U}$ takes the form

$$\hat{g} = g_{ab} dx^a dx^b ,$$

with $\frac{\partial}{\partial u}$ – tangent to the generators of $\mathcal{E}_1$. It is easily seen that X is tangent to $\mathcal{E}$, so that on $\mathcal{U}$ the Killing vector field X can be written in the form

$$X^\mu \partial_\mu = X^0 \partial_0 + Z^a \partial_a .$$

From the Killing equation $\mathcal{L}_X g = 0$ we obtain

$$\frac{\partial Z^a}{\partial u} = 0 , \qquad \mathcal{L}_Z(g_{ab} dx^a dx^b) = -X^0 \frac{\partial g_{ab}}{\partial u} dx^a dx^b . \qquad (5.1)$$

Let ω_{ab}, σ_{ab} and $\hat{\theta}$ be the vorticity, shear and expansion of $\mathcal{E}_1$, as defined in [4, p. 88]. We have $\omega_{ab} = 0$ as the generators of $\mathcal{E}_1$ are normal to $\mathcal{E}_1$. According to Hawking and Ellis [4, Prop. 9.31] one also has $\hat{\theta} = \sigma_{ab} = 0$, which together with (5.1) implies that

$$\mathcal{L}_Z(g_{ab} dx^a dx^b) = 0 . \qquad (5.2)$$

It follows that $Z \equiv Z^a \partial_a$ is a Killing vector field on S^2. By hypothesis we have $Z \neq 0$. It is well known that the orbits of every Killing vector on S^2 are periodic. Let T denote the (smallest positive) period of Z. Let $\Psi = \phi_T$, where as before ϕ_s denotes the flow of X.

Let $G \subset \mathcal{D}\text{iff}(\mathcal{O})$ be the group generated by $\Psi|_{\mathcal{O}}$. Lemma 5.2 shows that $\hat{M} \equiv \mathcal{O}/G$ is a manifold. A result of Nomizu, which we present in Appendix A, shows that Ψ is analytic, so that $\hat{M}$ is an analytic manifold. Let Π be the natural projection map. It follows from point 2 of Lemma 5.2 that $\hat{M}$ is diffeomorphic to $\Sigma \times S^1$, where Σ is any level set of the time function t of Lemma 5.2. Note that our assumption that the space–time is time–orientable (see footnote 1) implies that $\mathcal{E}_1$ is two–sided. This, in turn, shows that $\mathcal{S} \cap \mathcal{E}_1$ is two–sided in $\mathcal{S}$. The spherical topology of K_1 implies then that we can choose $\mathcal{O}$ so that Σ is simply connected. Moreover, by construction, $\Pi\mathcal{E}_1$ is a hypersurface diffeomorphic to $S^2 \times S^1$ which is ruled by closed null geodesics. Theorem 1 of [36, p. 404] shows that there exists a neighborhood $\mathcal{V}$ of $\Pi\mathcal{E}_1$ on which a Killing vector field $\hat{Z}$ is defined, which is tangent to the generators of $\Pi\mathcal{E}_1$. Let $\hat{\mathcal{V}} \subset \mathcal{O}$ be any open set satisfying $\hat{\mathcal{V}} \cap \mathcal{E}_1 \neq \emptyset$ such that $\Pi|_{\hat{\mathcal{V}}}$ is a diffeomorphism between $\hat{\mathcal{V}}$ and $\Pi\hat{\mathcal{V}}$, with $\Pi\hat{\mathcal{V}} \subset \mathcal{V}$. We define $Z|_{\hat{\mathcal{V}}} = (\Pi|_{\hat{\mathcal{V}}}^{-1})_* \hat{Z}$; clearly $Z|_{\hat{\mathcal{V}}}$ is a Killing vector field on $\hat{\mathcal{V}}$ tangent to the generators of $\mathcal{E}_1 \cap \hat{\mathcal{V}}$. The asymptotic hypotheses on α and k needed in Theorem 4.2 are satisfied by [10, Prop. 1.9], and the ADM four–momentum of $\mathcal{S}$ is timelike by [29]. Our claims follow now by Theorem 4.2. $\square$

The most unsatisfactory feature of the rigidity theorem is the hypothesis of analyticity of the metric in a neighborhood of the event horizon, for which we have no justification. In this context it is worthwhile mentioning an example of a black hole vacuum space–time (with cosmological constant) considered recently by Bičák and Podolský [37]. In that paper Bičák and Podolský show that there exist (real analytic) Robinson–Trautman spacetimes which can be smoothly but *not* analytically extended through an event horizon to another (real analytic) Robinson–Trautman space–time: while the metric is smooth everywhere, it is analytic everywhere except on the event horizon.

