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Constant mean curvature foliations in cosmological spacetimes

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Abstract Foliations by constant mean curvature hypersurfaces provide a possibility of defining a preferred time coordinate in general relativity. In the following various conjectures are made about the existence of foliations of this kind in spacetimes satisfying the strong energy condition and possessing compact Cauchy hypersurfaces. Recent progress on proving these conjectures under supplementary assumptions is reviewed. The method of proof used is explained and the prospects for generalizing it discussed. The relations of these questions to cosmic censorship and the closed universe recollapse conjecture are pointed out.

1 Conjectures on constant mean curvature hypersurfaces

The purpose of this paper is to put forward some conjectures on the existence of hypersurfaces of constant mean curvature in spatially compact spacetimes and to present some recent results which show that these conjectures are true in certain special cases. The spacetimes considered are globally hyperbolic solutions of the Einstein equations with vanishing cosmological constant which possess a compact Cauchy hypersurface and satisfy the strong energy condition, i.e. the condition that $T_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2}T^\alpha_\alpha \geq 0$ for any unit timelike vector u^α. Following the terminology of Bartnik [1], I refer to these as *cosmological spacetimes.* If S is a spacelike hypersurface in a spacetime $(M, g_{\alpha\beta})$, its induced metric and second fundamental form will be denoted by g_{ab} and k_{ab} respectively. The mean curvature of S is the trace $\mathrm{tr}k = g^{ab}k_{ab}$. The hypersurface S is said to have constant mean curvature (CMC) if the function $\mathrm{tr}k$ on S is constant.

There are a number of well-known properties of CMC hypersurfaces in cosmological spacetimes (see e.g. [1], [2]). The first concerns uniqueness: in a cosmological spacetime there exists at most one compact hypersurface with a given non-zero value of $\mathrm{tr}k$. In the case of a maximal hypersurface ($\mathrm{tr}k = 0$) the statement is not quite so strong. In that case there is at most one compact hypersurface with the given value of $\mathrm{tr}k$ unless the spacetime is static with timelike Killing vector t^α and $R_{\alpha\beta}t^\alpha t^\beta = 0$. For solutions of the Einstein equations coupled to reasonable matter the latter situation is rare. For instance, if the matter satisfies the non-negative pressures condition ($T_{\alpha\beta}x^\alpha x^\beta \geq 0$ for any spacelike vector x^α) and the dominant energy condition, then a spacetime of this type is necessarily flat. The next property is that if there exists one compact CMC hypersurface in a cosmological spacetime, there exists a foliation of a neighbourhood of that hypersurface by compact CMC hypersurfaces. The mean curvature varies in a monotone way from slice to slice.

These properties suggest one possible motivation for studying foliations by CMC hypersurfaces. If such a foliation exists globally in a given spacetime a function t can be defined by the property that its value at each point of a given leaf of the foliation is equal to the mean curvature of that leaf. This defines a scalar function on spacetime whose gradient is everywhere timelike or zero. In fact it can be shown that where t is non-zero its gradient is timelike. Only on a maximal hypersurface can the gradient vanish and even in that case it does not usually happen. Thus, leaving aside possible problems with exceptional maximal hypersurfaces, the CMC foliation provides a global time coordinate which is invariantly defined by the geometry. This is very useful if one wishes to consider the Einstein equation as an evolution equation, for instance when investigating cosmic censorship [3]. It may also be of relevance for the problem of time in quantum gravity (cf. [4]). The exceptional maximal hypersurfaces, where t is not a good time coordinate have the properties that $k_{ab} = 0$ and $R_{\alpha\beta}n^\alpha n^\beta = 0$, where n^α is the unit normal vector to the hypersurfaces. These conditions are reminiscent of those for non-uniqueness of maximal hypersurfaces mentioned earlier. As in that case, if the non-negative pressures and dominant energy conditions are satisfied then the spacetime must be vacuum. Nevertheless there are many vacuum examples, e.g. the Taub-NUT solution with $m = 0$.

