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Autor: Heusler, Markus

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No-Hair Theorems and Black Holes with Hair

By Markus Heusler

Institute for Theoretical Physics
The University of Zurich
CH–8057 Zurich, Switzerland

Abstract. The critical steps leading to the uniqueness theorem for the Kerr-Newman metric are examined in the light of the new black hole solutions with Yang-Mills and scalar hair. Various methods – including scaling techniques, arguments based on energy conditions, conformal transformations and divergence identities – are reviewed, and their range of application to selfgravitating scalar and non-Abelian gauge fields is discussed. In particular, the no-hair theorem is extended to harmonic mappings with arbitrary Riemannian target manifolds.

1 Introduction

The uniqueness theorem for the asymptotically flat, stationary black hole solutions of the Einstein-Maxwell equations is, by now, established quite rigorously (see, e.g. [1] and [2]). Some open gaps, notably the electrovac staticity theorem [3] and the topology theorem (see [4] and the following lecture by P. Chruściel) have been closed recently. The theorem, conjectured by Israel, Penrose and Wheeler in the late sixties (see [5] for a historical account), implies that all stationary electrovac black hole spacetimes are characterized by their mass, angular momentum and electric charge. This beautiful result – together with the striking analogy between the laws of black hole physics and the laws of equilibrium thermodynamics – provided support for the expectation that *all* stationary black hole solutions can be described in terms of a small set of asymptotically measurable quantities.

Tempting as it was, the hypothesis was disproved in 1989, when several authors presented a counterexample within the framework of SU(2) Einstein-Yang-Mills (EYM) theory [6]:

Although the new solution was static and had vanishing Yang-Mills charges, it was different from the Schwarzschild black hole and, therefore, *not* characterized by its total mass. In fact, a whole variety of new black hole configurations violating the generalized no-hair conjecture were found during the last few years. These include, for instance, black holes with Skyrme [7], dilaton [8] or Yang-Mills-Higgs hair [9]. The diversity of new solutions gives rise to a reexamination of the logic to the proof of the uniqueness theorem. In particular, one needs to investigate whether there are steps in the uniqueness proof which are not sensitive to the details of the matter contents. In order to do so, we shall now briefly recall some of the main issues which are involved in the uniqueness program.

2 The Uniqueness Program

At the basis of the reasoning lies Hawking's *strong rigidity* theorem [10], [11]. It relates the concept of the event horizon to the independently defined – and logically distinct – local notion of the Killing horizon: Requiring that the fundamental matter fields obey well behaved hyperbolic equations and that the stress-energy tensor satisfies the weak energy condition, the theorem asserts that the event horizon of a *stationary* black hole spacetime is a Killing horizon. This also implies that either the null-generator Killing field of the horizon coincides with the stationary Killing field or spacetime admits at least one axial Killing field. (The original proof of the rigidity theorem was based on an analyticity assumption which has, for instance, no justification if the domain of outer communications admits regions where the stationary Killing field becomes null or spacelike; see the lecture by P. Chruściel.) The strong rigidity theorem implies that stationary black hole spacetimes are either axisymmetric or have a nonrotating horizon. The classical uniqueness theorems were, however, established for spacetimes which are either *circular* or *static*. Hence, in both cases, one has to prove in advance that the Frobenius *integrability conditions* for the Killing fields are satisfied as a consequence of the symmetry properties and the matter equations.

The *circularity* theorem, due to Kundt and Trümper [12] and Carter [13], [14], implies that the metric of a vacuum or electrovac spacetime can, without loss of generality, be written in the well-known Papapetrou (2+2)-split. As we shall show in section 5, the generalization of the theorem is straightforward for selfgravitating scalar mappings [15] (i.e., Higgs fields, harmonic mappings, Skyrme fields, etc.). The circularity theorem does, however, *not* hold for the EYM system (without imposing additional constraints on the gauge fields by hand).

The *staticity* theorem, establishing the hypersurface orthogonality of the stationary Killing field for electrovac black hole spacetimes with nonrotating horizons, is more involved than the circularity problem. First, one has to establish *strict* stationarity, that is, one needs to exclude ergoregions. This problem, first discussed by Hajicek [16] and Hawking and Ellis [11], was solved only recently by Sudarsky and Wald [3], assuming a foliation by maximal slices [17]. Once ergoregions are excluded, it remains to prove that the stationary Killing field satisfies the Frobenius integrability conditions. This was achieved by Hawking [10], extending a theorem due to Lichnerowicz [18] to the *vacuum black hole* case. As already mentioned, Sudarsky and Wald were eventually able to solve the staticity problem for *elec-*

trovac black holes by using the generalized version of the first law of black hole physics [3]. Like the circularity theorem, the staticity theorem is easily extended to scalar fields [15], whereas the problem is again open for selfgravitating non-Abelian gauge fields.

The main task of the uniqueness problem is to show that the static electrovac black hole spacetimes (with nondegenerate horizon) are described by the Reissner-Nordström metric, whereas the circular ones (i.e., the stationary and axisymmetric ones with integrable Killing fields) are given by the Kerr-Newman metric.

In the static case it was Israel who, in his pioneering work [19], [20], was able to establish that both static vacuum and electrovac black hole spacetimes are *spherically symmetric*. Israel's ingenious method, based on integral identities and Stokes' theorem, triggered a series of papers devoted to the uniqueness problem (see, e.g. [21], [22]). More recently, Bunting and Masood-ul-Alam [23] (see also [25]) found a new proof of the Israel theorem, taking essential advantage of the positive mass theorem [26], [27].

The uniqueness theorem for stationary and axisymmetric black holes is mainly based on the Ernst formulation of the Einstein (-Maxwell) equations [28]. The key result consists in Carter's observation that the field equations can be reduced to a 2-dimensional boundary-value problem [29]. A (most amazing) identity due to Robinson [30] then establishes that all vacuum solutions with the same boundary and regularity conditions are identical. The uniqueness problem for the electrovac case remained open until Mazur [31] and Bunting [32] independently succeeded in deriving the desired divergence identities in a systematic way: The Mazur identity is based on the observation that the Ernst equations describe a nonlinear sigma-model on the coset space G/H, where G is a connected Lie group and H is a maximal compact subgroup of G. In the electrovac case one finds $G/H = SU(1,2)/S(U(1) \times U(2))$. Within Mazur's approach, the Robinson identity turns out to be the explicit form of the sigma-model identity for the vacuum case, $G/H = SU(1,1)/U(1)$.

3 Selfgravitating Soliton Solutions

One of the reasons why it was not until 1989 that black hole solutions with selfgravitating gauge fields were discovered was the widespread belief that the EYM equations admit *no soliton* solutions either. There were, at least, four good reasons in support of this hypothesis.
• First, there are *no purely gravitational solitons*, that is, the only globally regular, asymptotically flat, static vacuum solution of Einstein's equations with finite energy is Minkowski spacetime. Using Stokes' theorem for a spacelike hypersurface Σ, this result is obtained from the positive mass theorem [26], [27] and the Komar expression for the total mass of an asymptotically flat, stationary spacetime with Killing field (1-form) k, say:

$$M = -\frac{1}{8\pi G}\int_{S^2_\infty} *dk = -\frac{1}{4\pi G}\int_\Sigma *R(k) = 0\,. \tag{3.1}$$

Here we have also used the vacuum Einstein equations and the Ricci identity $d{*}dk = 2{*}R(k)$ (where the 1-form $R(k)$ is defined by $R(k)_\mu \equiv R_{\mu\nu}k^\nu$).

• Second, both Deser's energy argument [33] and Coleman's scaling method [34] show that there exist *no flat spacetime solitons in pure YM* theory.
• Third, the *Einstein-Maxwell* equations admit *no soliton* solutions. This follows immediately from Stokes' theorem

$$\int_{\partial\Sigma} *(k \wedge \alpha) = - \int_{\Sigma} (d^{\dagger}\alpha) * k \tag{3.2}$$

for an arbitrary, invariant 1-form α ($L_k \alpha = 0$), and from the static Maxwell equations ($V \equiv -\langle k\,, k\rangle \equiv -k^{\mu}k_{\mu}$)

$$d^{\dagger}\left(\frac{E}{V}\right) = 0\,, \quad d^{\dagger}\left(\frac{B}{V}\right) = 0\,, \quad E = d\phi\,, \quad dB = 0\,, \tag{3.3}$$

for the electric ($E \equiv -i_k F$) and the magnetic ($B \equiv i_k * F$) component of the field strength 2-form F (see, e.g. [2]). (Here $d^{\dagger} \equiv *d*$ denotes the coderivative operator and $*k = i_k \eta$ is the volume 3-form on Σ.) First choosing $\alpha = \frac{E}{V}$ gives $-4\pi Q \equiv \int_{\partial\Sigma} *(k \wedge \frac{E}{V}) = 0$. Then setting $\alpha = \phi\frac{E}{V}$ yields $0 = -4\pi Q \phi_{\infty} = \int_{\Sigma} \frac{\langle E\,,E\rangle}{V}$, and therefore $E = 0$. A similar argument shows that the magnetic field vanishes as well.
• Finally, Deser [35] has shown that the 3-*dimensional* EYM equations admit *no soliton solutions* either. His argument relies on the fact that the magnetic 1-form, $B = i_k * F$, has only one nonvanishing component in $2 + 1$ dimensions.

All this shows that it was conceivable to conjecture a nonexistence theorem for selfgravitating EYM solitons (in $3+1$ dimensions). On the other hand, none of the above examples takes care of the full nonlinear structure of both the gauge and the gravitational fields. It is therefore, with hindsight, not too surprising that EYM solitons actually do exist – although it came, of course, unexpected when Bartnik and McKinnon (BK) presented the first particle-like solution in 1988 [36]. Since the BK solutions are spherically symmetric, it is instructive to consider the implications of scaling methods for static, spherically symmetric matter models coupled to gravity.

4 Scaling Techniques

Scaling arguments provide an efficient tool for proving nonexistence theorems in flat spacetime. Moreover, experience shows that they also give reliable hints concerning the possible existence of such solutions. In relativity, scaling techniques are, of course, limited to highly symmetric situations, since they are based on the existence of distinguished coordinates. In this section we shall, therefore, restrict ourselves to *spherically symmetric*, selfgravitating field theories. (Here we present a slightly improved version of the arguments given in [37].)

The effective action for static, spherically symmetric soliton configurations is

$$\mathcal{A}\,[m, S, \psi] = \int_0^{\infty} (L_{\text{mat}} - S\,m')\; dr\,, \tag{4.1}$$

where ψ stands for the matter fields with effective matter Lagrangian $L_{\text{mat}}[m, S, \psi]$, and $m(r)$ and $S(r)$ parametrize the static spacetime metric,

$$\boldsymbol{g} = -S^2 N\, dt^2 + N^{-1}\, dr^2 + r^2\, d\Omega^2 , \quad \text{with } N(r) \equiv 1 - \frac{2\, m(r)}{r} . \tag{4.2}$$

The complete set of static field equations is equivalent to the Euler-Lagrange equations for m, S and the matter fields.