6   Bifurcation surfaces

Let us start with some terminology. Recall, first, that the surface gravity of a null hypersurface to which a Killing vector field ζ is tangent is defined by the equation

$$\nabla^a(\zeta^b \zeta_b) = -2\kappa\zeta^a \ .$$

A black hole is called degenerate when $\kappa \equiv 0$, and non–degenerate when κ has no zeros.

Consider, next, the set $\mathcal{N} = \{X^\mu X_\mu = 0, X \neq 0\}$ in the Kruszkal–Szekeres–Schwarzschild space–time, where X is the standard "$\partial/\partial t$" Killing vector. $\mathcal{N}$ has four connected components $\mathcal{N}_a$, $a = 1, \ldots, 4$, and for each a the set $\overline{\mathcal{N}_a} \backslash \mathcal{N}_a$ consists of the same two–sphere, namely the set of points $\mathcal{S}$ where X vanishes. $\mathcal{S}$ is called the bifurcation surface of the bifurcate horizon $\mathcal{N} \cup \mathcal{S}$. Whenever we have a non–degenerate Killing horizon, it is extremely convenient for technical reasons to have the property that this horizon comprises a compact bifurcation surface. For example, this hypothesis is made throughout the classification theory of static (non–degenerate) black holes (*cf. e.g.* [7, 8, 10]). The problem is, that while we have good control of the geometry of the domain of outer communications, various unpleasant things can happen at its boundary. In particular, in [38] it has been shown that there might be

an obstruction for the extendability of a domain of outer communications in such a way that the extension comprises a compact bifuraction surface. Nevertheless, as far as applications are concerned, it suffices to have the following: Given a space–time (M, g) with a domain of outer communications $\langle\langle M_{\text{ext}}\rangle\rangle$ and a non–degenerate Killing horizon, there exists a space–time (M', g'), with a domain of outer communications $\langle\langle M'_{\text{ext}}\rangle\rangle$ which is isometrically diffeomorphic to $\langle\langle M_{\text{ext}}\rangle\rangle$, such that all non–degenerate Killing horizons in (M', g') contain their bifurcation surfaces. Rácz and Wald have shown [38], under appropriate conditions, that this is indeed the case:

Theorem 6.1 (I. Rácz & R. Wald, 96) *Let (M, g_{ab}) be a stationary, or stationary–rotating space–time with Killing vector field X and with an asymptotically flat region M_{ext}. Suppose that $J^+(M_{\text{ext}})$ is globally hyperbolic with asymptotically flat Cauchy surface Σ which intersects the event horizon $\mathcal{N} = \partial\langle\langle M_{\text{ext}}\rangle\rangle \cap J^+(M_{\text{ext}})$ in a compact cross–section. Suppose that X is tangent to the generators of $\mathcal{N}$ and that the surface gravity of every connected component of $\mathcal{N}$ is a non–zero constant. Then there exists a space–time (M', g'_{ab}) and an isometric embedding*

$$\Psi : \langle\langle M_{\text{ext}}\rangle\rangle \to \langle\langle M'_{\text{ext}}\rangle\rangle \subset M' \ ,$$

where $\langle\langle M'_{\text{ext}}\rangle\rangle$ is a domain of outer communications in M', such that:

1. *There exists a one–parameter group of isometries of (M', g'_{ab}), such that the associated Killing vector field X' coincides with $\Psi_* X$ on $\langle\langle M'_{\text{ext}}\rangle\rangle$.*

2. *Every connected component of $\partial\langle\langle M'_{\text{ext}}\rangle\rangle$ is a Killing horizon which comprises a compact bifurcation surface.*

3. *There exists a "wedge–reflection" isometry about every connected component of the bifurcation surface.*

It should be emphasized that neither field equations, nor energy inequalities, nor analyticity have been assumed above. In [38] conditions are given which guarantee that κ is constant on every connected component of a Killing horizon (recall that this property holds for *e.g.* electro–vacuum black holes).

It might be worthwhile to mention that the above conclusion about existence of "wedge–reflection" isometries has been established in [38] for connected event horizons only. This claim can, however, easily be generalized to the multiple black hole case using an obvious extension of the construction in [38], together with the Kuratowski–Zorn lemma.

