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The Group of Large Diffeomorphisms in Classical and Quantum Gravity

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Abstract We describe results on mapping-class groups of oriented 3-manifolds with one regular end. These groups naturally act on the state spaces of classical and quantum gravity.

1 Introduction

In its canonical formulation, General Relativity is considered as constrained Hamiltonian system on $T^*(\mathrm{Riem}(\Sigma))$, the (trivial) cotangent bundle over the space of Riemannian metrics on a 3-manifold Σ. For the description of approximately isolated systems we are interested in orientable Σ with a single regular end. The last condition means that there exists a compact set $K \subset \Sigma$ diffeomorphic to $R^3 - B^3$, where B^3 is a closed 3-ball. This is equivalent to saying that the one-point-compactification $\bar{\Sigma} := \Sigma \cup \infty$ is a closed manifold. The action of the diffeomorphism group of Σ on $T^*(\mathrm{Riem}(\Sigma))$ is just obtained by canonically lifting its obvious action on $\mathrm{Riem}(\Sigma)$. However, the so-called diffeomorphism constraints generate only the action of a normal subgroup thereof, namely the identity-component of asymptotically trivial diffeomorphisms. Orbits of this group in $T^*(\mathrm{Riem}(\Sigma))$ assemble physically identical states. However, the reduced phase space still carries an action of a residual symmetry group which contains the discrete group of mapping classes of Σ. In terms of the fiducial compactified manifold $\bar{\Sigma}$ the mapping class group is isomorphic to the quotient $\mathcal{S}(\bar{\Sigma}) := D_F(\bar{\Sigma})/D_F^0(\bar{\Sigma})$, where $D_F(\bar{\Sigma})$ is the group of diffeomorphisms of $\bar{\Sigma}$ that fix the frames at ∞, and $D_F^0(\bar{\Sigma})$ is its identity-component. This group is generically infinite and non-abelian. It acts on the classical as well as quantum state spaces after they have been reduced by the action of the constraints. Its interpretation is either that of a residual gauge group (redundancy) or a physically meaningful symmetry, depending on whether its associated action (by automorphisms) on the algebra of physical observables is trivial or not. Besides issues of interpretation, these infinite discrete groups also pose several technical problems concerning their implementation in classical as well as quantum gravity. See e.g. [1] for an explicit model in 2+1-dimensional quantum gravity. Also, $\mathcal{S}(\bar{\Sigma})$ is isomorphic to the fundamental group of the reduced configuration space [2]. All this motivates to explore the general structure of $\mathcal{S}(\bar{\Sigma})$ and its dependence on $\bar{\Sigma}$.

2 General 3-Manifolds

Any 3-manifold can be decomposed into smaller pieces by cutting it along embedded 2-spheres. This procedure comes to an end if the pieces themselves cannot be decomposed further except by cutting out 3-disks or cylinders $[0,1] \times S^2$. This happens after a finite number of steps for compact manifolds. If the manifold is orientable the elementary pieces are uniquely determined up to permutation. In this way one can write Σ as a finite connected sum of uniquely determined so-called prime-manifolds (primes). A prime is either $S^1 \times S^2$ (called 'the handle') or has trivial second homotopy group. The latter ones are called irreducible primes and we denote them by P_i. (A list of primes and their properties may be found in [3].) Hence we write

$$\bar{\Sigma} = \left(\biguplus_{i=1}^{n} P_i \right) \uplus \left(\biguplus_{i=1}^{l} S^2 \times S^1 \right) , \tag{2.1}$$