These general results do not immediately give any information about the existence question, i.e. the question of whether a given cosmological spacetime admits a CMC hypersurface. If a spacetime does admit a CMC hypersurface one can ask which real numbers occur as the mean curvature of such a hypersurface in a given spacetime. Some light is thrown on the latter question by a theorem on barrier hypersurfaces. Two compact hypersurfaces S_1 and S_2 in a cosmological spacetime are called barriers if the maximum M of the mean curvature of S_1 is less than the minimum m of the mean curvature of S_2. In that case it can be shown that S_2 lies to the future of S_1 and that for any H with $M < H < m$ there exists a CMC hypersurface with mean curvature H. In particular this rules out gaps in the set of values taken on by the mean curvature of CMC hypersurfaces in a given spacetime. This set of values must be an interval. Apart from the exceptional static case mentioned above, it is an open interval I. The question which remains is what the endpoints of this interval are in a given spacetime, i.e. the question of the range of the mean curvature.

It is easy to produce examples of spacetimes not containing a CMC hypersurface by cutting pieces out of a given spacetime. For instance, one could start with a $k = 1$ Friedman model, whose homogeneous hypersurfaces define a global CMC foliation with the mean curvature taking on all real values. A connected subset of this spacetime which does not contain any homogeneous hypersurface but does admit a compact Cauchy hypersurface obviously does not contain any compact CMC hypersurface. For any such hypersurface would be a compact CMC hypersurface in the original spacetime and so, by uniqueness, would have to be one of the homogeneous hypersurfaces, a contradiction. The way to avoid such trivial examples is to consider only maximal globally hyperbolic spacetimes, i.e. spacetimes which are the maximal Cauchy development of initial data defined on some compact spacelike hypersurface. The answer to the existence question in the maximal globally hyperbolic case is not always positive. Bartnik [1] exhibited a cosmological spacetime which is maximal globally hyperbolic but does not admit any compact CMC hypersurface or even any compact hypersurface where the mean curvature is everywhere of a single sign.

Even if a CMC hypersurface does exist the range of the mean curvature need not consist of all real numbers. This follows from the following well-known argument. The Hamiltonian constraint implies that the scalar curvature of the induced metric of a maximal hypersurface in a spacetime satisfying the weak energy condition must be non-negative. On the other hand, results of Gromov and Lawson [5] imply that not all manifolds admit metrics with non-negative scalar curvature. In fact, in a certain vague sense, most compact three-dimensional manifolds do not do so. Each Riemannian metric on a compact three-dimensional manifold is conformal to a metric with scalar curvature 1, 0 or -1. For each metric only one of these three cases can occur. This is a special case of the Yamabe theorem. According to which case occurs Riemannian metrics fall into three classes, denoted by Y_+, Y_0 and Y_- respectively. Three-dimensional manifolds can be classified into three disjoint types, which are characterized by the following properties [6]. Manifolds of type Y_- admit no metrics of non-negative scalar curvature. Manifolds of type Y_0 admit flat metrics and no non-flat metrics of non-negative scalar curvature. Finally, manifolds of type Y_+ admit metrics of all three Yamabe classes. The above argument shows that, provided the weak energy condition holds, a maximal hypersurface must be a manifold of type Y_- or Y_0. Moreover, if it is of type Y_0 the second fundamental form and the energy density must vanish on that hypersurface. In the presence of the dominant energy condition this implies that the spacetime must be flat. Thus to produce an example of a maximal globally hyperbolic spacetime which contains a CMC hypersurface but no maximal hypersurface, it suffices to do the following. Consider the maximal Cauchy development of data of non-zero constant mean curvature on a manifold of Yamabe type Y_- with a matter model which satisfies the weak energy condition. It would also suffice to take non-vacuum data on a manifold of type Y_0 and a matter model satisfying the dominant energy condition. A simple example of this is given by a $k = 0$ Friedman model, compactified so as to have the spatial topology of a three-dimensional torus.