7 Uniqueness of static, degenerate black holes

An important part of the classification program of stationary space–times is the classification of static black holes. Until recently the existing theory covered only non–degenerate black

holes (*cf.* [1, 32, 6, 7, 8, 10]). For the sake of completeness it is clearly of interest to include the degenerate ones in the classification. Recall, moreover, that degenerate black holes play some role in quantum theories [39], *e.g.* in string theory (*cf. e.g.* [12]). It is widely expected that the only such space–times are the "standard Majumdar–Papapetrou black holes". More precisely, consider a metric g and an electro–magnetic potential A of the form [40, 41]

$$g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \tag{7.1}$$

$$A = u^{-1}dt\,, \tag{7.2}$$

with some nowhere vanishing, say positive, function u. Einstein–Maxwell equations read then

$$\frac{\partial u}{\partial t} = 0\,, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\,. \tag{7.3}$$

A space–time will be called a standard MP space–time if the coordinates x^μ of (7.1)–(7.2) cover the range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$, and if the function u has the form

$$u = 1 + \sum_{i=1}^{I} \frac{m_i}{|\vec{x} - \vec{a}_i|}\,, \tag{7.4}$$

for some positive constants m_i. It has been shown by Hartle and Hawking [42] that every standard MP space–time can be analytically extended to an electro–vacuum space–time with a non–empty black hole region, and with a domain of outer communications which is non–singular in the sense described in Theorem 7.2 below. It is of interest to enquire whether the standard Majumdar–Papapetrou metrics describe all possible regular black holes with a metric of the form (7.1). For the purpose of classification of black holes it is actually necessary to allow metrics of the form (7.1) for which the coordinates are not necessarily global ones: we shall say that a space–time (M, g) is locally a MP space–time if for every point p there exists a coordinate system defined in a neighborhood of this point such that (7.1)–(7.2) holds. We have the following result of M. Heusler [43]:

Theorem 7.1 (M. Heusler, 96) *Let (M, g) be a static electro–vacuum space–time, and suppose that there exists in M an asymptotically flat simply connected slice $\mathcal{S} \subset \langle\langle M_{\mathrm{ext}} \rangle\rangle$ with compact interior and compact boundary $\partial\mathcal{S} \subset \mathcal{E} = \partial\langle\langle M_{\mathrm{ext}} \rangle\rangle \cap J^+(M_{\mathrm{ext}})$. (Here asymptotic flatness is defined in the sense of eq. (2.2) with $\alpha > 0$ and $k \geq 1$, together with eq. (3.3).) For every connected component $\partial\mathcal{S}_a$ of $\partial\mathcal{S}$ we set*

$$Q_a = -\frac{1}{4\pi} \int_{\partial\mathcal{S}_a} *F,$$

where F is the electro–magnetic field two–form. If all the components of $\mathcal{E}$ are degenerate, and if $Q_a Q_b \geq 0$ for all pairs of indices a, b, then (M, g) is locally a MP space–time.

It would be of interest to exclude the possibility of existence of non–connected static black holes with degenerate horizons and with $Q_a Q_b < 0$ for some pair of indices a and b. The hypothesis of staticity can be replaced by that of stationarity if one further assumes

that the event horizon is connected, that the global angular momentum of Σ_{ext} vanishes and that there are no ergo–regions, see [9].

In the case of a connected black hole the idea of proof of Theorem 7.1 is rather simple. In such a case the following mass–squared identity has been established in [9], see also [43] and Section 12 of the accompanying lectures of Heusler [44]:

$$Q^2 + \left(\frac{\kappa A}{4\pi}\right)^2 = M^2, \tag{7.5}$$

where A is the area of ∂S. To prove this identity one can use Carter's result that there are no ergo–regions in static domains of outer communications, together with simple connectedness of the d.o.c. Let us mention that the above identity has already been shown by Israel under the hypothesis that the horizon comprises its bifurcation surface (in particular it cannot be degenerate). Eq. (7.5) gives $|Q| = M$ when κ vanishes, which implies an identity which, together with an observation of Israel and Wilson shows that the metric is, locally, a *Majumdar–Papapetrou* metric.

A classification theorem for a class of static degenerate black holes will follow from the above and from the following result [45]:

Theorem 7.2 (P.C. & N. Nadirashvili, 95) *Consider an electro–vacuum space–time* (M, g) *with a non–empty black hole region* $\mathcal{B}$. *Suppose that there exists in* M *an asymptotically flat space–like hypersurface* Σ *with compact interior and with boundary* $\partial \Sigma \subset \mathcal{B}$. *(Here asymptotic flatness is defined in the sense of eq. (2.2) with* $\alpha > 0$ *and* $k \geq 1$, *together with eq. (3.3).) Assume moreover that on the closure of the domain of outer communications* $\langle\langle M_{\text{ext}} \rangle\rangle$ *there exists a Killing vector field* X *with complete orbits diffeomorphic to* $\mathbb{R}$, X *being timelike in an asymptotic region of* Σ. *If* (M, g) *is locally a MP space–time in the sense described above, then* $\langle\langle M_{\text{ext}} \rangle\rangle$ *is isometrically diffeomorphic to a standard MP space–time.*

Remark: Let us mention that, in particular, under the hypotheses above, every connected component of the event horizon has to carry charge, and all charges have to be of the same sign.