Let us now assume that we are given a solution $m(r)$, $S(r)$, $\psi(r)$ with boundary conditions $m(0) = m_0$, $m(\infty) = m_\infty$ etc. Then each member of the 1-parameter family

$$m_\lambda(r) \equiv m(\lambda r) , \quad S_\lambda(r) \equiv S(\lambda r) , \quad \psi_\lambda(r) \equiv \psi(\lambda r) \tag{4.3}$$

assumes the same boundary values at $r = 0$ and $r = \infty$, and the action $\mathcal{A}_\lambda \equiv \mathcal{A}[m_\lambda, S_\lambda, \psi_\lambda]$ must therefore have a critical point at $\lambda = 1$, $[d\mathcal{A}/d\lambda]_{\lambda=1} = 0$. Since the purely gravitational part does not depend on λ (since $\int_0^\infty dr\, S_\lambda(r)[\partial\, m_\lambda(r)/\partial r] = \int_0^\infty d\rho\, S(\rho)\, [\partial m(\rho)/\partial \rho]$, with $\rho = \lambda r$), we obtain the following *necessary* condition for the existence of soliton solutions:

$$\left[\frac{d}{d\lambda} \int_0^\infty L_{\text{mat}}\, [m_\lambda(r), S_\lambda(r), \psi_\lambda(r)]\; dr\right]_{\lambda=1} = 0 . \tag{4.4}$$

(It is worth noticing that gravity *does* enter the argument, although the purely gravitational part of the action does not appear in the above formula.)

As a first example, we consider the purely magnetic SU(2) configuration used by Bartnik and McKinnon [36] (Witten ansatz) with gauge potential 1-form A,

$$A = [w(r) - 1]\, [\tau_\varphi\, d\vartheta \;-\; \tau_\vartheta \sin\vartheta\, d\varphi] , \tag{4.5}$$

where $\tau_\vartheta \equiv \partial_\vartheta \tau_r$, $\tau_\varphi \sin\vartheta \equiv \partial_\varphi \tau_r$ and $\tau_r \equiv (2i|\vec{r}|)^{-1} \vec{r} \cdot \vec{\sigma}$. The effective matter Lagrangian then becomes (see, e.g. [38])

$$L_{\text{mat}} = S\, [N T_{YM} + V_{YM}] , \quad \text{where } T_{YM} \equiv \frac{1}{2} w'^2 , \quad V_{YM} \equiv \frac{(1 - w^2)^2}{4\, r^2} . \tag{4.6}$$

A short computation shows that the condition (4.4) now implies that every solution must satisfy

$$\int_0^\infty dr\, S\, [N T_{YM} + V_{YM}] = \int_0^\infty dr\, \frac{2m}{r}\, S\, T_{YM} . \tag{4.7}$$

In flat spacetime the same argument yields $\int dr\, [T_{YM} + V_{YM}] = 0$, which shows that no solitons exist. If, however, gravity is taken into account, then the virial type relation (4.7) does *not exclude* selfgravitating soliton solutions any more.

A second, even simpler example are scalar (Higgs) fields with non-negative potentials $P[\phi]$. In this case we find

$$L_{\text{mat}} = S\, [N T_H + V_H] , \quad \text{where } T_H \equiv \frac{1}{2} r^2 \phi'^2 , \quad V_H \equiv r^2 P[\phi] . \tag{4.8}$$

The scaling argument now yields the condition $\int dr\, S\, [T_H + 3V_H] = 0$, which is – up to the positive function $S(r)$ – the same relation as one obtains in flat spacetime. Hence, as is well known, there are no spherically symmetric scalar solitons (with non-negative potentials), independently of whether or not gravity is taken into account. In fact, the above technique can be applied to spherically symmetric *harmonic* mappings with arbitrary Riemannian target manifolds (see also the next section).

The above scaling argument can also be used to exclude – or indicate – the existence of *black hole* solutions. To this end, one has to replace the lower boundary in the action integral (4.1) by the horizon distance r_H. In order to have fixed boundary conditions (at $r = r_H$ and $r = \infty$) we consider the modified 1-parameter family

$$m_\lambda(r) \equiv m(r_H + \lambda \cdot (r - r_H)), \quad S_\lambda(r) \equiv S(r_H + \lambda \cdot (r - r_H)) \tag{4.9}$$

(and similarly for the matter field amplitudes). The same argument as before shows that every solution is subject to the condition (4.4), where now the functions m_λ etc. are given by eq. (4.9) and the lower boundary in the integral is $r = r_H$. Using the effective YM and scalar field Lagrangians (4.6) and (4.8) we obtain the virial relations

$$\int_{r_H}^{\infty} dr\, S\, \left\{ [1 + \frac{2m}{r}\left(\frac{r_H}{r} - 2\right)]\, T_{YM} \,+\, [1 - \frac{2r_H}{r}]\, V_{YM} \right\} \,=\, 0 \tag{4.10}$$

and

$$\int_{r_H}^{\infty} dr\, S\, \left\{ [\frac{2r_H}{r}\left(1 - \frac{m}{r}\right) - 1]\, T_H \,+\, [\frac{2r_H}{r} - 3]\, V_H \right\} \,=\, 0\,, \tag{4.11}$$

respectively. It is obvious that the factors in front of the (non-negative) YM quantities T_{YM} and V_{YM} in eq. (4.10) are indefinite. Hence, in the EYM case, the scaling argument does *not* exclude black hole solutions with hair. In contrast to this, both factors in front of the (non-negative) quantities T_H and V_H in eq. (4.11) *do* have a fixed, negative sign. (Use $[\ldots](r_H) = 0$ and $[\ldots]'(r) \le 0\, \forall r \ge r_H$ (since $m' \ge 0$ as a consequence of the dominant energy condition) to show that the bracket in front of the kinetic term T_H is nonpositive.) Hence, the virial relation (4.11) is violated, unless $T_H = V_H = 0$, which excludes spherically symmetric black holes with nontrivial scalar fields. In fact, (a slightly more complicated version of) this argument provided the first proof of the no-hair theorem for spherically symmetric scalar fields with *arbitrary* non-negative potentials [37]. In the meantime, Sudarsky [41] and Bekenstein [42] have presented different proofs of the spherically symmetric scalar no-hair theorem. There exists, yet, another proof which is exclusively based on a mass bound for spherically symmetric black holes and the circumstance that scalar fields (with harmonic action and non-negative potentials) violate the strong energy condition [43]. The following section is devoted to a brief outline of this argument.

5 Uniqueness Theorems and Mass Bounds

The total mass of a stationary, asymptotically flat spacetime with Killing field k is given by the Komar formula (3.1). Applying Stokes' theorem and the Ricci identity for Killing fields, one obtains the Smarr formula

$$M = \frac{1}{4\pi}\kappa\mathcal{A} - \frac{1}{4\pi}\int_{\Sigma} *R(k)\,, \tag{5.1}$$

where κ and $\mathcal{A}$ denote the surface gravity and the area of the horizon, respectively, and where we have assumed that the latter is nonrotating (i.e., generated by k). Now using the identity $d\omega = *(k \wedge R(k))$ for the derivative of the twist ($\omega \equiv \frac{1}{2} * [k \wedge dk]$), as well as the general identity (6.7) below, it is not hard to derive the formula (see, e.g. [2] or [43])

$$M = \frac{1}{4\pi}\kappa\mathcal{A} + \frac{1}{4\pi}\int_{\Sigma}\left[\frac{R(k,k)}{V} - 2\,\frac{\langle\omega\,,\omega\rangle}{V^2}\right] * k\,, \tag{5.2}$$

where, as before, $V \equiv -\langle k\,,k\rangle \equiv -k^{\mu}k_{\mu}$. If the domain of outer communications is *strictly* stationary, then k is nowhere spacelike and ω is nowhere timelike. The violation of the *strong* energy condition (SEC) (by which we mean $R(k,k) \leq 0$ throughout the domain of outer communications) then implies an *upper* bound for the total mass:

$$M \leq \frac{1}{4\pi}\kappa\mathcal{A}, \quad \text{if } \boldsymbol{T}(k,k) - \frac{1}{2}\boldsymbol{g}(k,k)\,\mathrm{tr}\boldsymbol{T} \leq 0\,. \tag{5.3}$$

If no horizons are present, this inequality reduces to $M \leq 0$, which is in contradiction to the *positive mass* theorem [26], [27] – unless $M = 0$. Since the latter is based on the *dominant* energy condition (DEC), we conclude that there exist *no soliton* solutions in selfgravitating field theories which are subject to the DEC, but violate the SEC at every point.

For static *black hole* solutions of matter models which violate the SEC we have $M \leq \frac{1}{(4\pi)}\kappa\mathcal{A}$. Hence, in order to extend the above soliton argument to the black hole case, a derivation of the converse inequality, $M \geq \frac{1}{(4\pi)}\kappa\mathcal{A}$, on the basis of the DEC is needed. However, this was not achieved until now. In fact, the Penrose conjecture [44], $M^2 \geq \mathcal{A}/(16\pi)$, which is closely related to the above inequality, was also proven only under additional geometrical assumptions [45] or in the spherically symmetric case [46], [47] (see also [48], [49] and [43]). In the general case, the strongest result consists in the extension of the positive mass theorem by Gibbons *et al.* [50] to *black hole* spacetimes. It is, however, not hard to establish the desired bound, $M \geq \frac{1}{(4\pi)}\kappa\mathcal{A}$, for spherically symmetric configurations (see below). This yields the conclusion that all static, spherically symmetric black hole solutions with matter satisfying the DEC and violating the SEC coincide with the Schwarzschild metric.

Before we apply this result to scalar fields, we owe the derivation of $M \geq \frac{1}{(4\pi)}\kappa\mathcal{A}$ for static, spherically symmetric black hole spacetimes with matter being subject to the DEC. First, using the parametrization (4.2) of the metric, the DEC implies that both $m(r)$ and $S(r)$ are monotonically increasing functions, since $m' = \frac{1}{2}r^2 G_{00} \geq 0$ and $S'/S = \frac{1}{2}rN^{-1}[G_{00} + G_{11}] \geq 0$. Second, the local mass function $M(r)$, defined by

$$M(r) \equiv -\frac{1}{8\pi}\int_{S_r^2} *dk = mS + NS'r^2 - m'Sr\,, \tag{5.4}$$

assumes the values M and $\frac{1}{4\pi}\kappa\mathcal{A}$ for $r \to \infty$ and $r = r_H$, respectively. Using $m' \geq 0$, $S' \geq 0$, asymptotic flatness and $N(r_H) = 0$, we now find

$$M \geq lim_{r\to\infty}(m\,S) \quad \text{and} \quad \frac{1}{4\pi}\kappa\mathcal{A} \leq m(r_H)\,S(r_H)\,. \tag{5.5}$$

Finally, taking again advantage of the monotonicity of m and S, eq. (5.5) yields the desired result.