where the connected-sum operation, $\uplus$, consists of removing $n + 2l$ 3-disks from a 3-sphere (called the base) and removing a 3-disk from each P_i whose n 2-sphere-boundaries are identified with the first n 2-sphere-boundaries of the base. The l handles are then attached by taking l cylinders, $[0,1] \times S^2$, and identifying the $2l$ 2-sphere-boundaries with the remaining boundaries of the base (see e.g. [4][3]). We stress at this point that the prime $S^1 \times S^2$ is distinguished in that it is attached by two rather than just one 2-sphere. In the way just sketched we intend to think of $\bar{\Sigma}$ as a configuration of $n + l$ elementary objects on a common base. The objects may fall into classes of identical ones (diffeomorphic primes). Finally we mention that the fundamental group of $\bar{\Sigma}$ is the free product of the fundamental groups of the primes:

$$\pi_1(\bar{\Sigma}, \infty) = \left(\mathop{\ast}_{i=1}^{n} \pi_1(P_i) \right) * \left(\mathop{\ast}_{1}^{l} \mathbf{Z} \right) . \tag{2.2}$$

3 Mapping Class Groups in General

One studies $\mathcal{S}(\bar{\Sigma})$ by considering the group homomorphism

$$h_F: \mathcal{S}(\bar{\Sigma}) \to \mathrm{Aut}(\pi_1(\bar{\Sigma}, \infty)), \quad h_F([\phi])([\gamma]) := [\phi \circ \gamma], \tag{1.1}$$

where γ is a loop based at ∞, $[\gamma]$ its homotopy class, $\phi \in D_F(\bar{\Sigma})$ and $[\phi]$ its class in $\mathcal{S}(\bar{\Sigma})$. The strategy is to obtain $\mathcal{S}(\bar{\Sigma})$ from 1.) $\mathrm{Ker}(h_F)$ = kernel of h_F, 2.) $\mathrm{Im}(h_F)$ = image of h_F, 3.) a prescription to extend $\mathrm{Im}(h_F)$ by $\mathrm{Ker}(h_F)$. Given the connected sum decomposition of $\bar{\Sigma}$, it is indeed possible to explicitly present $\mathcal{S}(\bar{\Sigma})$ for a large class of 3-manifolds. The generators of this presentation can be represented by some elementary diffeomorphisms adapted to the decomposition (2.1). This is possible because: 1.) for connected sums in which for each prime homotopic diffeomorphisms are also isotopic (presently no counterexamples are known) $\mathrm{Ker}(h_F)$ is generated by rotations of so-called spinorial primes [3] and twists of handles [5][6], 2.) explicit and adapted presentations for the automorphism groups of free products are known [7] which can be used to present $\mathrm{Im}(h_F)$, 3.) $\mathcal{S}(\bar{\Sigma})$ is a fairly obvious semi-direct product of $\mathrm{Ker}(h_F)$ and $\mathrm{Im}(h_F)$. The crucial step is 2.), i.e. to present $\mathrm{Im}(h_F)$. Its generators fall into 3 classes: a.) internal diffeomorphisms, whose support is inside the primes (up to isotopy), b.) exchange diffeomorphisms exchanging diffeomorphic primes, c.) so-called slide-diffeomorphisms which mix points interior and exterior to the primes. Roughly speaking, each prime can be slid a full turn within a closed tube whose axis-loop generates an element of the fundamental group of another prime. See e.g. [3] for details. Together with $\mathrm{Ker}(h_F)$, a.) and b.) generate a subgroup which is just the usual semi-direct product of internal and external symmetries of a set of $n+l$ 'particles'. In [6] we therefore called it the 'particle group' G^P. On the other hand, the slides generate a normal subgroup G^S and one may show that $G^P \cap G^S = \{1\}$ iff $l = 0$ (i.e. no handles) [6]. In this case $\mathcal{S}(\bar{\Sigma})/G^S \cong G^P$ and $\mathcal{S}(\bar{\Sigma})$ is a semi-direct product $G^S \times_\theta G^P$ with some homomorphism $\theta : G^P \mapsto \mathrm{Aut}(G^S)$ that can be written down explicitly. However, if handles are present $G^P \cap G^S$ is non-trivial and G^P is not a quotient of $\mathcal{S}(\bar{\Sigma})$. For more than two handles G^S is a perfect group [6]. A presentation and discussion of $\mathcal{S}(\bar{\Sigma})$ for the case where $\bar{\Sigma}$ is the connected sum of handles was given in the appendix of [3].