PROOF: By [29] the ADM mass of Σ is timelike, so that Theorem 1 of [45] applies (*cf.* also Appendix B where one of the steps of the proof of that result is justified in more detail than in [45]). It remains to show that the set $\hat{M}$ defined in [45] coincides with $\langle\langle M_{\text{ext}} \rangle\rangle$. This will be established by arguments rather similar to those used in Section 3 of [24]. By [45] we have $\hat{M} \subset \langle\langle M_{\text{ext}} \rangle\rangle$. Suppose, for contradiction, that $\partial \hat{M} \cap \langle\langle M_{\text{ext}} \rangle\rangle \neq \emptyset$, thus there exists a point $p \in \partial \hat{M}$ with $p \in \langle\langle M_{\text{ext}} \rangle\rangle$. Let r be a point in $\langle\langle M_{\text{ext}} \rangle\rangle$ such that there exists a causal future directed curve γ from r to p. We wish to show that there exists T such that $\phi_T(p) \in I^+(\Sigma)$. Indeed, if $r \in I^+(\Sigma)$ we are done, otherwise there exists T such that $\phi_T(r) \in I^+(\Sigma)$, and our claim follows as $\phi_T(p) \in \phi_T(I^+(r)) = I^+(\phi_T(r)) \subset I^+(\Sigma)$. Now $\hat{M}$ is ϕ_T invariant by its definition, so is therefore $\partial \hat{M}$, so that $\phi_T(p) \in \partial \hat{M} \cap I^+(\Sigma)$. We note the following:

Lemma 7.3 *The standard MP space–times $(M, g_{\mu\nu})$ are globally hyperbolic, with the slices $t = const$ being Cauchy surfaces.*

PROOF: Let γ be a causal curve in M. We can parameterize γ by t, so that $\gamma(t) = (t, \vec{x}(t))$. The condition of causality gives

$$\left|\frac{d\vec{x}}{dt}\right|_\delta \leq u^{-2} , \tag{7.6}$$

where $|.|_\delta$ denotes the norm with respect to the flat metric δ_{ij}. Let $\{\vec{a}_i\}_{i=1}^N$ be the singular set of φ. The hypersurfaces $t = const$ would fail to be Cauchy in M if and only if $|\vec{x}(t)|$ would go to infinity in finite time, or if $\vec{x}(t)$ would reach one of the $\vec{a}_i$'s in finite time. The former can be immediately excluded by the asymptotic flatness of the metric, to exclude the latter note that we have $\varphi \geq \frac{m_i}{r_i}$, where $r_i = |\vec{x} - \vec{a}_i|$. Eq. (7.6) thus gives

$$\frac{dr_i}{ds} \leq \frac{r_i^2}{m_i^2}.$$

This last inequality implies

$$\left|\frac{1}{r_i}(t_1) - \frac{1}{r_i}(t_2)\right| \leq \frac{|t_1 - t_2|}{m_i^2},$$

and the result follows.   □

Returning to the proof of Theorem 7.2, let γ be a future directed causal curve from p to M_{ext}. By Lemma 7.3 γ intersects every level set of t. Consider the asymptotically flat hypersurface $\Sigma \cap \hat{M}$. By [46] $\Sigma \cap \hat{M}$ is a graph of a function u, over a subset Ω of $\{t = 0\}$, such that u approaches a constant as r tends to ∞. Moreover Ω contains the complement of some ball. As $\Sigma \cap \hat{M}$ is spacelike Ω is open. By the interior compactness property of Σ it follows that Ω is closed, hence $\Omega = \{t = 0\}$. Consider any future directed causal curve γ in M such that r tends to infinity on γ towards the future. Clearly $\varphi_t(\Sigma \cap \hat{M})$ intersects γ for t large enough. As γ is timelike the set of t such that γ intersects $\varphi_t(\Sigma \cap \hat{M})$ is open. By the interior compactness property of Σ together with global hyperbolicity of $\hat{M}$ it is closed. It follows that γ intersects $\Sigma \cap \hat{M}$ at some point q. We then have $q \in I^+(p) \subset I^+(\Sigma)$ which contradicts achronality of Σ, and Theorem 7.2 follows.   □