As an application of the above uniqueness results we consider scalar fields or, more generally, *harmonic* mappings ϕ from spacetime $(M, \boldsymbol{g})$ into *Riemannian* target manifolds $(N, \boldsymbol{G})$. The action is

$$\mathcal{A} = \int \left(-\frac{1}{16\pi} R + \frac{1}{2} \langle d\phi \, , d\phi \rangle_G + P[\phi] \right) * 1 \, , \tag{5.6}$$

where we have also allowed for a nonnegative potential $P[\phi]$. In local coordinates the harmonic density becomes $\langle d\phi \, , d\phi \rangle_G \equiv G_{AB} \langle d\phi^A \, , d\phi^B \rangle = G_{AB}[\phi(x)] g^{\mu\nu}(x) \phi^A{}_{,\mu} \phi^B{}_{,\nu}$. (If the target manifold is a linear space, then this reduces to the ordinary kinetic term for dim(N) decoupled scalar fields.) The important observation is that the stress-energy tensor,

$$\boldsymbol{T} = G_{AB}[\phi] \, d\phi^A \otimes d\phi^B - \boldsymbol{g} \left(\frac{1}{2} \langle d\phi \, , d\phi \rangle_G + P[\phi] \right) \tag{5.7}$$

violates the SEC in the above sense, since

$$\boldsymbol{T}(k,k) - \frac{1}{2} \boldsymbol{g}(k,k) \, \mathrm{tr} \boldsymbol{T} = - V \, P[\phi] \le 0 \, , \tag{5.8}$$

where we have required that ϕ is a stationary mapping, $d\phi^A(k) = L_k \phi^A = 0$. We are now able to conclude that selfgravitating harmonic mappings with non-negative potentials and Riemannian target spaces admit
- no soliton solutions,
- no static, spherically symmetric black hole solutions other than Schwarzschild.

In order to get rid of the requirement of spherical symmetry in the black hole case, one needs either a proof for $M \ge \frac{1}{4\pi} \kappa \mathcal{A}$ or a different argument. A very powerful alternative is provided by divergence identities, as Bekenstein originally used in this context [51]. This method gives, however, only conclusive results for *convex* potentials and target spaces with *nonpositive* sectional curvature (see, e.g. [52]). We shall return to this problem and its partial resolution in sections 7 & 9.

6 Staticity and Circularity

The integrability theorems for the Killing fields provide the link between the strong rigidity theorem and the assumptions on which the classical uniqueness theorems are based. In the nonrotating case one has to show that a stationary domain of outer communications (with Killing field k, say) is static,

$$* \, (k \wedge dk) = 0 \, , \tag{6.1}$$

whereas in the stationary and axisymmetric situation (with Killing fields k and m, say) one must establish the circularity conditions:

$$* \, (m \wedge k \wedge dk) = 0 \, , \quad * \, (k \wedge m \wedge dm) = 0 \, . \tag{6.2}$$

(The Hodge duals are taken for later convenience.) As mentioned in the introduction, the vacuum staticity and circularity theorems were proven by Carter and Hawking, respectively. Whilest Carter also succeeded in establishing the electrovac circularity theorem in the early seventies (see, e.g. [14]), it took, however, some effort until the corresponding staticity issue was settled as well [3].

In this section we shall first show that – under fairly mild assumptions – the Frobenius integrability conditions (6.1) and (6.2) are equivalent to the (local) *Ricci conditions*,

$$* \, (k \wedge R(k)) = 0 \,, \tag{6.3}$$

and

$$* \, (m \wedge k \wedge R(k)) = 0 \,, \quad * \, (k \wedge m \wedge R(m)) = 0 \,, \tag{6.4}$$

respectively. This equivalence will then enable us

- to prove the vacuum staticity theorem for not necessarily connected horizons;
- to prove the staticity and circularity theorems for scalar mappings (e.g. Higgs fields);
- to understand why the electrovac circularity theorem does *not* generalize to YM fields.

We start by noting that the Ricci conditions (6.3) and (6.4) are the derivatives of the Frobenius conditions (6.1) and (6.2), respectively: The staticity condition (6.1) is, of course, equivalent to the vanishing of the twist ω_k assigned to the stationary Killing field ($\omega_k \equiv \frac{1}{2} * [k \wedge dk]$). Hence,

$$\frac{1}{2} \, d * (k \wedge dk) \, = \, d\omega_k \, = \, \frac{1}{2} * \left(k \wedge d^\dagger dk\right) \, = \, * \, (k \wedge R(k)) \,, \tag{6.5}$$

where $d^\dagger \equiv *d*$, and where we have used the Ricci identity for Killing fields in the last step (see, e.g. [2] for details). As for the derivatives of the circularity conditions (6.2), we note that $*(m \wedge k \wedge dk)$ can be written as $-2i_m\omega_k \, (= -2m_\mu \omega_k^\mu)$. We therefore find

$$\frac{1}{2} \, d * (m \wedge k \wedge dk) \, = \, -d \, (i_m \omega_k) \, = \, i_m d\omega_k \, = \, i_m * (k \wedge R(k)) \, = \, *[m \wedge k \wedge R(k)] \,. \tag{6.6}$$

Here we have also used the fact that the Killing fields commute [39]. (See [40] for a powerful proof of $[k, m] = 0$ which does not require the existence of an axis.)

The above connection shows that the Frobenius conditions imply the Ricci conditions. In the stationary and axisymmetric case, the *converse* statement is also easily established, since the vanishing of the 1-form $d(i_m\omega_k)$ implies that the *function* $i_m\omega_k$ is constant. Since, in addition, $i_m\omega_k$ vanishes on the rotation axis, the circularity condition (6.2) follows from the Ricci condition (6.4) in every domain of spacetime intersecting the symmetry axis. In the nonrotating case the situation is more difficult, since $d\omega_k = 0$ does not automatically imply that the twist 1-form itself vanishes. However, using $d(\frac{k}{V}) = \frac{2}{V^2} * (k \wedge \omega)$ (write $*(k \wedge \omega) = -i_k * \omega$ to obtain this), it is not hard to prove the useful identity

$$d \left(\omega \wedge \frac{k}{V}\right) \, = \, d\omega \wedge \frac{k}{V} \, - \, 2\frac{\langle \omega \, , \omega \rangle}{V^2} * k \,, \tag{6.7}$$

where, as usual, $V \equiv -\langle k\,,k\rangle$. Since asymptotic flatness and the general properties of Killing horizons imply that $\omega \wedge \frac{k}{V}$ vanishes at infinity and at (each component of) the Killing horizon, we can apply Stokes' theorem for a spacelike hypersurface to conclude that the integral over the l.h.s. vanishes. Hence, $d\omega = 0$ implies $\int_\Sigma V^{-2}\langle \omega\,, \omega\rangle * k = 0$ and therefore – provided that the domain of outer communications is *strictly* stationary ($V \geq 0$) – also $\omega = 0$. This shows that $\omega = 0$ follows from $d\omega = 0$ whenever the horizon is nonrotating and the domain is strictly stationary. In particular, no restrictions concerning the connectedness of the horizon enter the above argument. (The original proof of the vacuum staticity theorem was based on the fact that $d\omega = 0$ implies the local existence of a potential, and was therefore subject to stronger topological restrictions.) Hence, under the (weak) assumptions used above, it is sufficient to establish the *Ricci* staticity and circularity conditions (6.3), (6.4) in order to conclude that the *Frobenius* integrability conditions (6.1), (6.2) are satisfied.

We shall now demonstrate that the circularity property is a consequence of the field equations for both scalar mappings and *Abelian* gauge fields, whereas the situation is completely different for the EYM system.

6.1 Scalar Mappings

Consider a selfgravitating mapping ϕ from spacetime $(M, \boldsymbol{g})$ into a target manifold $(N, \boldsymbol{G})$ with matter Lagrangian $\mathcal{L} = \mathcal{L}(\boldsymbol{g}, \boldsymbol{G}, \phi, d\phi)$. If $\mathcal{L}$ does not depend on derivatives of the spacetime metric, then the stress-energy tensor becomes

$$\boldsymbol{T} = 2\,\frac{\partial \mathcal{L}}{\partial \boldsymbol{g}^{-1}} - \mathcal{L}\cdot \boldsymbol{g} = f_{AB}\, d\phi^A \otimes d\phi^B - \mathcal{L}\cdot \boldsymbol{g}\,, \tag{6.8}$$

where the $f_{AB} = f_{AB}(\boldsymbol{g}, \boldsymbol{G}, \phi, d\phi)$ depend on the explicit form of the Lagrangian. Let ϕ be invariant under the 1-parameter group of transformations generated by a Killing field ξ, say, $L_\xi \phi = 0$. Then the stress-energy 1-form with respect to ξ, $T(\xi)$, becomes proportional to ξ itself, $T(\xi) = -\mathcal{L}\cdot\xi$, and thus

$$T(\xi) \wedge \xi = 0\,. \tag{6.9}$$

Hence, a stationary mapping ϕ fulfills the Ricci staticity condition (6.3), whereas the Ricci circularity conditions (6.4) are satisfied for stationary and axisymmetric scalar mappings. By virtue of the equivalence between these conditions and the Frobenius integrability conditions (6.1) and (6.2), we are able to conclude that stationary black hole spacetimes coupled to stationary scalar mappings are either static or circular.

6.2 Gauge Fields

As mentioned earlier, the *staticity* theorem for Maxwell fields has been established only recently [3]. In contrast to this, the corresponding circularity problem was solved by Carter already in the early seventies. Here we restrict ourselves to the *circularity* issue, which we will discuss for both electromagnetic *and* YM fields. Our main objective is to understand why – in contrast to the Maxwell case – circularity is not a consequence of the field equations when gravity is coupled to *non-Abelian* gauge fields. In order to do so, we first derive an expression for the 2–form $\xi \wedge T(\xi)$, where ξ stands either for the stationary or the axial Killing field.

We start by recalling that a gauge potential A is symmetric with respect to the action of ξ if there exists a Lie algebra valued function ψ (for every Killing field ξ), such that

$$L_\xi A \;=\; D\psi \tag{6.10}$$

(see, e.g. [53]). Using $L_\xi(dA + A \wedge A) = dL_\xi A + [A\,, L_\xi A] = DL_\xi A = D^2\psi$, the symmetry equation for the field strength (and every other gauge-covariant p-form) becomes

$$L_\xi F \;=\; [F\,,\psi]\,, \tag{6.11}$$

(which, as expected, reduces to $L_\xi F = 0$ in the Abelian case).