Suppose now that $(M, g_{\alpha\beta})$ is a maximal globally hyperbolic cosmological spacetime which admits at least one compact CMC hypersurface. If $\mathrm{tr}k$ is non-zero on this hypersurface, it can be assumed without loss of generality to be negative. For the sign of tr$k$ depends on the choice of orientation of the normal vector to the hypersurface. As a result of the arguments given above, if the spacetime is not flat and has a Cauchy hypersurface of type Y_- or Y_0 and satisfies the dominant energy condition the interval I of values attained by the mean curvature of compact CMC hypersurfaces must be a subset of $(-\infty, 0)$. In the case of a manifold of type Y_+ the argument gives no restrictions.

Are there restrictions coming from other sources? In fact the choice of matter model is of importance here. A simple example which indicates the nature of the problem is that of dust. Solutions of the Einstein equations coupled to dust frequently develop shell-crossing singularities. Whether a global CMC foliation exists in these spacetimes depends on whether the foliation runs into these singularities in finite time or whether it avoids them. Foliations by CMC hypersurfaces are known (chiefly from experience with numerical calculations) to have a tendency to avoid spacetime singularities. On the other hand it is doubtful whether shell-crossing singularities should really be considered as spacetime singularities or just as some mathematical pathology of the matter model. It is therefore plausible that CMC hypersurfaces should not take these singularities seriously and collide with them. It will be shown elsewhere that this is in fact the case. Given $\epsilon > 0$ there exist initial data for the Einstein equations coupled to dust with mean curvature $H < 0$ such that the maximal Cauchy development of this initial data contains no compact CMC hypersurface of mean curvature H' with $|H - H'| \geq \epsilon$. Thus to get good general theorems it is necessary to make some detailed assumptions on the matter model used. In the following the simplest case, that of vacuum, will be emphasized but results for other types of matter will be mentioned when they exist.

Conjecture 1 Let $(M, g_{\alpha\beta})$ be a maximal globally hyperbolic vacuum spacetime with compact CMC Cauchy hypersurface S. Then if S is of type Y_+ the spacetime admits a foliation of compact CMC hypersurfaces with the mean curvature taking on all real values. If S is of type Y_-, or of type Y_0 and the spacetime is non-flat, then it admits a foliation of compact CMC hypersurfaces taking on all values in the interval $(-\infty, 0)$.

There are no known counterexamples to this conjecture. There is however one piece of evidence which casts doubt on the form of the conjecture just given. Numerical work of Abrahams and Evans [7] suggests that localized zero mass singularities may occur in solutions of the Einstein vacuum equations. It is not obvious whether singularities of this type would be avoided by CMC foliations. More generally, it is doubtful whether all naked singularities would be avoided. Thus there may be a close relationship between the existence of global CMC foliations and cosmic censorship. Violations of the simple version of cosmic censorship which says that, for reasonable matter, no naked singularities arise in the evolution of regular initial data, could lead to violations of Conjecture 1. To take account of this the conjecture could be modified so as to require the conclusions only to hold for generic initial data.

This conjecture says nothing about which part of the spacetime is covered by the CMC foliation. This is the subject of the next conjecture.

Conjecture 2 If a spacetime satisfies the conditions of Conjecture 1 and the Cauchy hypersurface S is of type Y_+ then the CMC foliation covers the entire spacetime. If S

is of type Y_- or Y_0 and admits a spacelike Killing vector field without fixed points, then once again the CMC foliation covers the entire spacetime. Moreover the spacetime is future geodesically complete.