Theorems 2.1, 7.1 and 7.2 (together with some standard arguments which we shall not reproduce here) give the following result:

Theorem 7.4 *Consider a static electro–vacuum black–hole space–time (M, g) with a globally hyperbolic domain of outer communications. Suppose that M contains an asymptotically flat achronal hypersurface Σ which has compact interior and a compact connected boundary $\partial\Sigma$ located on the event horizon. (Here asymptotic flatness is defined in the sense of eq. (2.2) with $\alpha > 0$ and $k \geq 1$, together with eq. (3.3).) If the event horizon is degenerate, then the domain of outer communications is isometrically diffeomorphic to that of an extreme Reissner–Nordström black hole.*

8 Uniqueness of stationary, axisymmetric, non–degenerate black holes

Another key ingredient of the classification of regular black holes is that of classification of the stationary and axisymmetric black holes. It is known that the Kerr–Newman black holes exhaust the family of stationary–axisymmetric, connected, non–degenerate, and appropriately regular electro–vacuum black holes [47, 5] satisfying

$$M^2 > Q^2 + a^2 \ . \tag{8.1}$$

Here M is the total ADM mass of the black hole, Q its total electric charge and aM its total angular momentum. Some new results concerning the non–connected case have been recently obtained by Weinstein [33] (*cf.* also [48, 49, 50]), let us discuss those shortly.

Consider thus a stationary–axisymmetric electro–vacuum black hole space–time, and suppose that its domain of outer communications $\langle\langle M_{\text{ext}}\rangle\rangle$ contains an asymptotically flat space–like hypersurface Σ with compact interior and compact boundary $\partial\Sigma_a \subset \mathcal{E} \equiv \partial\langle\langle M_{\text{ext}}\rangle\rangle \cap J^+(M_{\text{ext}})$. Let $\partial\Sigma_a$, $a = 1,\dots,N$ be the connected components of $\partial\Sigma$. Let $\tau = g_{\mu\nu}X^\mu dx^\nu$, where X^μ is the Killing vector field which asymptotically approaches the unit normal to Σ_{ext}. Similarly set $\zeta = g_{\mu\nu}Y^\mu dx^\nu$, Y^μ being the Killing vector field associated with rotations. Define

$$q_a = -\tfrac{1}{4\pi} \int_{\partial\Sigma_a} *F \ , \tag{8.2}$$

$$m_a = -\tfrac{1}{8\pi} \int_{\partial\Sigma_a} *d\tau \ , \tag{8.3}$$

$$L_a = -\tfrac{1}{4\pi} \int_{\partial\Sigma_a} *d\zeta \ . \tag{8.4}$$

Here F is the electro–magnetic field two–form. Note that (8.3) is the standard Komar integral associated with X^μ, and L_a is the Komar integral associated with the Killing vector Y^μ. It is therefore natural to think of L_a as the angular momentum of each connected component of the black hole. Set

$$\mu_a = m_a - 2\omega_a L_a \ , \tag{8.5}$$

where ω_a is the rotation velocity of the a'th black hole. Weinstein shows that one necessarily has $\mu_a > 0$. Let $r_a > 0$, $a = 1,\dots,N-1$, be the distance along the axis between neighboring black holes as measured with respect to an auxiliary (unphysical) metric, *cf.* [33] for details. Let finally λ_a, $a = 1,\dots,N$ be the constants introduced in [33]. The definition of the λ_a's involves various auxiliary potentials, let us simply note that in vacuum we have $\lambda_a = L_a$. We have the following result of Weinstein [33]:

Theorem 8.1 (G. Weinstein, 96) *Consider a stationary-axisymmetric electro–vacuum space–time (M, g) with a globally hyperbolic domain of outer communications $\langle\langle M_{\text{ext}}\rangle\rangle$. Suppose that M contains an asymptotically flat hypersurface Σ with compact interior and compact boundary $\partial\Sigma = \cup_{i=1}^N \partial\Sigma_a$, where each of the of the N connected components $\partial\Sigma_a$ of $\partial\Sigma$ satisfies $\partial\Sigma_a \subset \mathcal{E} \equiv \partial\langle\langle M_{\text{ext}}\rangle\rangle \cap J^+(M_{\text{ext}})$. (Here asymptotic flatness is defined in the sense of eq. (2.2) with $\alpha > 0$ and $k \geq 1$, together with eq. (3.3).) Suppose moreover that every connected component of $\mathcal{E}$ is non–degenerate. Then:*

1. The inequality (8.1) holds.

2. The metric on $\langle\langle M_{\text{ext}}\rangle\rangle$ is uniquely determined (up to isometry) by the $4N - 1$ parameters

$$(\mu_1, \ldots, \mu_N, \lambda_1, \ldots, \lambda_N, q_1, \ldots, q_N, r_1, \ldots, r_{N-1}) \tag{8.6}$$

described above, with $r_a, \mu_a > 0$.