It is convenient to introduce the electric, $E \equiv -i_\xi F$, and the magnetic, $B \equiv i_\xi * F$, component of F with respect to the Killing field ξ. (If ξ denotes the timelike and hypersurface orthogonal Killing field of a *static* spacetime then E and B are the ordinary electric and magnetic 1–forms.) Since the inner product, $i_\xi\alpha$, of ξ with an arbitrary p–form α is obtained from $i_\xi\alpha = -*(\xi \wedge *\alpha)$, we have (with $*^2\alpha = -(-1)^p\alpha$)

$$E \;=\; *(\xi \wedge *F)\;, \qquad B \;=\; *(\xi \wedge F)\;. \tag{6.12}$$

An important consequence of the symmetry condition (6.10) is the fact that – for every Killing field ξ – there exists an *electric potential* W: Since $E = -i_\xi(dA + A \wedge A) = -L_\xi A + di_\xi A + [A\,, i_\xi A] \;=\; -D\psi + D(i_\xi A)$, one finds

$$E \;=\; DW\,, \quad \text{with } W \;\equiv\; i_\xi A \;-\; \psi\,. \tag{6.13}$$

Now using the definitions (6.12) and some basic identities for Killing fields (see, e.g. [2]) it is not difficult to verify that the source-free YM equations $D * F = 0$, together with the Bianchi identity $DF = 0$, are equivalent to the following set of equations

$$*\,D * \left(\frac{E}{N}\right) \;=\; 2\,\frac{\langle\omega\,,B\rangle}{N^2}\,, \qquad E \;=\; DW\,, \tag{6.14}$$

$$*\,D * \left(\frac{B}{N}\right) \;=\; -\,2\,\frac{\langle\omega\,,E\rangle}{N^2}\,, \qquad DB \;=\; -*DE\,, \tag{6.15}$$

where N and ω are the norm and the twist, respectively, assigned to the Killing field ξ, $N \equiv \langle\xi\,,\xi\rangle$, $\omega = \frac{1}{2}*(\xi \wedge d\xi)$.

We are now able to express the 2–form $\xi \wedge T(\xi)$ in terms of the electric potential W and the magnetic 1-form B. In order to do so, we use the YM stress-energy tensor

$$T_{\mu\nu} \;=\; \frac{1}{4\pi}\,\mathrm{tr}\left\{F_{\mu\sigma}F_\nu{}^\sigma \;-\; \frac{1}{2}g_{\mu\nu}\langle F\,,F\rangle\right\}\,, \tag{6.16}$$

and the above definitions for E and B to obtain

$$\xi \wedge T(\xi) = \frac{1}{4\pi} * \mathrm{tr}\,\{E \wedge B\} \,. \tag{6.17}$$

Now using the YM equations $E = DW$, $DB = - * D^2 W = - * [F\,, W]$ and the fact that $\mathrm{tr}\,\{W[F\,, W]\}$ vanishes, we find $\mathrm{tr}\,\{E \wedge B\} = \mathrm{tr}\,\{DW \wedge B\} = \mathrm{tr}\,\{D(WB)\} - \mathrm{tr}\,\{WDB\} = d(\mathrm{tr}\,\{WB\})$. We therefore obtain the result

$$* \,(\xi \wedge T(\xi)) = -\frac{1}{4\pi} d\,(\mathrm{tr}\,\{W\,B\}) = -\frac{1}{4\pi} d\,(\mathrm{tr}\,\{W\,i_\xi * F\}) \,. \tag{6.18}$$

By virtue of the general identity (6.5) and Einstein's equations, the l.h.s. is (up to a factor of $\frac{1}{8\pi}$) exactly the derivative of the twist assigned to ξ. Since the r.h.s. is an exact differential-form as well, we can integrate the equation and introduce a "combined twist potential" U, such that

$$dU = \omega + 2\,\mathrm{tr}\,\{WB\} \,. \tag{6.19}$$

Hence, we have expressed the twist in terms of an integration function U and the gauge fields. For the stationary Killing field k, the Frobenius staticity condition therefore becomes

$$\omega_k = dU - 2\,\mathrm{tr}\,\{W_k\, i_k * F\} = 0\,, \tag{6.20}$$

where W_k is the electric potential with respect to k; see eq. (6.13).

In a similar way one can integrate the right hand sides of the Ricci *circularity* conditions in order to obtain an expression for the Frobenius conditions (6.1), (6.2). Since the latter can be written as $i_m \omega_k = 0$ and $i_k \omega_m = 0$, we can also use the above formula for the twist, which yields the desired expressions

$$i_m \omega_k = -2\,\mathrm{tr}\,\{W_k\,(*F)(k,m)\} = 0\,, \quad i_k \omega_m = 2\,\mathrm{tr}\,\{W_m\,(*F)(k,m)\} = 0\,, \tag{6.21}$$

where $(*F)(k,m) = i_m i_k * F$. (Here we have used the circumstance that the constants of integration vanish.) Carter has derived the connection between the circularity conditions and the gauge fields in the Abelian case. His formula involves magnetic potentials, which locally exist as a consequence of the *Maxwell* equations $dB_k = dB_m = 0$. The above formulae (6.20), (6.21) show that even in the *non-Abelian* case – where the magnetic potentials do not exist any more – one can integrate the Ricci conditions and obtain an explicit expression for the Frobenius conditions in terms of the gauge fields. (The same result can be established if Higgs fields are taken into account as well.)

A very important peculiarity of the *Abelian* case is the fact that both $F(k,m)$ and $(*F)(k,m)$ vanish as a consequence of the symmetry properties and Maxwell's equations. This is seen as follows: Since k and m commute, the operator $d i_m i_k$ can be cast into the form

$$d\,i_m i_k = i_k L_m - i_m L_k + i_m i_k\, d\,. \tag{6.22}$$

Applying this to F and $*F$, and using the fact that the Hodge dual and the Lie derivative with respect to a Killing field commute, we immediately find (with $L_k F = L_m F = 0$)

$$d\,[F(k,m)] = i_m i_k dF = 0\,, \quad d\,[(*F)(k,m)] = i_m i_k d * F = 0\,. \tag{6.23}$$

Hence, the functions $F(k,m)$ and $(*F)(k,m)$ are constant and vanish on the rotation axis. We therefore have $F(k,m)=0$ and $(*F)(k,m)=0$ where, by virtue of eq. (6.21), the latter property implies that the Frobenius circularity conditions are fulfilled. This is Carter's electrovac circularity theorem, establishing the integrability property for the Killing fields on the basis of the Maxwell equations and the symmetry properties of the gauge fields.

In contrast to this, circularity does *not* follow from the field equations in the non-Abelian case. That is, the quantities $\mathrm{tr}\,\{W_k(*F)(k,m)\}$ and $\mathrm{tr}\,\{W_m(*F)(k,m)\}$ do not vanish as a consequence of $DF=0$, $D*F=0$ and the symmetry equations $L_kA=D\psi_k$, $L_mA=D\psi_m$. In particular, the diagonal boxes of the fieldstrength tensor are no longer automatically zero. Instead of eq. (6.23) we now have

$$\begin{aligned} D\,[F(k,m)] &= [E_k\,,W_m] - [E_m\,,W_k]\,, \\ D\,[(*F)(k,m)] &= [B_m\,,W_k] - [B_k\,,W_m]\,. \end{aligned} \tag{6.24}$$

(Use eq. (6.22) and $L_mW_k=[W_k\,,\psi_m]$ to derive this. Also note that $F(k,m)=[W_k\,,W_m]$, since $i_mi_kF = -i_mE_k = -i_mDW_k = -L_mW_k - [i_mA\,,W_k] = -[W_k\,,\psi_m]+[W_k\,,i_mA] = [W_k\,,W_m]$.)

The above arguments also imply that the usual (2+2)-split of the metric of a stationary and axisymmetric spacetime imposes additional constraints on the YM fields. Writing the metric in the circular Papapetrou form and considering the corresponding ansatz for the gauge potential,

$$\boldsymbol{g} = g_{tt}\,dt\otimes dt + 2\,g_{t\varphi}\,dt\otimes d\varphi + g_{\varphi\varphi}\,d\varphi\otimes d\varphi + \tilde{\boldsymbol{g}}\,,$$

$$A = W_k\,dt + W_m\,d\varphi + \tilde{A}\,, \tag{6.25}$$

the integrability conditions require that the scalar $\tilde{*}\tilde{F}$ is orthogonal to W_k and W_m with respect to the inner algebra product,

$$\mathrm{tr}\left\{W_\xi\,\tilde{*}\tilde{F}\right\} = 0\,, \quad \text{for}\ \xi = k,m\,. \tag{6.26}$$

A detailed discussion of the circularity issue for non-Abelian gauge fields will be presented elsewhere [54].

7 The No-Hair Theorem for Harmonic Mappings

Selfgravitating scalar mappings from spacetime $(M,{}^{(4)}\boldsymbol{g})$ into Riemannian manifolds $(N,\boldsymbol{G})$ admit both soliton and black hole solutions with hair. A particular example is provided by the Skyrme model [7], which can be described as a static mapping into S^3 [55] with matter action $\int(e_2[\phi]+e_4[\phi])*1$, where $e_2[\phi]\equiv\frac{1}{2}G_{AB}[\phi]\langle d\phi^A\,,d\phi^B\rangle$ is the harmonic part of the action and $e_4[\phi]\equiv\frac{1}{4}G_{AC}G_{BD}\langle d\phi^A\wedge d\phi^B\,,d\phi^C\wedge d\phi^D\rangle$. If, however, only the *harmonic* part is taken into account, then the situation changes dramatically and one obtains the following uniqueness theorem:

The Kerr metric is the unique solution with nondegenerate Killing horizon ($\kappa \neq 0$) amongst all stationary black hole solutions of selfgravitating *harmonic* mappings into *Riemannian* target manifolds $(N, \boldsymbol{G})$, with action

$$\mathcal{A} = \int \Big(-\frac{1}{16\pi} R + \frac{1}{2} G_{AB}[\phi] \langle d\phi^A , d\phi^B \rangle \Big) * 1 . \tag{7.1}$$

The proof of this theorem involves the following three main steps: First, one has to show that all stationary black hole solutions to the above action are either static ($k \wedge dk = 0$) or circular ($m \wedge k \wedge dk = k \wedge m \wedge dm = 0$). In section 6 we have argued that this is indeed the case, i.e., that the Killing fields fulfill the integrability conditions. It therefore remains to prove that

- there are no static black hole solutions other than Schwarzschild,
- there are no circular black hole solutions other than Kerr.

Under the additional assumption of spherical symmetry we have already presented a proof of the first assertion in section 5 – playing off the strong and the dominant energy conditions against each other. Here we shall only give a brief outline of the rationale leading to the above results, and refer to [2] for details.

