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Numerical Approach to the Global Structure of Spacetimes

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Abstract. In most approaches to numerical relativity it is very difficult to determine global properties of spacetimes since typically only finite regions of spacetime are evaluated. But global properties are of major interest for many fields of general relativity, e. g. cosmic censorship (existence and location of event horizons) and gravitational radiation (whose definition and determination requires knowledge of the geometry at null infinity). In this talk I will briefly outline an approach using conformal techniques to calculate global properties. The advantages of this method are demonstrated using two examples, the determination of event horizons and the decay of radiation. The calculations have been performed with a spherically symmetric spacetime using a conformal scalar field as radiation and matter model.

1 Introduction and description of model

General relativity as a theory for the structure of spacetime not only predicts local deviations from theories on a fixed background, like Newtonian physics and electrodynamics, but also dramatic changes of the structure in the large. Small deviations from the theories on fixed backgrounds can be determined by comparing certain quantities in finite volumes of spacetime — in the following called local properties. Unfortunately for numerical relativity, the answer to many questions concerning the structure of spacetime requires knowledge over an infinite volume of spacetime — in the following called global properties. Typical global properties are issues related to cosmic censorship like existence and location of event horizons.

Furthermore due to the nonlinearity of general relativity the definition and determination of gravitational radiation requires knowledge about infinitely far points (null infinity) and therefore gravitational radiation is also a global property.

In this talk I present results from a numerical approach which allows one to calculate global properties. The approach makes use of conformal techniques. More details can be found in [6].

The numerical solution of a relativistic initial value problem without symmetries is still an unsolved problem, the difficulties range from the correct treatment of boundaries to issues concerning the "right" choice of coordinates and last but not least the required, but not available, computational resources. In spherically symmetric spacetimes the choice of coordinates is relatively easy and the required computational resources are provided by a small workstation. As spherically symmetric spacetimes do not admit gravitational radiation a conformal scalar field has been chosen as matter and radiation model. The field equations are:

$$(1-\frac{1}{4}\kappa\tilde{\phi}^2)\,\tilde{R}_{ab} = \left(\kappa\,(\tilde{\nabla}_a\tilde{\phi})(\tilde{\nabla}_b\tilde{\phi}) - \frac{1}{2}\kappa\,\tilde{\phi}\,\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} - \frac{1}{4}\kappa\,\tilde{g}_{ab}(\tilde{\nabla}^c\tilde{\phi})(\tilde{\nabla}_c\tilde{\phi})\right) \tag{1.1}$$

$$\tilde{\Box}\tilde{\phi} - \frac{\tilde{R}}{6}\,\tilde{\phi} = 0. \tag{1.2}$$

Equation (1.1) is equivalent to $\tilde{G}_{ab} = \kappa\,\tilde{T}_{ab}$ with

$$\tilde{T}_{ab} = (\tilde{\nabla}_a\tilde{\phi})(\tilde{\nabla}_b\tilde{\phi}) - \frac{1}{2}\,\tilde{\phi}\,\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + \frac{1}{4}\,\tilde{\phi}^2\,\tilde{R}_{ab} - \frac{1}{4}\,\tilde{g}_{ab}\left((\tilde{\nabla}^c\tilde{\phi})(\tilde{\nabla}_c\tilde{\phi}) + \frac{1}{6}\,\tilde{\phi}^2\,\tilde{R}\right). \tag{1.3}$$

The $\tilde{}$ labels quantities of the physical spacetime.

Equation (1.2) is form invariant under the rescalings $g_{ab} = \Omega^2\,\tilde{g}_{ab}$ and $\phi = \Omega^{-1}\,\tilde{\phi}$, the field equation $\tilde{\Box}\tilde{\phi} = 0$ for the massless Klein-Gordon scalar field is not. This was the reason for not choosing the massless Klein-Gordon scalar field with its simpler field equation in physical spacetime. In [5] it has been shown that the initial value problems for these matter models are mathematically equivalent.

As coordinates $(t, r, \vartheta, \varphi) = (u + v, v - u, \vartheta, \varphi)$ with double null coordinates (u, v) have been chosen. (ϑ, φ) coordinatize the orbit of the symmetry group and are omitted further on. It can be shown that this coordinate system covers the whole domain of dependence of the initial value surface [4].

2 Description of formalism

To define asymptotical flatness and to describe radiation Penrose developed a formalism in which the physical spacetime $(\tilde{M}, \tilde{g}_{ab})$ is mapped to the interior of an "unphysical" spacetime (M, g_{ab}) with boundary $\mathcal{I}$, see e. g. [3]. This construction is very similar to the mapping of the plane of complex numbers to the Riemannian sphere.