Let us note that for connected black holes point 1 of Theorem 8.1 removes the condition (8.1) from Mazur's theorem [5]. The fact that (8.1) is not necessary has been known to some authors [51, 52], but we are not aware of any previous published proof.

It is known that for some sets of parameters (8.6) the solutions will have "strut singularities" between some pairs of neighboring black holes [50, 53]. In the statement of Theorem 8.1 we have for simplicity assumed smoothness of the domain of outer communications. The conclusion of Theorem 8.1 remains valid when strut singularities are allowed in the metric.

As far as existence of *vacuum* non–connected rotating black holes is concerned, we have the following result, also due to Weinstein [50]:

Theorem 8.2 (G. Weinstein, 94) *For every $N \geq 2$ and for every set of parameters*

$$(\mu_1, \ldots, \mu_N, L_1, \ldots, L_N, r_1, \ldots, r_{N-1}) , \tag{8.7}$$

with $\mu_a, r_a > 0$, there exists a vacuum space–time (M, g) satisfying the hypotheses of Theorem 8.1, except perhaps for "strut singularities" on the axis between some neighboring black holes.

The existence of the "struts" for all sets of parameters as above is not known, and is the main open problem in our understanding of stationary–axisymmetric electro–vacuum black holes.

For the electro–vacuum case, an equivalent of Theorem 8.2 is true [33] modulo a "regularity" result on the singular set for the harmonic maps with prescribed singularities into the complex hyperbolic plane, which has not been established yet (*cf.* [54] for some related results). More precisely, Weinstein has shown [33] that for any set of parameters (8.6) with $\mu_a, r_a > 0$ there exists a space–time (M, g) satisfying the hypotheses of Theorem 8.1, except perhaps for singularities on the axis of rotation. One expects that those singularities, if any, will again be "strut singularities" between some neighboring black holes, but no rigorous proof of this fact is available so far. It would be of interest to fill this gap.

9   Differentiability of event horizons

One of the hypotheses of Theorem 5.1, in addition to analyticity of space–time and of the metric, is that of analyticity of the event horizon. This is a logically independent requirement, as is clearly demonstrated by the following result [55]:

Theorem 9.1 (P.C. & G. Galloway, 96) *There exist vacuum analytic space–times with a non–empty black hole region and with a* nowhere differentiable *event horizon $\mathcal{E}$, in the sense that no open subset of $\mathcal{E}$ is a differentiable manifold.*

The example constructed in [55] is admittedly artificial, as it is obtained by removing an appropriate subset out from Minkowski space–time. There are certainly several desirable and independent sets of supplementary hypotheses which one could impose to exclude this kind of space–times. It is nevertheless worrisome, because it shows that high differentiability of global constructs such as event horizons, Cauchy horizons, etc., should not be taken for granted and should be justified in situations under consideration. In particular, Theorem 9.1 leads immediately to the question, whether the area theorem hold for event horizons with low differentiability. Because of the widely accepted relationship of the area theorem with the laws of thermodynamics, it would be of interest to analyze this problem.

10   Conclusions

We have discussed various recent developments in our understanding of space–time with black holes. From what has been said it should be clear that several important questions remain open. We shall end this paper with two lists of problems which, we believe, are worthwhile being investigated. The main problems are:

1. Prove the Rigidity Theorem 5.1 without the hypothesis of analyticity of the space–time and of the event horizon, or construct a counterexample.

2. Show that a sufficiently regular multi black–hole space–time must be a Majumdar–Papapetrou space–time, or construct a counterexample.

The following problems below would clearly be superseded by solutions of their counterparts above. Nevertheless one could try to solve the problems below, as a step towards solving the more difficult ones:

1. Remove the hypothesis of analyticity of the event horizon in the Rigidity Theorem 5.1.

2. Show that Weinstein's non–connected (stationary, axi–symmetric, electro–vacuum) black holes are singular (*cf.* [50, 53] for some partial results).

3. Show that the only sufficiently regular Israel–Wilson–Perjes black holes are in the Majumdar–Papapetrou class (*cf.* [42, 56] for some partial results).