7.1 The Static Case

With respect to the static metric

$$^{(4)}\boldsymbol{g} = -S^2 dt^2 + \boldsymbol{g} , \tag{7.2}$$

the Einstein equations obtained from the action (7.1) become

$$\Delta^{(g)} S = 0 , \quad R^{(g)} = \kappa \, G_{AB} \, g^{ij} \, \phi^A{}_{,i} \, \phi^B{}_{,j} , \tag{7.3}$$

$$R_{ij}^{(g)} - S^{-1} \nabla_j^{(g)} \nabla_i^{(g)} S = \kappa \, G_{AB} \, \phi^A{}_{,i} \, \phi^B{}_{,j} , \tag{7.4}$$

where Latin indices refer to the spatial metric $\boldsymbol{g}$. In addition, we have the matter equations for the scalar fields ϕ^A, $\Delta^{(g)} \phi^A + S^{-1}(dS | d\phi^A) + \Gamma^A_{BC}(\phi)(d\phi^B | d\phi^C) = 0$. Like in the new proof of the static vacuum [23] and electrovac [56] [24] uniqueness theorems, the strategy is to show that the 3-dimensional spacelike manifold $(\Sigma, \boldsymbol{g})$ is conformally flat. In fact, the conformal factor which yields the desired result is the same as in the vacuum case,

$$\Omega_\pm = \frac{1}{4}(1 \pm S)^2 . \tag{7.5}$$

More precisely, we consider the manifold $\hat{\Sigma}$ which is composed of the two copies $\hat{\Sigma}_+$ and $\hat{\Sigma}_-$ of Σ, pasted together along their boundaries $\mathcal{H}_+$ and $\mathcal{H}_-$. Furnishing $\hat{\Sigma}_+$ and $\hat{\Sigma}_-$ with the metrics

$$\hat{\boldsymbol{g}}_\pm = \Omega_\pm^2 \, \boldsymbol{g} , \tag{7.6}$$

one can show that the metric and the second fundamental form match continuously on $\mathcal{H}_\pm$. Moreover, $(\hat{\Sigma}_+, \hat{\boldsymbol{g}}_+)$ is asymptotically flat with vanishing mass and non-negative Ricci curvature, $\hat{R} \geq 0$: Using the expansions $\boldsymbol{g} = (1+\frac{2M}{r})\boldsymbol{\delta}+\mathcal{O}(r^{-2})$ and $\Omega_+ = 1-\frac{M}{r}+\mathcal{O}(r^{-2})$ one finds $\hat{\boldsymbol{g}}_+ = \Omega_+\boldsymbol{g} = 1 + \mathcal{O}(r^{-2})$, which shows that $(\hat{\Sigma}_+, \hat{\boldsymbol{g}}_+)$ has vanishing mass. Computing the conformal transformation of the Ricci scalar gives

$$\begin{aligned}\frac{\Omega_\pm^4}{2}\hat{R} &= \frac{\Omega_\pm^2}{2}R^{(g)} - 2\,\Omega_\pm\,\Delta^{(g)}\Omega_\pm + \langle d\Omega_\pm\,, d\Omega_\pm\rangle^{(g)} \\ &= \frac{\Omega_\pm^2}{2}R^{(g)} - \Omega_\pm\,(S \pm 1)\Delta^{(g)}S = \frac{\Omega_\pm^2}{2}\,\kappa\langle d\phi\,, d\phi\rangle_G \geq 0\,,\end{aligned} \tag{7.7}$$

where we have used eqs. (7.3) and the fact that the target metric $\boldsymbol{G}$ is Riemannian.

Since $(\hat{\Sigma}, \hat{\boldsymbol{g}})$ constructed in this way is an asymptotically flat, complete, orientable 3-dimensional Riemannian manifold with non-negative Ricci curvature and vanishing mass, it is – as a consequence of the positive mass theorem – isometric to $(\mathbb{R}^3, \boldsymbol{\delta})$. Hence, the spatial geometry of the domain of outer communications is conformally flat, $\hat{R} = 0$, which implies that the scalar fields assume constant values. The field equations (7.3), (7.4) therefore reduce to the vacuum Einstein equations. Finally, the fact that the static vacuum solutions with conformally flat 3-geometry are spherically symmetric concludes the proof of the static no-hair theorem for harmonic mappings [15].

7.2 The Stationary and Axisymmetric Case

Like in the static situation, the uniqueness problem for stationary and axisymmetric black hole solutions of selfgravitating harmonic mappings can be reduced to the corresponding vacuum problem. This is due to the following observations:

• First, the twist is closed, since *harmonic* mappings have the special property that the Ricci 1-form (with respect to either of the Killing fields) vanishes,

$$R(k) = R(m) = 0\,. \tag{7.8}$$

(Use $L_k\phi^A = L_m\phi^A = 0$ and eq. (5.7) with $P[\phi] = 0$ to obtain this.) One can therefore introduce the same Ernst potential as in the vacuum case,

$$\mathrm{E} = -X + iY\,, \quad \text{where } X \equiv \langle m\,, m\rangle\,,\ dY \equiv 2\omega = *(m \wedge dm)\,. \tag{7.9}$$

• Second, the general identity $X\Delta X - \langle dX\,, dX\rangle + 4\langle\omega\,,\omega\rangle + 2XR(m,m) = 0$ reduces to the following equation for E

$$\frac{1}{\rho}\,d^{\dagger(\gamma)}(\rho\, d\mathrm{E}) = \frac{1}{X}\,\langle d\mathrm{E}\,, d\mathrm{E}\rangle^{(\gamma)}\,. \tag{7.10}$$

This is again a consequence of $R(m) = 0$ and the fact that a circular spacetime admits a foliation by 2-dimensional integrable surfaces orthogonal to the Killing fields k and m, implying that the metric can be written in the Papapetrou (2+2) form

$$^{(4)}\boldsymbol{g} = \boldsymbol{\sigma} + \frac{1}{X}\boldsymbol{\gamma} = -\frac{\rho^2}{X}\,dt^2 + X\,(d\varphi + A\,dt)^2 + \frac{1}{X}\,\boldsymbol{\gamma}\,, \tag{7.11}$$

where the function A is related to the twist potential Y by $dA = \rho X^{-2} *^{(\gamma)} dY$.

• Third, the determinant $\rho = \sqrt{-\det(\boldsymbol{\sigma})}$ is subject to the equation $X\rho\Delta^{(\gamma)}\rho = \langle k\,,k\rangle R(m,m) + \langle m\,,m\rangle R(k,k) - 2\langle k\,,m\rangle R(k,m)$ which, like in the vacuum case, reduces to

$$\Delta^{(\gamma)}\rho \;=\; 0\,. \tag{7.12}$$

This shows that ρ is a harmonic function on the 2-dimensional Riemannian manifold $(\Gamma, \boldsymbol{\gamma})$. One can therefore introduce Weyl coordinates, that is, one can use ρ and its conjugate harmonic function z, say, as coordinates on $(\Gamma, \boldsymbol{\gamma})$ (see [57] for an elegant and complete proof).

The above observations imply that eq. (7.9) for the twist and the norm of the axial Killing field decouples from the remaining field equations and is, in fact, identical to the *vacuum* Ernst equation,

$$\frac{1}{\rho}\,\underline{\nabla}(\rho\,\underline{\nabla}\mathrm{E}) \;+\; \frac{(\underline{\nabla}\mathrm{E}|\underline{\nabla}\mathrm{E})}{X} \;=\; 0\,, \tag{7.13}$$

where $\underline{\nabla} \;=\; (\partial_\rho, \partial_z)$. Since the latter describes a regular boundary-value problem on a fixed 2-dimensional background space, one can apply the vacuum Mazur [31] (Robinson [30]) identity to conclude that the Ernst potential is uniquely determined by the boundary conditions. Also using $\frac{1}{\rho}A_{,\rho} = X^{-2}Y_{,z}$ and $\frac{1}{\rho}A_{,z} = -X^{-2}Y_{,\rho}$ shows that the metric $\boldsymbol{\sigma}$ of the orbit manifold is identical with the corresponding vacuum solution. It remains to discuss the orthogonal set of equations for the scalar fields and the metric $\boldsymbol{\gamma}$.

Like the Ernst equation, the field equations for the harmonic scalar field ϕ,

$$\frac{1}{\rho}\,\underline{\nabla}(\rho\,\underline{\nabla}\,\phi^A) \;+\; \Gamma^A_{BC}\,(\underline{\nabla}\phi^B|\underline{\nabla}\phi^C) \;=\; 0\,, \tag{7.14}$$

involve no unknown metric functions. The only remaining metric function $h(\rho, z)$ – defined by $\boldsymbol{\gamma} = e^{2h}(d\rho^2 + dz^2)$ – is therefore obtained by quadrature from the Ernst potential E and the solution ϕ to the above matter equation: Writing $h = h^0 + 8\pi G\tilde{h}$, it remains to integrate

$$\frac{1}{\rho}h^0{}_{,\rho} = \frac{1}{4X^2}\left[\mathrm{E}_{,\rho}\,\bar{\mathrm{E}}_{,\rho} - \mathrm{E}_{,z}\,\bar{\mathrm{E}}_{,z}\right]\,,\quad \frac{1}{\rho}h^0{}_{,z} = \frac{1}{4X^2}\left[\mathrm{E}_{,\rho}\,\bar{\mathrm{E}}_{,z} + \mathrm{E}_{,z}\,\bar{\mathrm{E}}_{,\rho}\right] \tag{7.15}$$

and

$$\frac{1}{\rho}\tilde{h}_{,\rho} = \frac{G_{AB}[\phi]}{2}\left[\phi^A{}_{,\rho}\,\phi^B{}_{,\rho} - \phi^A{}_{,z}\,\phi^B{}_{,z}\right]\,,\quad \frac{1}{\rho}\tilde{h}_{,z} = \frac{G_{AB}}{2}\left[\phi^A{}_{,\rho}\,\phi^B{}_{,z} + \phi^A{}_{,z}\,\phi^B{}_{,\rho}\right]\,. \tag{7.16}$$

The last step in the uniqueness proof is to show that the matter equations (7.14) and (7.16) admit only the trivial solution $\phi^A = \phi^A_0$, $\tilde{h} = 0$. Using Stokes' theorem, this is established as a consequence of asymptotic flatness, the fall-off conditions

$$\phi^A = \phi^A_\infty + \mathcal{O}(r^{-1})\,,\quad \phi^A{}_{,r} = \mathcal{O}(r^{-2})\,,\quad \phi^A{}_{,\vartheta} = \mathcal{O}(r^{-1}) \tag{7.17}$$

and the requirement that $G_{AB}[\phi]$ and the derivatives of the scalar fields with respect to the Boyer-Lindquist coordinates r and ϑ remain finite at the boundary of the domain of outer communications (see [58] for details). This finally establishes the uniqueness of the Kerr metric amongst the stationary and axisymmetric black hole solutions of selfgravitating harmonic mappings.

8 Integral Identities

Integral identities play an outstanding role in the derivation of various uniqueness results. In fact, both Israel's uniqueness theorem for nonrotating vacuum black holes [19] and Robinson's theorem for the corresponding rotating situation [30] were based on ingenious applications of Stokes' law. Moreover, the fact that the electric potential depends only on the gravitational potential – a result which opened the way for the static electrovac uniqueness theorem – was also obtained by this technique [20]. Finally, it was a divergence identity for sigma-models on symmetric spaces which provided the key to the uniqueness proof for the Kerr-Newman metric [31], [32]. Moreover, many nonexistence results for *soliton* configurations are also based on integral formulae. For instance, the integrated version of the Ricci identity for a stationary Killing field, $d * dk = 2 * R(k)$, provides the link between the total mass, $M = -8\pi G \int_\infty *dk$, and the volume integral over the stress-energy 1-form $\boldsymbol{T}(k) - \frac{1}{2}\mathrm{tr}(\boldsymbol{T})k$. In combination with the positive mass theorem, this excludes the existence of purely gravitational solitons other than flat spacetime.

