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Quasi-Einstein metrics and their renormalizability properties

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Abstract. We construct a family of kählerian quasi-Einstein metrics with an isometry group $U(n)$ acting linearly on the holomorphic coordinates. Suitable restrictions on the parameters give rise to complete non-compact as well as compact metrics whose geometrical structure is studied in detail. And we discuss The two loop renormalizability properties of the bosonic σ -models.

1 Introduction

Quasi-Einstein metrics are defined by the constraints

$$Ric_{\mu\nu} = \lambda g_{\mu\nu} + \frac{1}{2}(D_\mu v_\nu + D_\nu v_\mu). \quad (1.1)$$

In section 2, using Kähler geometry and an isometry group $U(n)$, we reduce (1.1) to a non-linear differential equation. Since this equation cannot be integrated to get the Kähler potential we switch to non-holomorphic coordinates which are useful to display explicitly the distance.

In section 3 we give two classes of complete non-compact metrics and we describe the compact metrics of the family.

In section 4 we discuss the two-loop renormalizability of the bosonic σ -models built on kählerian quasi-Einstein metrics. Our main result is that in the chosen isometry class the requirement of two-loop renormalizability selects uniquely CP^n , which is not only Einstein but symmetric!

2 Quasi-Einstein Kähler metrics

2.1 Isometry group $U(n)$

The idea to obtain explicit quasi-Einstein metrics is to use holomorphic coordinates $\{z_i, i = 1, \dots, n\}$ in $\mathbb{C}^n \setminus \{0\}$ and to choose for isometry group a $U(n)$ for which $\{z_i\}$ transform according to its fundamental representation. This implies that the Kähler potential K depends solely on $s = \bar{z} \cdot z = \sum_{i=1}^n \bar{z}_i z_i$. It follows that the metric is

$$\frac{1}{2} g = g_{i\bar{j}} dz^i d\bar{z}^j = A d\bar{z} \cdot dz + A' |\bar{z} \cdot dz|^2, \quad A(s) = \frac{dK(s)}{ds}.$$

The Ricci tensor reads

$$Ric_{i\bar{j}} = -\partial_{i\bar{j}}^2 \ln D, \quad D = \det(g_{i\bar{j}}) = A^{n-1}(sA)'.$$

The restrictions put on the metric by the quasi-Einstein requirement (1.1) imply ([1])

$$v_i = c \partial_i(sA), \quad A^{n-1}(sA)' = e^{-\lambda K - (Re c)sA}. \quad (2.1)$$

where c is an a priori complex integration constant.

2.2 The coordinates choice

In view of the complexity of (2.1), there is little hope to get the explicit Kähler potential $K(s)$ and the distance if we insist on using holomorphic coordinates. We define

$$z_i = \sqrt{s} \xi_i \quad i = 1, \dots, n$$

and

$$\xi = \frac{e^{i\tau}}{1 + \bar{u} \cdot u} (1, u), \quad \tau \in [0, 2\pi], \quad \{u_i, i = 1, \dots, n-1\}.$$

This gives for the final form of our distance

$$\frac{1}{2} g = \frac{(dt)^2}{4\varrho} + \varrho (d\tau + \theta)^2 + t \frac{h}{2} (CP^{n-1})$$

with

$$\varrho = s \frac{dt}{ds}, \quad t = s K', \quad \theta = \frac{1}{1 + \bar{u} \cdot u} \frac{\bar{u} \cdot du - u \cdot d\bar{u}}{2i}.$$

The differential equation (2.1) becomes

$$\frac{d}{dt} (e^{ct} t^{n-1} \varrho(t)) = e^{ct} (n t^{n-1} - \lambda t^n). \quad (2.2)$$

The vector v is given by the holomorphic 1-form $v = c dt$ for some real constant c .

3 Complete and compact metrics

Integrating (2.2) gives

$$t^{n-1} \varrho = \frac{e^{-ct} \int_0^{ct} e^u \left(n u^{n-1} - \frac{\lambda}{c} u^n \right) du}{c^n} - B e^{-ct}, \quad n = 1, 2, \dots \quad (3.1)$$

3.1 Complete non-compact metrics

The first one appears for $B = 0, c > 0, \lambda < 0$.

At $t = 0$, setting $t = r^2$ the distance becomes

$$\frac{1}{2} g \approx dr^2 + r^2 \left[(d\tau + \theta)^2 + \frac{h}{2} (CP^{n-1}) \right] = dr^2 + r^2 d\bar{\xi} \cdot d\xi$$

which shows that $r = 0$ is a “nut”, [4].