Asymptotical flatness is by definition equivalent to the existence of the rescaling $g_{ab} = \Omega^2\tilde{g}_{ab}$ together with requirements on the properties of Ω. $\mathcal{I}$ can be identified with infinitely far points and is a null hypersurface consisting of two disjoint parts $\mathcal{I}^-$ and $\mathcal{I}^+$. Null geodesics start at $\mathcal{I}^-$ and end at $\mathcal{I}^+$, therefore $\mathcal{I}$ is also called null infinity. Under certain conditions $\mathcal{I}$

can be extended to include three more points, i^0, i^- and i^+, representing the endpoint of all spacelike geodesics (spacelike infinity i^0) and the start and the endpoint of timelike geodesics (past and future timelike infinity).
In the used initial value formulation I give data on a hypersurface Σ which intersects $\mathcal{I}^+$ and is spacelike with respect to g_{ab}. Σ corresponds to an everywhere spacelike hypersurface $\tilde{\Sigma}$ in $\tilde{M}$ which approaches asymptotically an outgoing null cone. This kind of initial value problem is called a hyperboloidal initial value problem. Actually one would like to solve a "normal" initial value problem, but a normal spacelike hypersurface ends at i^0 and for spacetimes with non-vanishing mass there are unsolved technical problems in treating i^0.
Asymptotical flatness of the initial data is reflected in regularity conditions on the data at the intersection of Σ with $\mathcal{I}^+$ (represented by a point in (t,r) coordinates). Σ can be extended beyond the intersection with $\mathcal{I}^+$, but the choice of data does not influence the evolution on the part M corresponding to $\tilde{M}$ and is therefore irrelevant for the physics. For that reason the physics does not depend on the treatment of the outer grid boundary if the numerical scheme converges and the outer boundary is beyond $\mathcal{I}$.
Although Penrose's formalism provides an elegant language to formulate and describe global properties of spacetimes, it is not well suited for initial value problems. The field equations (1.1) and (1.2) describing the physical spacetime transform under the rescaling to a set of equations which are singular for $\Omega = 0$. Hence they cannot easily be used to solve an initial value problem in the unphysical spacetime. Friedrich found a new system of equations, which are regular on $\mathcal{I}$ by interpreting the rescaled Einstein equation as an equation for the rescaling factor and adding new equations derived from the Bianchi identity. This system is equivalent to the Einstein equation and a symmetric hyperbolic subsystem of evolution equations can be extracted [2, 5].
In the coordinates used and under the assumption of spherical symmetry the evolution equations are also semilinear. The characteristics of the resulting first order system have slope ± 1 and 0.

3 Numerical calculations

For both examples the data are chosen such that (i) ϕ has compact support on Σ and $\tilde{\Sigma}$ and (ii) the scalar field would purely ingoing[1] for $\kappa = 0$ (a condition on $\dot{\phi}$). The setting $\kappa = 0$ corresponds to a scalar field on Minkowski background. For $\kappa = 0$ all the wave moves towards the center, passes the center, moves outward and finally crosses $\mathcal{I}$ (the thick line in figure 1, "I" marks the region with non-vanishing ϕ). The fact that the wave moves over $\mathcal{I}$ is also phrased "escapes to null infinity".
For a nonlinear model, obtained by setting $\kappa = 1$, a non-vanishing scalar field in region II and III is expected for the same initial setting of ϕ and $\dot{\phi}$. This is a pure backscattering effect. As the coordinate system is obtained by reference to the characteristics of the equation (double null coordinates and semilinearity), this comparison of linear and nonlinear models is well defined. Region IVa and IVb have vanishing scalar field also in the nonlinear case.

[1] the direction is of course reverted when the wave goes through the center

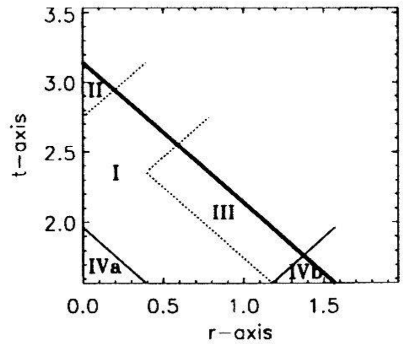

Figure 1: qualitative picture of the structure in the large

The location of $\mathcal{I}$ is known by analytic considerations, it is the ingoing null line starting at the point with $\Omega = 0$ on the initial value surface. The comparison with the calculated $\Omega = 0$ contour provides an error estimate.
For weak initial data the spacetime evolved from the initial data possesses the same conformal structure as Minkowski space, $\mathcal{I}$ intersects the axis and this intersection point has the required properties for regular future timelike infinity (infinite proper time for an observer in physical space time, correct behaviour of Ω and its derivatives). For strong initial data this intersection point is "hidden behind a singularity".