For spacetimes admitting a stationary – not necessarily static – isometry (with Killing field k, say), it turns out to be convenient to use the following version of Stokes' theorem: Consider a 1-form α which is invariant under the action of k, $L_k\alpha = 0$. Then, using the fact that the Lie derivative with respect to a Killing field commutes with the Hodge dual, one finds $0 = *L_k\alpha = L_k*\alpha = i_k d*\alpha + d i_k * \alpha$. Since $d^\dagger\alpha$ is a 0-form, we have $i_k d*\alpha = -(d^\dagger\alpha) i_k\eta$ and $i_k * \alpha = - * (k \wedge \alpha)$. Hence, integrating over a spacelike hypersurface Σ with boundary $\partial\Sigma$, Stokes' formula becomes

$$\int_{\partial\Sigma} * (k \wedge \alpha) \;=\; - \int_\Sigma (d^\dagger\alpha)\, i_k\eta\,. \tag{8.1}$$

Using the field equations (matter or gravitational) for $d^\dagger\alpha$ one now obtains, for instance, relations between asymptotic charges and quantities defined on the horizon. In the best case one can combine the field equations such that the sum of the boundary terms on the l.h.s. vanishes and the sum of the integrands on r.h.s. has a fixed sign. The method will be explained in the last section for the Einstein-Maxwell equations. A well-known example of this kind is Bekenstein's no-hair theorem for scalar fields.

9 The Bekenstein Argument

In 1972 Bekenstein showed that there are no black holes with nontrivial massive scalar fields [51]. His strategy was to multiply the matter equation $(\Delta - m^2)\phi = 0$ by ϕ, integrating by parts and using Stokes' theorem for the resulting identity. More generally, one can consider an arbitrary function $f[\phi]$ of the scalar field and choose $\alpha = f d\phi$ in eq. (8.1). The scalar field equation (with potential $P[\phi]$),

$$\Delta\phi \;=\; -\, d^\dagger d\phi \;=\; P_{,\phi}\;, \tag{9.1}$$

then yields

$$d^\dagger\alpha = d^\dagger(f\,d\phi) = -\langle df\,,d\phi\rangle + f\,d^\dagger d\phi = -f_{,\phi}\langle d\phi\,,d\phi\rangle - fP_{,\phi}\,.$$

Now using Stokes' theorem (8.1) and the fact that the scalar field assumes a constant value on the horizon and at spacelike infinity, we find

$$4\pi\,(f[\phi_\infty]\,s_\infty - f[\phi_H]\,s_H) = \int_\Sigma (f_{,\phi}\langle d\phi\,,d\phi\rangle + f\,P_{,\phi})\,i_k\eta\,, \tag{9.2}$$

where we have defined $s_{\infty(H)} \equiv \frac{1}{4\pi}\int_{S^2_\infty(H)} *(k\wedge d\phi)$. Using general properties of Killing horizons, it is not difficult to see that s_H vanishes. Hence, provided that $f[\phi_H]$ is finite, the horizon term on the l.h.s. does not contribute. Our aim is to choose $f[\phi]$ such that the asymptotic term vanishes as well. Since asymptotic flatness implies that the scalar charge s_∞ is finite, it is sufficient to require $f[\phi_\infty]=0$. Following Bekenstein [51], we set $f[\phi]=\phi-\phi_\infty$. The above formula then gives

$$\int_\Sigma [\langle d\phi\,,d\phi\rangle + (\phi-\phi_\infty)\,P_{,\phi}]\,i_k\eta = 0\,,\quad \text{if } |\phi_H-\phi_\infty|<\infty, \tag{9.3}$$

where we recall that the horizon contribution was argued away only for $|f[\phi_H]|<\infty$. Considering a convex, non-negative potential with $P_{,\phi}(\phi_\infty)=0$, the integrand is non-negative and the above relation shows that ϕ must assume the constant value ϕ_∞. (Also note that $d\phi$ is nowhere timelike since $\langle k\,,d\phi\rangle = L_k\phi = 0$ and since the domain is required to be strictly stationary.) It is worth pointing out that the Bekenstein black hole solution for a free scalar field [59] does not contradict the above argument, since the latter applies only to scalar fields which remain finite on the horizon.

A slightly different version of the above result is obtained by setting $f[\phi]=P_{,\phi}$, i.e., $\alpha = f\,d\phi = dP$ [2]. Using again asymptotic flatness and the fact that the curvature invariant $R_{\mu\nu}R^{\mu\nu}$ must remain finite on the horizon enables one to conclude that

$$\int_\Sigma \left[P_{,\phi\phi}\langle d\phi\,,d\phi\rangle + (P_{,\phi})^2\right] i_k\eta = 0\,. \tag{9.4}$$

This shows that there are only trivial scalar field configurations for black hole solutions with nondegenerate, nonrotating horizon and strictly stationary domain of outer communications.

I am not aware of a uniqueness theorem for the Schwarzschild metric for scalar fields with *arbitrary*, non-negative potentials – unless spherical symmetry is imposed (see sections 4 & 5). In this connection, it is probably worth noticing that the Bekenstein argument makes no essential use of Einstein's equations. In an attempt to include more information one can, for instance, take the R_{00} equation into account as well,

$$d^\dagger\left(\frac{dV}{V}\right) = -\frac{2}{V}R(k,k) = 2\kappa\,P[\phi]\,. \tag{9.5}$$

Choosing $\alpha = g(\phi,V)\,\frac{dV}{V}$ and $\alpha = f(\phi,V)\,d\phi$ in Stokes' theorem (8.1) yields the two integral equations

$$\begin{aligned} 8\pi\left(g_\infty M - g_H\frac{\kappa\mathcal{A}}{4\pi}\right) &= \int_\Sigma\left[g_{,V}\frac{\langle dV\,,dV\rangle}{V} + g_{,\phi}\frac{\langle d\phi\,,dV\rangle}{V} - \kappa\,g\,P\right] i_k\eta\,,\\ 4\pi\,(f_\infty\,s_\infty - f_H\,s_H) &= \int_\Sigma [f_{,\phi}\langle d\phi\,,d\phi\rangle + f_{,V}\langle dV\,,d\phi\rangle + f\,P_{,\phi}]\,i_k\eta\,, \end{aligned} \tag{9.6}$$

where we have also used the Komar formula, the identity $*(k \wedge \frac{dV}{V}) = - * dk - 2\frac{k}{V} \wedge \omega$ and the fact that the boundary integral of $\frac{k}{V} \wedge \omega$ vanishes. One can now try to combine these formulae in order to derive relations between the boundary terms. A proof of the scalar uniqueness theorem for arbitrary non-negative potentials along these or other lines would close a longstanding gap.

The technique presented in this section can also be applied to selfgravitating harmonic mappings (generalized scalar fields). This yields uniqueness theorems for mappings into target manifolds with nonpositive sectional curvature (see [52]). Since this restriction was absent in the uniqueness proof presented in section 7.1., the conformal techniques are, in this case, more powerful than the divergence identities.

10 The Israel Theorem

This celebrated theorem establishes that all static black hole solutions of Einstein's vacuum equations are spherically symmetric [19]. Israel was able to obtain this result – and its extension to electrovac spacetimes – by considering a particular foliation of the static 3-dimensional hypersurface Σ: Requiring that $S \equiv \sqrt{-\langle k\,, k\rangle}$ is an admissible coordinate (see also [21]), the spacetime metric can be written in the (1+1+2)-split

$$^{(4)}\boldsymbol{g} \;=\; -S^2\, dt \otimes dt \,+\, \rho^2\, dS \otimes dS \,+\, \tilde{\boldsymbol{g}}\,, \tag{10.1}$$

where both ρ and the metric $\tilde{\boldsymbol{g}}$ depend on S and the coordinates of the 2-dimensional surfaces with constant S. With respect to the tetrad fields $\theta^0 = Sdt$ and $\theta^1 = \rho dS$ one finds

$$G_{00} + G_{11} \;=\; \frac{1}{\rho}\left[\frac{K}{S} - \frac{\partial K}{\partial S} - \frac{\rho}{2}K^2\right] \,-\, \frac{2}{\sqrt{\rho}}\,\tilde{\Delta}\sqrt{\rho} \,-\, \left[\frac{\langle\tilde{\nabla}\rho\,,\tilde{\nabla}\rho\rangle}{2\rho^2} + \overset{\circ}{K}_{ab}\overset{\circ}{K}{}^{ab}\right], \tag{10.2}$$

$$G_{00} + 3\,G_{11} \;=\; \frac{1}{\rho}\left[3\frac{K}{S} \,-\, \frac{\partial K}{\partial S}\right] - \tilde{R} \,-\, \tilde{\Delta}\ln\rho \,-\, \left[\frac{\langle\tilde{\nabla}\rho\,,\tilde{\nabla}\rho\rangle}{\rho^2} + 2\,\overset{\circ}{K}_{ab}\overset{\circ}{K}{}^{ab}\right], \tag{10.3}$$

where $K_{ab} = -(2\rho)^{-1}\partial\tilde{g}_{ab}/\partial S$ is the extrinsic curvature of the embedded surface $S = const.$ in Σ, and $\overset{\circ}{K}_{ab} \equiv K_{ab} - \frac{1}{2}\tilde{g}_{ab}K$ is the tracefree part of K_{ab}. Using the Poissson equation

$$\frac{\partial\rho}{\partial S} \;=\; \rho^2\,(K \,-\, \rho S\, R_{00})\,, \tag{10.4}$$

and the vacuum Einstein equations, one obtains the following inequalities from eqs. (10.2) and (10.3):

$$\frac{\partial}{\partial S}\left(\frac{\sqrt{\tilde{g}}}{\sqrt{\rho}}\frac{K}{S}\right) \;\leq\; -2\,\frac{\sqrt{\tilde{g}}}{S}\,\tilde{\Delta}\sqrt{\rho}\,, \qquad \frac{\partial}{\partial S}\left(\frac{\sqrt{\tilde{g}}}{\rho}\,[KS + \frac{4}{\rho}]\right) \;\leq\; -\,S\,\sqrt{\tilde{g}}\,(\tilde{\Delta}\ln\rho + \tilde{R})\,, \tag{10.5}$$

where equality holds if and only if $\overset{\circ}{K}_{ab} = \tilde{\nabla}\rho = 0$. The strategy is now to integrate the above estimates over a spacelike hypersurface extending from the horizon ($S = 0$) to S^2_∞ ($S = 1$).