The first part of this conjecture is a consequence of Conjecture 1. This is proved using the Raychaudhuri equation as in Hawking's singularity theorem. The motivation for the other parts of the conjecture will now be explained. It is connected with the idea of formation of black holes. The picture is that in the case where S is of type Y_+, we have a universe which recollapses. Thus to the extent that black holes form, they will coalesce with the cosmological singularity as the latter is approached. On the other hand, in the case where S is of type Y_-, the spacetime expands for ever and black holes which form during the expansion can be expected to have some features in common with black holes in asymptotically flat spacetimes. Now it is well known that in the Schwarzschild solution, for instance, slices of non-positive mean curvature which come from outside the black hole cannot approach the singularity [8]. Thus such slices cannot cover the maximally extended Schwarzschild spacetime. By analogy it seems plausible that when black holes form in a universe which expands indefinitely, the cosmological CMC foliation will only be able to penetrate the black hole regions to a limited extent and so will not cover the maximal Cauchy development of an appropriate Cauchy hypersurface. The part of Conjecture 2 concerned with spacetimes possessing a fixed-point free spacelike Killing vector comes from the intuition that collapse to a black hole is necessarily a localized phenomenon, while the presence of a spacelike Killing vector without fixed points forces any collapse which happens to be spread out in at least one direction.

Conjecture 2 implies the following conjecture on solutions of the constraints.

Conjecture 3 A solution of the vacuum constraints on a compact manifold S of type Y_- or Y_0 which admits a symmetry without fixed points does not contain a future trapped surface T with the property that the connected components of the inverse image of T in a non-compact covering manifold $\tilde{S}$ are compact. In particular, it does not admit a future trapped surface T with spherical topology.

The proof that Conjecture 2 implies Conjecture 3 is to apply the Penrose singularity theorem to the maximal Cauchy development of the data on the covering space obtained by pulling back data on S. If one wanted to disprove Conjecture 2 a good starting point might be to try to construct initial data violating the conclusion of Conjecture 3.

The idea behind Conjecture 2 is closely related to the closed universe recollapse conjecture[9]. Define the *lifetime* of a cosmological spacetime to be the supremum of the lengths of all timelike curves. This may a priori be finite or infinite. The closed universe recollapse conjecture says that if the topology of the Cauchy hypersurface is of type Y_+ then the lifetime is finite. Note that, in contrast to the case of the above conjectures on CMC hypersurfaces, singularities arising as a result of the behaviour of the matter fields tend to help this conjecture to be true rather than to hurt it. The case of this conjecture for vacuum spacetimes admitting a compact CMC hypersurface is implied by Conjecture 1, using the Raychaudhuri argument from the Hawking singularity theorem once more. A related conjecture going in the opposite direction, and which does require restrictions on the matter fields, is that a cosmological solution of the vacuum equations, for instance, with a Cauchy hypersurface of type Y_0 or Y_- should always have infinite lifetime. In the case of spacetimes which admit a CMC Cauchy hypersurface this would follow from Conjecture 2. However it also makes sense in the absence of CMC hypersurfaces.

Note for comparison that in the case of the Einstein equations with negative cosmological constant Λ the above topological obstruction to the existence of a maximal hypersurface does not occur. Thus Conjectures 1 and 2 may be modified in that case to say that for $\Lambda < 0$ there is always a global CMC foliation with the mean curvature taking all real values. Related to this is the fact that for $\Lambda < 0$ a version of the closed universe recollapse conjecture can be proved rather easily. For this sign of the cosmological constant the lifetime of any globally hyperbolic solution of the Einstein-matter equations satisfying the strong energy condition is bounded by $\pi\sqrt{-3/\Lambda}$. This statement, which follows from Theorem 11.9 of [10], does not even require the assumption of a compact Cauchy hypersurface. The bound is sharp since it becomes an equality for a suitable globally hyperbolic part of anti-de Sitter space.

What happens in the case that the CMC foliation does not cover the spacetime? Looking at the Schwarzschild solution again makes it tempting to speculate that in the case that the region covered by CMC hypersurfaces is not the whole spacetime its boundary is a smooth maximal hypersurface which is non-compact. It would consist in general of many connected components, the number of these corresponding to the number of black holes present at late times. Each component would have an asymptotically cylindrical geometry and would resemble the limiting maximal hypersurface in the Schwarzschild spacetime.