4. Extend Weinstein's Classification Theorem 8.1 to include degenerate components of the event horizon.

A   Analyticity of isometries

We have the following unpublished result of Nomizu, which we prove here for completeness. The analyticity of isometries of analytic manifolds is a direct consequence of this result:

Theorem A.1 *Suppose M and M' are real analytic manifolds each provided with a real analytic affine connection. Then a (smooth) diffeomorphism f from M onto M' which preserves the affine connections is real analytic.*

PROOF: To prove the result, it suffices to note that for M with an analytic affine connection ∇, the exponential mapping $\exp_p : T_pM \to M$ is real analytic on a neighborhood of 0 in T_pM, say U (so that $\exp_p(U)$ is a normal neighborhood of p). We have for each $p \in M$

$$f(\exp_p(X)) = \exp_{f(p)}(AX) \quad \text{for every } X \in V ,$$

where A is the differential $f_*(p)$ of f at p and V is an open neighborhood of 0 in T_pM. The analyticity of f follows immediately from this equation.   □

B   A comment to the classification theorem of non–singular MP black holes

In [45], in the paragraph following eq. (12) it was claimed without justification that the functions x^a defined there provide a global coordinate system on the manifold $\hat{M}$ defined there. While this claim is correct, some work is needed to justify that. We shall present here the missing elements of the proof. We use the notation and the definitions of [45]. Let us denote by Ψ the map from $\hat{M}$ to $\mathbb{R}^4$ defined by the functions x^a. A standard asymptotic analysis of the equations (11)–(12) of [45] shows that Ψ is a diffeomorphism between $\Sigma_{\text{ext}} \times \mathbb{R}$ and $(\mathbb{R}^3 \setminus K) \times \mathbb{R}$, for some compact set K. Here one might need to replace the constant R which defines Σ_{ext} by a larger constant.

Consider an affinely parameterized geodesic γ of the metric $h_{\mu\nu}$ defined by eq. (10) of [45]. From the definition of the x^a's together with the covariant constancy of the e^a_ν's we

have

$$\frac{dx^a(\gamma(s))}{ds} = e^a_\nu(\gamma(s))\frac{d\gamma^\nu(s)}{ds} \Longrightarrow \frac{D^2 x^a(\gamma(s))}{ds^2} = 0,$$

which shows that

$$x^a(\gamma(s)) = e^a_\nu(\gamma(s_0))\frac{d\gamma^\nu(s_0)}{ds}s + x^a(\gamma(s_0)). \tag{B.1}$$

We have the following lemma:

Lemma B.1 *Let $\gamma : [a,b) \to \hat{M}$ be an affinely parametrized geodesic of the metric $h_{\mu\nu}$ defined by eq. (10) of [45]. If there exists $\varepsilon > 0$ such that $u|_\gamma > \varepsilon$, then γ can be extended in $\hat{M}$ to a geodesic defined on an interval $[a, b+\delta)$, for some $\delta > 0$.*

PROOF: If $\overline{\gamma([a,b))} \subset \text{int}(\Sigma_{\text{ext}}) \times \mathbb{R}$ the result is obvious by properties of geodesics in Minkowski space–time together with the fact, mentioned above, that Ψ is a diffeomorphism in the asymptotic region. Note that the geodesic γ cannot approach the boundary of $\hat{M}$, at which u vanishes, we may thus assume that $\gamma([a,b)) \subset \Sigma_c \times \mathbb{R}$, where Σ_c is compact. Because t is a linear function of s on γ (see eq. (B.1)), we have in fact $\gamma([a,b)) \subset \Sigma_c \times [t(\gamma(a)), t(\gamma(a)) + C(b - a)]$, for some constant C, which is a compact set. Let $s_i \in [a,b)$ be any sequence such that $s_i \to b$, let p be an accumulation point of $\gamma(s_i)$. Let (τ, y^i) be local MP coordinates defined in a neighbourhood of p, in those coordinates γ is a straight line which accumulates at p and clearly can be extended as claimed. □

Consider the exponential map $\exp_q$ of the metric h,

$$\exp_q : T_q\hat{M} \supset \mathcal{O}_q \to \hat{M}$$

defined on $\mathcal{O}_q$. Lemma B.1 shows that $\mathcal{O}_q$ is of the form $\mathbb{R} \times \mathcal{U}_q$ for some open star–shaped set $\mathcal{U}_q \subset \mathbb{R}^3$. Moreover every maximally extended affinely parametrized geodesic in $\hat{M}$ starting at q is of the form $\{\exp_q(\alpha r, r\vec{n})| r \in [0, r_{\vec{n}})\}$ for some $\alpha \in \mathbb{R}$, with either $r_{\vec{n}} = \infty$ or $\liminf_{r \to r_{\vec{n}}} u(\exp_q(\alpha r, r\vec{n})) = 0$. Let us write a vector field $Y \in T_q\hat{M}$ as a linear combination of the basis vectors $e^{a\mu}(q)$, as defined in [45], $Y = Y_a e^{a\mu}(q)$. It follows from (B.1) that

$$x^a \circ (\exp_q(Y_b e^{b\mu}(q)) = Y_a + x^a(q) . \tag{B.2}$$

In other words,

$$\Psi \circ \exp_q : \mathcal{O}_q \to \mathbb{R}^4 \quad \text{is a translation.} \tag{B.3}$$