An Example

Let us briefly look at the case where $\bar{\Sigma}$ is the connected sum of n real projective spaces RP^3. In passing we mention that time-symmetric asymptotically flat initial data can be constructed by the method of images. The data correspond to n black holes of arbitrary individual (Penrose-) masses momentarily at rest. For a single prime, the evolution may be obtained by an appropriate identification of the Kruskal spacetime via a free Z_2 action. So the manifold $\bar{\Sigma}$ is in fact quite relevant to construct initial data for n holes without internal infinities.

For the case at hand $\mathrm{Ker}(h_F) = \{1\}$. Also, a distinguishing feature of the RP^3 prime is that it has no internal symmetries [2], i.e., $\mathcal{S}(RP^3) = \{1\}$, so that $G^P \cong P_n$, where

P_n is the permutation group for n objects. $\pi_1(\bar{\Sigma})$ is the n-fold free product of $\mathbf{Z}_2$'s. The $n(n-1)$ generators for the slide subgroup G^S are $\{\mu_{ij}\}$ for $1 \leq i \neq j \leq n$, where μ_{ij} is a slide of prime j through prime i [7]. Here and in the following distinct indices always represent distinct values. The relations among the μ_{ij} are of three types: 1.) $\mu_{ij}^2 = 1$, 2.) $\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}$ and $\mu_{ki}\mu_{kj} = \mu_{kj}\mu_{ki}$, 3.) $\mu_{ki}\mu_{kj}\mu_{ij} = \mu_{ij}\mu_{ki}\mu_{kj}$ and the same with i, j exchanged [7]. Their geometric interpretation is fairly obvious. Finally, $\mathcal{S}(\bar{\Sigma})$ is the semi-direct product $\mathcal{S}(\bar{\Sigma}) = G^S \times_\theta P_n$ with $\theta : P_n \to \mathrm{Aut}(G^S)$, $\theta(\omega)(\mu_{ij}) := \mu_{\omega(i)\omega(j)} = \omega\mu_{ij}\omega^{-1}$, where we used ω in the obvious double meaning. This establishes the remaining relations between the generators of G^P and G^S and therefore the presentation for $\mathcal{S}(\bar{\Sigma})$.

Let us apply this to the case $n = 2$, which was already discussed in [8]. For $n = 2$ we have three involutive generators: two slides μ_{12} and μ_{21}, and one exchange ω. Since relations of the form 2.) 3.) are now absent we have $G^S \cong \mathbf{Z}_2 * \mathbf{Z}_2$. The only other relation is $\omega\mu_{12}\omega^{-1} = \mu_{21}$. We may drop it and at the same time eliminate μ_{21} from the presentation (a so-called Tietze transformation). This leaves us with two involutive generators (called ω and μ) and no relation between them, so that also $\mathcal{S}(\bar{\Sigma}) \cong \mathbf{Z}_2 * \mathbf{Z}_2$. In quantum gravity one would be interested in all irreducible representations of $\mathcal{S}(\bar{\Sigma})$. Here they are given by the obvious 1-dimensional ones: $\omega \mapsto \pm 1$, $\mu \mapsto \pm 1$ and the one-parameter $(0 < t < \pi)$ family of two-dimensional ones: $\omega \mapsto \tau_1 \sin t + \tau_3 \cos t$, where τ_i are the standard Pauli matrices.

For $n > 2$ one can use similar Tietze-transformations to obtain a presentation with only 3 generators, two exchanges (generating P_n), and one slide. Its explicit form is given and discussed in [6]. A major difference to $n = 2$ are the non-trivial relations given under 2.) and 3.) above.

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