Using the Gauss-Bonnet theorem, $\int_S \tilde{R}\,\tilde{\eta} = 8\pi$, one finds

$$\left[\int_S \frac{K}{\sqrt{\rho S}}\,\tilde{\eta}\right]_0^1 \le 0\,, \quad \left[\int_S \frac{KS + 4\rho^{-1}}{\rho}\,\tilde{\eta}\right]_0^1 \le -4\pi\,. \tag{10.6}$$

Finally, taking advantage of asymptotic flatness and the fact that on the horizon $K_{ab} = 0$ and $K/S = \rho\tilde{R}/2$, one obtains $8\pi\sqrt{M} - 4\pi\sqrt{\rho_H}$ for the first integrand, and $-4\mathcal{A}\rho_H^{-2}$ for the second one (where $\mathcal{A}$ denotes the area of the horizon). Since $1/\rho_H$ equals the surface gravity of the horizon, $1/\rho_H = \kappa$, the above inequalities eventually give the estimates

$$M \le \frac{1}{4\kappa}\,, \quad \frac{1}{4\pi}\mathcal{A}\kappa \ge \frac{1}{4\kappa}\,, \tag{10.7}$$

where we recall that equality holds if and only if both $\mathring{K}_{ab}$ and $\tilde{\nabla}\rho$ vanish. Since the Komar expression for the total mass of a static *vacuum* black hole spacetime yields $M = \frac{1}{4\pi}\kappa\mathcal{A}$ (see eq. (5.1)), we conclude that equality must hold in the above estimates. This is, however, equivalent to

$$K_{ab} - \frac{1}{2}\tilde{g}_{ab}K = 0\,, \quad \tilde{\nabla}\rho = 0\,. \tag{10.8}$$

Using this in eqs. (10.2) and (10.4) shows that both ρ and K depend only on S. Equation (10.3) then implies that $\tilde{R}$ is constant on the surfaces of constant S. Explicitly one finds $\rho = 4c(1-S^2)^{-2}$, $K = c^{-1}S(1-S^2)$ and $\tilde{R} = \frac{1}{2}c^{-2}(1-S^2)^2$, where c is a constant of integration. Hence, defining $r(S)$ by the relation $\tilde{R} = 2/r^2$, yields (with $c = M$)

$$S^2 = 1 - \frac{2M}{r}\,, \quad \rho^2 dS^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2\,, \quad \tilde{\boldsymbol{g}} = r^2 d\Omega^2\,, \tag{10.9}$$

which is the familiar form of the Schwarzschild metric.

11 The Mazur Identity

Before we give an outline of Mazur's uniqueness proof for the Kerr-Newman metric, we briefly recall some basic facts about the Einstein-Maxwell equations with a Killing field. We consider the axial Killing field m with norm $X = \langle m\,, m\rangle$ and twist $\omega_m = \frac{1}{2} * (m \wedge dm)$. Together with the symmetry condition $L_m F = 0$ for the electromagnetic field tensor (2-form) F, the Bianchi identity ($dF = 0$) and the Maxwell equation ($d * F = 0$) imply that there exist two potentials ϕ and ψ, such that $d\phi = -i_m F$ and $d\psi = i_m * F$. In addition, the identity $d\omega_m = *[m \wedge R(m)]$ implies $d\omega_m = -2d\phi \wedge d\psi$ and thus the existence of a twist potential Y, such that $\frac{1}{2}dY = \omega_m + \phi d\psi - \psi d\phi$. The $R(m,m)$ Einstein equation, $d^\dagger(\frac{dX}{X}) - \frac{4}{X^2}\langle \omega_m\,, \omega_m\rangle = \frac{2}{X}(\langle d\phi\,, d\phi\rangle + \langle d\psi\,, d\psi\rangle)$, and the remaining Maxwell equations, $d^\dagger(\frac{d\phi}{X}) = \frac{2}{X^2}\langle \omega_m\,, d\psi\rangle$ and $d^\dagger(\frac{d\psi}{X}) = -\frac{2}{X^2}\langle \omega_m\,, d\phi\rangle$, are then equivalent to the Euler-Lagrange equations for the Lagrangian

$$\mathcal{L}[\mathrm{E}, \phi] = \frac{|d\mathrm{E} + 2\bar{\Lambda}d\Lambda|^2}{X^2} + 4\,\frac{|d\Lambda|^2}{X} \tag{11.1}$$

where $|d\Lambda|^2 \equiv \langle d\Lambda\,, d\bar{\Lambda}\rangle$, and where the Ernst potentials E and Λ are given in terms of the four potentials X, Y, ϕ and ψ:

$$\mathrm{E} \,=\, -\,\left(X+\phi^2+\psi^2\right)\,+\,i\,Y\,, \quad \Lambda \,=\, -\,\phi\,+\,i\,\psi\,. \tag{11.2}$$

(It is worth noticing that there is no need to use Weyl coordinates and the (2+2)-split of the spacetime metric at this point: The structure of the Ernst system and the sigma-model identities associated with it concern the symmetries of the *target* manifold. The formulation requires only the existence of the above potentials, which can be introduced whenever spacetime admits at least *one* Killing field. The existence of a further Killing field (together with the integrability properties) is then responsible for the reduction of the equations to a boundary value problem on a fixed, 2-dimensional domain.)

The key to the uniqueness theorem for rotating electrovac black hole spacetimes is the observation that the above Lagrangian (11.1) describes a nonlinear sigma-model on the coset space $SU(1,2)/S(U(1){\times}U(2))$ [31], [60]. (In the vacuum case one obtains, instead, a mapping with target manifold $SU(1,1)/U(1)$. If the dimensional reduction is performed with respect to the stationary Killing field, then the target manifold becomes $SU(1,2)/S(U(1){\times}U(1,1))$.) In terms of the hermitian matrix Φ,

$$\Phi_{ab} \,=\, \eta_{ab} \,+\, 2\,\bar{v}_a v_b\,, \quad \text{with } (v_0, v_1, v_2) \,=\, \frac{1}{2\sqrt{X}}\,(\mathrm{E}-1\,, \mathrm{E}+1\,, 2\Lambda)\,, \tag{11.3}$$

(and $\eta = \mathrm{diag}(-1,+1,+1)$), the Lagrangian (11.1) becomes

$$\mathcal{L}[\Phi] \,=\, \frac{1}{2}\,Tr\,\langle J\,, J\rangle\,, \quad \text{with } J \,=\, \Phi^{-1}\,d\Phi\,. \tag{11.4}$$

The Mazur identity relates the Laplacian of the relative difference Ψ of two hermitian matrices $\Phi_{(1)}$ and $\Phi_{(2)}$ to a quadratic expression in $\triangle J$, where

$$\Psi \,\equiv\, \Phi_{(2)}\Phi_{(1)}^{-1} \,-\, \mathbb{1}\,, \quad \triangle J \,\equiv\, J_{(2)} \,-\, J_{(1)}\,. \tag{11.5}$$

Using the general equation $d^\dagger(f\alpha) \,=\, -\langle df\,, \alpha\rangle \,+\, f d^\dagger\alpha$ for the coderivative of a product of a function f with a 1–form α, and the properties $d\Phi \,=\, J^\dagger\Phi$, $d\Phi^{-1} \,=\, -\Phi^{-1}J^\dagger$ and $d\Psi = \Phi_{(2)}(\triangle J)\Phi_{(1)}^{-1}$, it is not hard to derive the identity

$$-\,d^\dagger d\,\mathrm{Tr}\,\Psi = -\mathrm{Tr}\left\{\Phi_{(2)}\,d^\dagger(\triangle J)\,\Phi_{(1)}^{-1}\right\} \,+\, \mathrm{Tr}\,\langle\Phi_{(1)}^{-1}\,\triangle J^\dagger\,, \Phi_{(2)}\,\triangle J\rangle\,. \tag{11.6}$$

If both $\Phi_{(1)}$ and $\Phi_{(2)}$ are *solutions* to the variational equations $d^\dagger J \,=\, 0$ for the action $\int \mathcal{L}[\Phi] * 1$, then $d^\dagger(\triangle J)$ vanishes and the first term on the r.h.s. of the Mazur identity (11.6) does not contribute.

Finally taking advantage of the full symmetries of spacetime, that is, using the circular metric (7.11), the coderivative of a stationary and axisymmetric 1–form α becomes $d^\dagger\alpha = \rho^{-1}\tilde{d}^\dagger(\rho\alpha)$, where quantities furnished with a tilde refer to the 2-dimensional Riemannian metric γ (see eq. (7.11)). The Lagrangian (11.4) now describes a nonlinear sigma-model on the symmetric space $SU(1,2)/S(U(1) \times U(2))$ with Riemannian base manifold (Γ, γ). The Ernst equations for the positive, hermitian matrix Φ become $\tilde{d}^{\dagger}(\rho J) = 0$. For two solutions $\Phi_{(1)}$ and $\Phi_{(2)}$, the Mazur identity (11.6) therefore yields

$$\int_{\partial S} \rho \,\tilde{*} d\,(Tr\,\Psi) \;=\; \int_{S} \rho \,\mathrm{Tr}\,\langle \Phi_{(1)}^{-1} \triangle J^{\dagger} \,, \Phi_{(2)} \triangle J \rangle \,\tilde{\eta}\,, \tag{11.7}$$

where $S \subset \Gamma$. The uniqueness theorem for the Kerr-Newman metric is now obtained as follows: First, on uses the boundary and regularity conditions to argue that $d(\mathrm{Tr}\Psi)$ vanishes on all parts of the boundary ∂S, that is, on the horizon, the rotation axis and in the asymptotic regime. The above formula then implies that the integral on the r.h.s must vanish as well. Since the embedding of the symmetric space G/H in G can be represented in the form $\Phi = gg^{\dagger}$, we may define $\mathcal{M} \equiv g_{(1)}^{-1} \triangle J^{\dagger} g_{(2)}$, which yields

$$\mathrm{Tr}\,\langle \Phi_{(1)}^{-1} \triangle J^{\dagger} \,, \Phi_{(2)} \triangle J \rangle \;=\; \mathrm{Tr}\,\langle \mathcal{M} \,, \mathcal{M}^{\dagger} \rangle \;\geq 0\,.$$

Here we have also used the facts that γ is a Riemannian metric, $\triangle J$ is spacelike and $\rho \geq 0$. This shows that the r.h.s. of eq. (11.7) can only vanish for $\triangle J = 0$, implying that two solutions $\Phi_{(1)}$ and $\Phi_{(2)}$ with identical boundary and regularity conditions are equal.

This concludes the outline of the uniqueness proof for the Kerr-Newman metric amongst the electrovac black hole solutions with nondegenerate event horizon and stationary and axisymmetric, asymptotically flat domain of outer communications. The corresponding result for the vacuum case was already established in 1975 by Robinson [30], using a rather complicated and ingeniously constructed identity. As it turned out, the Robinson identity is exactly the Mazur identity for the coset space $G/H = SU(1,1)/U(1)$ which is, in fact, the relevant symmetric space for the *vacuum* Ernst equations.