The region to the future of the limiting maximal hypersurface in the Schwarzschild spacetime can be covered by a CMC foliation whose leaves are non-compact, but have complete intrinsic geometry - the situation bears some resemblance to that in a cosmological spacetime (cf. the remarks in [8]). In the case of the Schwarzschild solution itself the CMC hypersurfaces can be compactified to have topology $S^2 \times S^1$, so that a cosmological spacetime of Kantowski-Sachs type is obtained. In more general cases (e.g. that of the Oppenheimer-Snyder solution) such a compactification would not be possible but the evolution might nevertheless be similar to that of a cosmological spacetime. Thus one may conjecture that in the case of vacuum cosmological spacetimes where the region C covered by compact CMC hypersurfaces is not all of spacetime, the complement of C can be covered by a unique foliation by complete non-compact CMC hypersurfaces.

A possible extension of Conjecture 2 is:

Conjecture 4 If a spacetime satisfies the conditions of Conjecture 1, the Cauchy hypersurface S is of type Y_- or Y_0 and the initial data are sufficiently close to those of a spacetime which is covered by a foliation by compact CMC hypersurfaces then the given spacetime also admits a global CMC foliation. Moreover the spacetime is future geodesically complete.

This conjecture, if true, would be a cosmological analogue of the famous theorem of Christodoulou and Klainerman [11] on the global nonlinear stability of Minkowski space.

2 Positive results

The simplest case in which to investigate the truth of these conjectures is that of spatially homogeneous spacetimes. The only classes of spatially homogeneous spacetimes which admit a compact Cauchy hypersurface are those of Bianchi types I and IX and the Kantowski-Sachs spacetimes. A much wider variety can be obtained by looking at *locally* spatially homogeneous spacetimes, i.e. those whose universal covers are spatially homogeneous. Then many other Bianchi types are possible. The specialization of Conjectures 1 and 2 to the locally spatially homogeneous case was proved in [12]. The manifolds obtained by compactifying spacetimes of Bianchi type IX and Kantowski-Sachs spacetimes are of type Y_+. All the other Bianchi types lead to manifolds of type Y_0 or Y_-. The result of [12] is as follows:

Theorem 1 Conjectures 1 and 2 hold for locally spatially homogeneous spacetimes, as do the analogous statements when the vacuum equations are replaced by the Einstein equations coupled to a perfect fluid with reasonable equation of state or collisionless matter satisfying the Vlasov equation.

The most difficult case of this theorem is that of Bianchi IX spacetimes where the proof relies on a result of Lin and Wald [13]. For inhomogeneous cosmological spacetimes the assumption of global symmetry still only allows a few types of group action and it is once again useful to look at local symmetry, where the symmetry group only acts on the universal covering space. The simplest situation is that of a three-dimensional group acting on two dimensional orbits. The corresponding spacetimes contain no gravitational radiation and so it is not surprising that in vacuum they are (locally) spatially homogeneous. For that reason the vacuum spacetimes in these classes are covered by Theorem 1. For spacetimes with matter the following theorem has been proved in [14] and [15] under an additional restriction.

Theorem 2 The analogues of Conjectures 1 and 2 when the vacuum equations are replaced by the Einstein equations coupled to collisionless matter satisfying the Vlasov equation or a massless scalar field hold for spherically symmetric spacetimes on $S^2 \times S^1$. Moreover for spacetimes with plane or hyperbolic symmetry the analogue of Conjecture 1 holds.

The additional restriction, namely that the mass function should be positive on the initial hypersurface in the case of hyperbolic symmetry, can be removed using the techniques of [16]. Plane and hyperbolic symmetry mean that the group action on the universal cover is an action with two dimensional orbits of the Euclidean group or the identity component of the isometry group of the hyperbolic plane, respectively. All these spacetimes possess three local Killing vectors. Note that $S^2 \times S^1$ belongs to the class Y_+, while the manifolds occurring in the cases of plane and hyperbolic symmetry belong to the classes Y_0 and Y_- respectively. The conclusions of Theorem 2 for spherical, plane and hyperbolic symmetry represent inhomogeneous generalizations of the Kantowski-Sachs, Bianchi I and Bianchi III cases of Theorem 1 respectively. There is also another kind of spherical symmetry. This corresponds to the action of the isotropy group of some vector on the unit sphere in R^4 in the group $SO(4)$ of rotations of R^4. There the Killing vectors have fixed points and this makes the problem much more difficult. If one tries to factor out by the isometry group, the equations on the resulting quotient space are singular. This symmetry is closely related to the kind of spherical symmetry familiar in the asymptotically flat case. There the equations written in polar coordinates are singular at the centre of symmetry.