In particular $\exp_q$ is injective for every $q \in \hat{M}$. We set

$$\phi_q = u \circ \exp_q : \mathcal{O}_q \to \mathbb{R} .$$

As discussed in [45] we have $\partial\phi_q/\partial t = 0$ so that, by a slight abuse of notation, we can consider ϕ_q as being defined on $\mathcal{U}_q$. Moreover, again as discussed in [45], ϕ_q satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi_q = 0, \tag{B.4}$$

$$|\text{grad } \phi_q^{-1}| \le C_1 . \tag{B.5}$$

Here both the gradient and its norm are taken with respect to the flat metric on $T_q\hat{M}$. We have the following version of Prop. 2 of [45], the proof of which can be obtained by trivial modifications of the proof given in [45]:

Proposition B.2 *Let $\mathcal{U}_q$ be an open subset of $\mathbb{R}^3$ and let ϕ_q be a function satisfying (B.4)–(B.5) on $\mathcal{U}_q$. (It follows from (B.5) that ϕ_q^{-1} can be extended by continuity to a function f on $\overline{\mathcal{U}_q}$.) Set*

$$\mathcal{S}_q = \{p \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q | f(p) = 0\} \ .$$

Then $\mathcal{S}_q$ is discrete.

Proposition B.2 and what has been said show that for every point $q \in \hat{M}$ the set $\mathcal{U}_q$ is $\mathbb{R}^3$ minus a finite set of half–rays, $\mathcal{U}_q = \mathbb{R}^3 \setminus (\cup_{i=1}^N \{r\vec{a}_{i,q}, r \in [1,\infty)\})$, with $\vec{a}_{i,q} \neq 0$.

We wish to show that $\Psi(\hat{M}) \supset \Omega \equiv \mathbb{R}^3 \setminus (\cup_{i=1}^N \{\vec{a}_i\})$, with $\vec{a}_i \equiv \vec{a}_{i,p}$. Here p is the point in $\hat{M}$ such that $x^a(p) = 0$; we know already that $\Omega \supset \mathbb{R}^3 \setminus (\cup_{i=1}^N \{r\vec{a}_i, r \in [1,\infty)\})$. Consider a point $(Y_a) = (0, r\vec{a}_i)$ with $r > 1$, for some i. We can find $r \in \Sigma_{\text{ext}}$ such that the triangle $T \subset \mathbb{R}^4$ with vertices 0, $x^a(r)$ and Y_a satisfies $(T \setminus \{s\vec{a}_i, s \in [1,\infty)\}) \cap \partial(\mathbb{R} \times \mathcal{O}_p) = \emptyset$. By a simple continuity argument it can be seen that $Y_i - x^i(r) \in \mathcal{U}_r$, so that by eq. (B.2) we obtain $x^a(\exp_r((Y_a - x^a)e^{a\mu}(r))) = Y_a$, except perhaps in the case in which there exists a point $\vec{a} \in \mathcal{S}_p$ such that $\vec{a} = C\vec{a}_{i,p}$, for some constant $C > 1$. That last possibility can be easily gotten rid of by replacing p by a nearby point p' in the construction of the map Ψ. It follows that $\Omega \subset \Psi(\hat{M})$.

For $Y \in \mathcal{O}_p$ we set $\Phi(Y) = \exp_p(Y)$, while for $Y \in \Omega \setminus \mathcal{O}_p$ we set $\Phi(Y) = \exp_r((Y_a - x^a)e^{a\mu}(r))$, where r is any point satisfying the requirements of the previous paragraph. By (B.2) Φ is well–defined (independent of the choice of r in the allowed class). Φ is smooth by smooth dependence of solutions of ODE's with smooth coefficients upon initial values. It is injective by eq. (B.2). The image of Φ is open in Ω because Φ is a local diffeomorphism. It is closed by Lemma B.1. It follows that Φ is surjective, hence a bijection, which finishes the proof of the missing step of the proof in [45].

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