For $t \rightarrow \infty$, setting $t = \frac{|\lambda|}{c} r^2$ the distance becomes

$$\frac{1}{2} g \approx dr^2 + r^2 \left\{ \left(\frac{|\lambda|}{c} \right)^2 |\bar{u} \cdot du|^2 + \frac{|\lambda|}{c} (d\bar{u} \cdot du - |\bar{u} \cdot du|^2) \right\}$$

showing that infinity is asymptotically flat (albeit not euclidean).

For $\lambda = 0$ the infinity is taubian, [2, p. 252].

The second one appears for $B > 0, c > 0, \lambda \leq 0$. But now t_n define by $\varrho(t_n) = 0$ is a “bolt” of twist $k = n + 1, n + 2, \dots$, [4].

3.2 Compact metrics

Let us suppose $c < 0, \lambda > |c|$ and $n \geq 2$. From (3.1) we can deduce there does exist a finite interval $[t_n^{(1)}, t_n^{(2)}]$ on which $\varrho(t)$ is positive. The constraints for $t = t_n^{(1)}$ and $t = t_n^{(2)}$ to be bolts of twist k are [5]

$$\begin{cases} \varrho(t_n^{(1)}) = 0, & \varrho'(t_n^{(1)}) = k, \\ \varrho(t_n^{(2)}) = 0, & \varrho'(t_n^{(2)}) = -k. \end{cases} \quad k = 1, 2, 3, \dots$$

which reduce, using the differential equation, $\varrho' + \left(c + \frac{n-1}{t} \right) \varrho = n - \lambda t$, to the transcendental equation

$$\int_{-k}^{+k} e^{-\xi u} u (n+u)^{n-1} du = 0, \quad \xi = \frac{|c|}{\lambda}, \quad k = 1, 2, \dots, n-1.$$

And this equation has a unique solution $\xi \in]0, 1[$.

4 Bosonic σ -models at two loops

For a given bosonic σ -model with classical action

$$\frac{1}{g^2} \int d^2x \frac{1}{2} g_{ij}(\varphi) \partial_\mu \varphi^i \partial_\mu \varphi^j$$

the one and two-loop on-shell divergences follow from Friedan [3] and are given by

$$\left(\frac{\hbar}{4\pi}\right) \frac{1}{\epsilon} \int d^2x Ric_{ij} \partial_\mu \varphi^i \partial_\mu \varphi^j + \left(\frac{\hbar}{4\pi}\right)^2 \frac{g^2}{\epsilon} \int d^2x \frac{1}{2} R_{is,tu} R_j^{s,tu} \partial_\mu \varphi^i \partial_\mu \varphi^j, \quad \epsilon = d - 2.$$

Two-loop renormalizability requires

$$X_{ij} = R_{is,tu} R_j^{s,tu} = 0 = D_i w_j + D_j w_i \quad \Rightarrow \quad w_i = c_2 \partial_i (sA)$$

and

$$X_{i\bar{j}} = R_{i\bar{k},l\bar{n}} R_{\bar{j}}^{\bar{k},l\bar{n}} = \lambda_2 g_{i\bar{j}} + D_i w_{\bar{j}} + D_{\bar{j}} w_i = \partial_{i\bar{j}}^2 (\lambda_2 K + (c_2 + \bar{c}_2) sA).$$

For the divergence $X_{i\bar{j}}$ to be absorbable it is therefore necessary that $X_{i\bar{j}}$ be Kähler. In the class of metrics considered in this article, the large isometry group enables one to compute the curvature tensor and

$$X_{i\bar{j}} = \mu(s) \delta_{ij} + \nu(s) \bar{z}_i z_j.$$

The Kähler condition requires therefore $\nu = \mu'$ which is integrated to $K(s) = a \ln(s + b)$. Among all kählerian metrics of dimension $n \geq 2$, with an isometry group $U(n)$ acting linearly on the holomorphic coordinates, only CP^n is one-loop and two-loop renormalizable.

References

- [1] G. Bonneau and F. Delduc, *Int. J. Mod. Phys. A*, **1** (1986) 997.
- [2] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Phys. Report.* **66** (1980) 214.
- [3] D. H. Friedan, *Ann. Phys.* **163** (1985) 318.
- [4] G.W. Gibbons and S. W. Hawking, *Commun. Math. Phys.* **66** (1979) 291.
- [5] N. Koiso, in “Recent topics in differential and analytic geometry”, *Advanced studies in pure Math.* **18-I** (1990) 327.