Theorems 1 and 2 together prove Conjecture 1 and its analogue for appropriate matter models in all known cases where there are at least three local Killing vectors and the nature of the orbits of the group action does not change from point to point. Under additional assumptions they also prove Conjecture 2. The next theorem generalizes these results to certain cases where there are only two Killing vectors[16].

Theorem 3 Conjecture 1 holds for spacetimes with local $U(1) \times U(1)$ symmetry, as do the analogous statements when the vacuum equations are replaced by the Einstein equations coupled to collisionless matter satisfying the Vlasov equation or a wave map.

The analogues of Conjectures 1 and 2 for these types of matter hold for locally $U(1) \times U(1)$-symmetric spacetimes with cosmological constant $\Lambda < 0$. The assumption of local $U(1) \times U(1)$ symmetry means, informally stated, that the group action on the universal covering space looks locally like the standard action of the group $U(1) \times U(1)$ on the torus $S^1 \times S^1 \times S^1$ by rotating two of the three factors. It includes the case of global $U(1) \times U(1)$ symmetry, where the Cauchy hypersurface is the torus with the standard action. A wave map (also known as a hyperbolic harmonic map or nonlinear σ-model) is a certain kind of mapping of spacetime into a complete Riemannian manifold. In the case where this Riemannian manifold is the real line the equation for wave maps reduces to the ordinary massless wave equation and so the matter models allowed in Theorem 3 include those covered by Theorem 2. In fact the plane symmetric case of Theorem 2 is a special case of Theorem 3, as are the Bianchi type II and Bianchi type VI_0 cases of Theorem 1. In should be noted that the special case of Theorem 3 for Gowdy spacetimes on $S^1 \times S^1 \times S^1$ was proved earlier by Isenberg and Moncrief[17]. The Gowdy spacetimes on the torus are those spacetimes with global $U(1) \times U(1)$ symmetry which satisfy the Einstein vacuum equations and the additional condition that the so-called twist constants vanish. There are other types of Gowdy spacetimes defined on S^3 and $S^2 \times S^1$ but their Killing vectors have fixed points. Although quite a bit is known about these spacetimes ([18],[19]), Conjectures 1 and 2 remain open in that case. The next obvious step would be to look at spacetimes with only one Killing vector, but no general results are known in that case.

The results which have been listed above are the most general known to the author. It is interesting to note that the analogue of Conjectures 1 and 2 in spacetime dimension 3 have been proved by Andersson and Moncrief[20]. In my opinion it is not reasonable to expect that they hold in dimensions greater than 4, at least without the additional genericity requirement.

3 Constant mean curvature foliations seen from the inside

One way of thinking about the question of CMC hypersurfaces is to start with a given spacetime and ask which part of it can be covered by a CMC foliation. The proofs of the above theorems make use of a different point of view, which may also be useful in more general situations. The idea is to construct the spacetime, which is the solution of a Cauchy problem, and a CMC foliation of it simultaneously. In other words, consider a spacetime which is globally foliated by CMC hypersurfaces with the mean curvature taking values in some interval. Describe the spacetime in terms of the time coordinate defined by the mean curvature. This gives rise to a certain set of partial differential equations defined on some time interval. To prove the above theorems what is needed is to obtain a global in time existence theorem for these equations. (The word global here refers to a solution which is defined on the entire time interval predicted by the conjecture under consideration.) The reason that the theorems are not enough to prove the part of Conjecture 2 concerning geodesic completeness is that in order to prove that it would be necessary to prove not only global existence but also some facts about the qualitative behaviour of the solution as the time parameter approaches its limiting value.