3.1 Location of singularities and event horizons

Figure 2 shows the upper corner of a supercritical spacetime. The amplitude of the initial scalar field pulse was 0.55, the critical parameter, where a singularity develops for the first time, is in the interval $]0.48, 0.49[$. The dashed thick line is $\mathcal{I}$ and the thick line is the calculated singularity. The singularity is spacelike near the center and approaches a null line near $\mathcal{I}$. A point of the grid is called singular if one or more of the following occur: (i) at least one variable of the system becomes too large, (ii) the principal part of the system of equations becomes singular or (iii) the values depend on points already marked singular. Due to the third condition and the fact that the scheme is run at a Courant factor of 1 it is possible to distinguish spacelike from timelike/nullike singularities. If in addition to the numerical singularity spacetime invariants like $R^{abcd}R_{abcd}$ blow up approaching the the numerical singularity, as in this case, the numerical singularity can be identified with a real singularity.
In all cases calculated the physical singularity was spacelike and covered by an apparent horizon (thin line in figure 2, the null expansion of the outgoing null direction vanishes here). The intersection of $\mathcal{I}$ with the singularity is a singular future timelike infinity (i^+), for an observer moving into i^+ an infinite amount of proper time passes. The outgoing null line ending at i^+ is the event horizon of the spacetime (dotted line). As null lines have slope ± 1, the determination of the event horizon is straightforward.
The results found agree with those derived by Christodoulou [1].

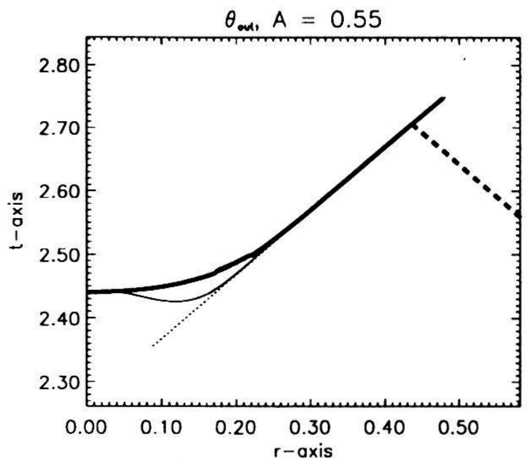

Figure 2: θ_{out} for $A = 0.55$

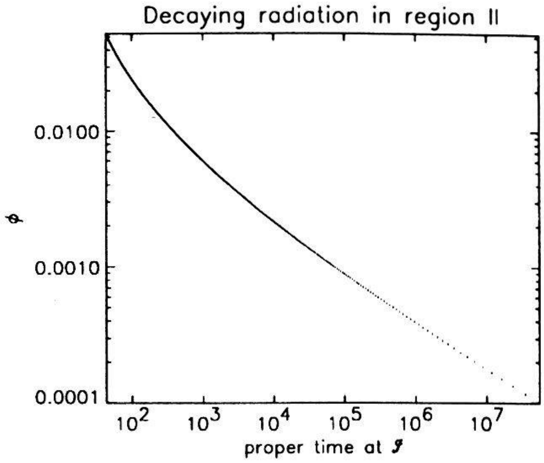

Figure 3: ϕ in region II

3.2 Decay of radiation

Figure 3 shows the radiation field ϕ on $\mathcal{I}$ in region II plotted against the proper time τ of an observer at $\mathcal{I}$ for an initial amplitude of 0.40 well below the critical value but already a model with strong gravitational fields. The expression for the proper time does not contain terms singular at $\mathcal{I}$ like Ω^{-1} although it includes the limit $\tilde{r} \to \infty$ for fixed $\tilde{t} - \tilde{r}$.
Region II starts at a proper time of $\approx 10^{1.5} \approx 30$, the time scale of the collapse event. The calculation delivered reliable results up to a proper time of order 10^8. Results are called reliable if the value for a given gridsize is on the plot indistinguishable from the value got from a Richardson extrapolation obtained from half and quarter gridsize calculations. As the proper time distance of the grid points decreases on the approach to $\mathcal{I}$ the dots in figure 3 become sparse. The very last points before i^+ become inaccurate in the defined sense and are not shown in figure 3. The plot shows a clear logarithmic fall-off $\phi \sim (\tau - \tau_0)^{0.34}$. I do not know of any other method which allows one to follow the evolution for such a long time scale.

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