12 An Electrovac Bogomol'nyi Equation

The static electrovac equations imply that the electric potential ϕ depends only on the norm V (gravitational potential) of the Killing field. Israel was able to draw this conclusion from Stokes' theorem, using a particular combination of the Einstein-Maxwell equations [20]. Here we present a systematic approach to divergence identities for electrovac black hole configurations with nonrotating horizon. We will not assume that the domain is static. Our method yields a set of relations between the charges and the values of the potentials on the horizon. Solving for the latter one can then derive the relation

$$M^2 \;=\; \left[\frac{1}{4\pi} \kappa \mathcal{A}\right]^2 \;+\; Q^2 \;+\; P^2\,, \tag{12.1}$$

where M, Q and P denote the total mass, the electric and the magnetic charge, respectively.

We start by recalling the Einstein-Maxwell equations in the presence of a stationary Killing field k, say, with norm $V \;\equiv\; -\langle k\,, k\rangle$ and twist $\omega \;\equiv\; \frac{1}{2} * (k \wedge dk)$. In terms of the electric potential, $d\phi = -i_k F$, and the magnetic potential, $d\psi = i_k * F$, the Maxwell equations become

$$d^\dagger\left(\frac{d\phi}{V}\right) = -2\,\frac{\langle\omega\,,d\psi\rangle}{V^2}\,, \qquad d^\dagger\left(\frac{d\psi}{V}\right) = 2\,\frac{\langle\omega\,,d\phi\rangle}{V^2}\,. \tag{12.2}$$

In addition, we consider the identity $d^\dagger(\omega/V^2) = 0$, which gives an equation for the coderivative of $\frac{dU}{V^2}$, where U denotes the twist potential, defined by $\omega = dU + \psi d\phi - \phi d\psi$. Finally, the coderivative of $\frac{dV}{V}$ (i.e., the $R(k,k)$ equation) becomes

$$d^\dagger\left(\frac{dV}{V}\right) = 4\,\frac{\langle\omega\,,\omega\rangle}{V^2} - 2\,\frac{\langle d\phi\,,d\phi\rangle + \langle d\psi\,,d\psi\rangle}{V}\,. \tag{12.3}$$

(Note that $d^\dagger = *d*$ is the coderivative with respect to the spacetime metric.) Before we apply Stokes' theorem, it is worthwhile noticing that each of the above equations can be written in the (current conservation) form $d^\dagger j = 0$. In fact, using for instance $d^\dagger(\phi\frac{\omega}{V^2}) = -\frac{\langle d\phi\,,\omega\rangle}{V^2}$ and $d^\dagger(\psi\frac{d\psi}{V}) = -\frac{\langle d\psi\,,d\psi\rangle}{V} + 2\psi\frac{\langle\omega\,,d\phi\rangle}{V^2}$, eqs. (12.2) and (12.3) are equivalent to

$$d^\dagger\left(\frac{d\phi}{V} - 2\psi\,\frac{\omega}{V^2}\right) = 0\,, \qquad d^\dagger\left(\frac{d\psi}{V} + 2\phi\,\frac{\omega}{V^2}\right) = 0 \tag{12.4}$$

and

$$d^\dagger\left(\frac{dV}{V} + 4U\,\frac{\omega}{V^2} - 2\phi\,\frac{d\phi}{V} - 2\psi\,\frac{d\psi}{V}\right) = 0\,, \qquad d^\dagger\left(\frac{\omega}{V^2}\right) = 0\,, \tag{12.5}$$

respectively. We can now integrate these equations over a (not necessarily static) hypersurface extending from the horizon to S^2_∞. Using Stokes' theorem and the fact that all potentials assume constant values on H and S^2_∞, one obtains linear combinations of the total mass M, the charges Q and P and the corresponding horizon quantities $M_H \equiv \frac{1}{4\pi}\kappa\mathcal{A}$, Q_H and P_H:

$$M,\,(M_H) = \frac{1}{8\pi}\int_{S^2_\infty,(H)} *\left(k\wedge\frac{dV}{V}\right)\,, \qquad Q,\,(Q_H) = -\frac{1}{4\pi}\int_{S^2_\infty,(H)} *\left(k\wedge\frac{d\phi}{V}\right)\,, \tag{12.6}$$

and similarly for P and P_H. (Here we have used the Komar expressions for the charges and the identities $*dk = -\frac{2}{V}(k\wedge\omega) - *(k\wedge\frac{dV}{V})$ and $F = k\wedge\frac{d\phi}{V} + *(k\wedge\frac{d\psi}{V})$, as well as the fact that the boundary integrals over the additional terms vanish; see [2] for details.) For simplicity, we shall also assume that the NUT charge vanishes, i.e., that $\int_{S^2_\infty} d(\frac{k}{V}) = 0$. This implies that the 1-form $\frac{\omega}{V^2}$ does not contribute to the boundary integrals, since $*(k\wedge\frac{\omega}{V^2}) = \frac{1}{2}d(\frac{k}{V})$. Every equation $d^\dagger j = 0$ gives now rise to a relation of the form $\int_\infty *(k\wedge j) = \int_H *(k\wedge j)$, which is evaluated by making the following substitutions: $(dV/V)_\infty \to 2M$, $(dV/V)_H \to 2M_H$, $(d\phi/V)_\infty \to -Q$, $(d\phi/V)_H \to -Q_H$, $(d\psi/V)_\infty \to -P$, $(d\psi/V)_H \to -P_H$ and $(d\omega/V^2)_\infty = (d\omega/V^2)_H \to 0$. Adopting a gauge where $\phi_\infty = \psi_\infty = U_\infty = 0$ and using $V_\infty = 1$ and $V_H = 0$, eqs. (12.4) and (12.5) now yield

$$Q = Q_H\,, \quad P = P_H\,, \quad M = M_H + \phi_H Q + \psi_H P\,, \tag{12.7}$$

where the last expression is, of course, the Smarr formula.

One may now ask whether there exist additional combinations of the 1-forms $\alpha^i \equiv (\frac{dV}{V}, \frac{\omega}{V^2}, \frac{d\phi}{V}, \frac{d\psi}{V})$ which – by virtue of the field equations (12.2) and (12.3) – can be cast into the form of local conservation laws. This is indeed the case. A relatively obvious possibility is

$$d^\dagger \left(\phi \frac{d\psi}{V} - \psi \frac{d\phi}{V} + (\phi^2 + \psi^2) \frac{\omega}{V^2} \right) = 0 \,, \tag{12.8}$$

which, after integration, gives rise to the formula

$$Q \, \psi_H = P \, \phi_H \,. \tag{12.9}$$

Together with the Smarr formula, this enables one to solve for the electromagnetic potentials in terms of the charges, the total mass and the horizon quantity $M_H = \frac{1}{4\pi}\kappa \mathcal{A}$,

$$\phi_H = Q \, \frac{M - M_H}{Q^2 + P^2} \,, \qquad \psi_H = P \, \frac{M - M_H}{Q^2 + P^2} \,. \tag{12.10}$$

In order to find all remaining differential identities in a systematic way, we consider the ansatz $j = \sum_{i=1}^4 f^i[V, U, \phi, \psi] \cdot \alpha^i$ and require that $d^\dagger j = 0$, i.e., that $\sum [f^i d^\dagger \alpha^i - \langle df^i \, , \alpha^i \rangle] = 0$. Writing $df = f_{,V} \, dV + f_{,U} \, (\omega - \psi d\phi + \phi d\psi) + f_{,\phi} \, d\phi + f_{,\psi} \, d\psi$, one obtains a set of (simple) partial differential equations for the four functions f^i of the four potentials. Solving these equations gives, in addition to eq. (12.8), three formulae of the desired form. The first one is

$$d^\dagger \left([...] \frac{\omega}{V^2} + U \frac{dV}{V} + \left[\phi^2 + \psi^2 - V\right] \left[\phi \frac{d\psi}{V} - \psi \frac{d\phi}{V}\right] - 2U \left[\phi \frac{d\phi}{V} + \psi \frac{d\psi}{V}\right] \right) = 0, \tag{12.11}$$

(where $[...] = \frac{1}{2}(\phi^2 + \psi^2)^2 + 2U^2 - \frac{1}{2}(V^2 + 1)$). This immediately yields $-2U_H(M_H + \phi_H Q + \psi_H P) + (\phi_H^2 + \psi_H^2)(\phi_H P - \psi_H Q) = 0$. Using the Smarr formula (12.7) and the symmetry property (12.9), this becomes $U_H M = 0$ and therefore

$$U_H = 0 \,, \tag{12.12}$$

which reflects the fact that we consider configurations with vanishing NUT charge. The second solution of the differential equations for the functions f^i gives rise to the conservation law

$$d^\dagger \left([...] \frac{\omega}{V^2} - \phi \frac{dV}{V} + \left[\phi^2 - 3\psi^2 + V - 1\right] \frac{d\phi}{V} + [4\phi\psi - 2U] \frac{d\psi}{V} \right) = 0 \,, \tag{12.13}$$

(where $[...] = 2\psi(1 + \phi^2 + \psi^2) - 4\phi U$). (Replacing $\psi \to \phi$ and $\phi \to -\psi$ in the above equation gives the third solution which does, however, not contain new information.) Evaluating the above current on the boundary and using again $Q_H = Q$ and $P_H = P$ immediately yields $2\phi_H M_H + (\phi_H^2 - 3\psi_H^2 - 1)Q + (4\phi_H \psi_H - 2U_H)P = 0$. Using the expressions (12.10) and (12.12) for the values of the potentials in terms of the charges gives the desired formula (12.1) or, equivalently,

$$T_H = \frac{2}{\mathcal{A}} \sqrt{M^2 - Q^2 - P^2} \tag{12.14}$$

for the Hawking temperature T_H. The reason why there exist nontrivial combinations of the Einstein-Maxwell equations (12.2), (12.3) which can be cast into the form $d^\dagger j = 0$ lies in the structure of the Ernst equations. We have already mentioned that – when formulated with respect to the stationary Killing field – the latter are equivalent to a nonlinear sigma-model with target manifold $SU(1,2)/S(U(1) \times U(1,1))$. The field equations are therefore obtained from an effective Lagrangian of the form (11.4) and assume the form $d^\dagger J^a_{\ b} = 0$, where $J^a_{\ b}$ is the matrix valued current 1-form $\Phi^{-1} d\Phi$. (The definitions of Φ and of the Ernst potential differ from the definitions given in eqs. (11.2), (11.3), since the latter were formed on the basis of the axial Killing field: Here we use $\mathrm{E} = (-V + \phi^2 + \psi^2) + iY$ and $\Phi_{ab} = \eta_{ab} - 2\bar{v}_a v_b$, where the Kinnersley vector is defined in terms of the Ernst potentials E and Λ as before.) The additional components of the equations $d^\dagger J^a_{\ b} = 0$ are, in fact, exactly the combinations (12.9), (12.11) and (12.13) of the Einstein-Maxwell equations constructed above.

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