The theorem to be proved has now been reformulated as a question of global existence. It can be reduced further to the task of obtaining sufficiently strong bounds on the behaviour of the solution on a finite interval of CMC time. The rough idea is as follows. Suppose a solution is given on the interval (t_1, t_2). If it is possible to obtain sufficiently strong control on the behaviour of the solution as t approaches t_2 (in other words to show that the solution is not becoming singular there) then it can be concluded that the solution extends to an interval (t_1, t_3) with $t_3 > t_2$. If this control can be obtained *whatever the value* of the time t_2 then considering the longest time interval on which a solution exists shows that global existence must hold. Thus, in order to prove global existence it suffices to obtain strong enough estimates on the solution on a finite time interval whose endpoint is different from that predicted by the conjectures. To prove future geodesic completeness or to determine in detail the nature of the initial singularity would require strong enough estimates on a time interval which is infinite or has an endpoint where the solution is expected to stop existing. This is a much more difficult task.

It will now be considered, what bounds can be obtained without any symmetry assumptions. It follows from the CMC condition that the lapse function α satisfies an elliptic equation (the lapse equation). Looking at this equation at a point where α takes its maximum on a given CMC hypersurface shows that $\alpha \leq 3/t^2$. Thus an upper bound for the lapse function away from maximal hypersurfaces is obtained. No such bound can be hoped for near a maximal hypersurface for the following reason (cf. [15]). In the case of those exceptional maximal hypersurfaces where the mean curvature fails to be a good time coordinate the lapse function blows up uniformly as the maximal slice is approached. (In the case $\Lambda < 0$ the estimate for α is improved to $\alpha \leq (\frac{1}{3}t^2 - \Lambda)^{-1}$ and this is the source of the stronger results which can be proved in that case.) In some highly symmetrical cases a positive lower bound for α on any finite time interval can also be obtained. (No collapse of the lapse on a finite time interval.) However this may well fail in general. On any finite interval where an upper bound on α is available the volume of the CMC hypersurfaces is bounded from above and from below by a positive constant. These are the only useful general bounds known to the author and so it seems that there is a long way to go in order to prove the conjectures by this route. In the proofs of Theorems 2 and 3 the next essential step is to use a generalization of a technique introduced by Malec and Ó Murchadha[21] and no way is known to generalize this method to spacetimes with less than 2 local Killing vectors.

To conclude, some remarks on the relationship of the conjectures above to cosmic censorship will be made. The idea of using a preferred time coordinate to reformulate the (strong) cosmic censorship hypothesis as a statement about global existence (and asymptotic behaviour) of solutions of a system of partial differential equations was put forward by Eardley and Moncrief[3]. Here the first step is to prove global existence. However, to prove something about strong cosmic censorship the asymptotic behaviour of the solution must be controlled. To make this idea concrete, consider the relationship of Theorems 1, 2 and 3 to cosmic censorship in the respective classes of spacetimes. In the situation of Theorem 1 it can be shown for non-vacuum spacetimes that the initial singularity is a curvature singularity, so that the specialization of strong cosmic censorship holds. However the asymptotic behaviour of the solution near the singularity is poorly understood, even in the Bianchi type I case [22]. In the situation of Theorem 2, it was shown by Rein [23] that the initial singularity is a curvature singularity for an open set of initial data (but not for all initial data). In the case of Gowdy spacetimes a similar result was proved by Chrusciel [24] while in the more special case of the polarized Gowdy spacetimes, detailed information can be obtained without additional restrictions on the initial data[19]. Clearly a lot remains to be done in this direction. Moreover, none of the results on inhomogeneous spacetimes just mentioned was proved using a CMC time coordinate. Instead a time coordinate was used which was specially adapted to the symmetry of the models. It would be desirable to prove versions of these results using CMC slicing in the hope of unifying, strengthening and generalizing them.

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