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## Classical and Quantum Aspects of 1+1 Gravity

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*Abstract.* We present a classification of all global solutions (with Lorentzian signature) for any general 2D dilaton gravity model. For generic choices of potential-like terms in the Lagrangian one obtains maximally extended solutions on arbitrary non-compact two-manifolds, including various black-hole and kink configurations. We determine all physical quantum states in a Dirac approach. In some cases the spectrum of the (black-hole) mass operator is found to be sensitive to the signature of the theory, which may be relevant in view of current attempts to implement a generalized Wick-rotation in 4D quantum gravity.

There is some good news and some bad news about the subject of this talk. To start with the bad news: I am going to speak about *pure* gravity-Yang-Mills systems. This means that matter fields are not included (with the exception of dilaton fields, which are not regarded as matter fields in our context). Furthermore, I am going to speak about  $(1 + 1)$  dimensional theories. As we seemingly live in a  $(3 + 1)$  dimensional universe, this might not be precisely what one eventually is interested in.

And here is the good news: We can give a comprehensive treatment of the subject, encompassing a large variety of different gravity models. We deduce a method to calculate

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all classical solutions of all of these models on the local as well as on the global level. And in some setting, made more precise later in this talk, we are able to calculate all physical quantum states and to analyze the quantum spectra of observables. We arrive at a family of quantum gravity theories, where we have complete control over the phase space of the underlying classical theory. This allows one to gain deep insights into the relation between features of the quantum theory and structures on the classical space-time.

The class of models considered comprises all 2D dilaton gravity theories [1] with a dilaton-dependent coupling to a Yang-Mills (YM) field:

$$L[g, \Phi, a] = \int_{\mathcal{M}} d^2x \sqrt{|\det g|} [U(\Phi)R + V(\Phi) + W(\Phi)\partial_\mu\Phi\partial^\mu\Phi + K(\Phi)\text{tr}(F_{\mu\nu}F^{\mu\nu})] . \quad (0.1)$$

Here  $g$  is the metric on the 2D space-time  $\mathcal{M}$ ,  $\Phi$  is the dilaton field,  $F = da + a \wedge a$  is the curvature of the YM-connection  $a$ , and  $U, V, W, K$  are some basically arbitrary functions specifying the model. In what will follow the gravity part can be extended further so as to include also terms giving rise to non-trivial torsion [2].

Our first important observation is that, in a first order formalism, the above gravity-Yang-Mills systems may be reformulated as Poisson- $\sigma$ -models [3]:

$$L = \int_{\mathcal{M}} A_i \wedge dX^i + \frac{1}{2} \mathcal{P}^{ij}(X(x)) A_i \wedge A_j . \quad (0.2)$$

Here the  $A_i$  and  $X^i$  are a multiplet of one-forms and functions on  $\mathcal{M}$ , respectively, and  $\mathcal{P}^{ij}$  is an antisymmetric  $X^i$ -dependent matrix, determined by  $U, V, W, K$ , satisfying the crucial equality  $\mathcal{P}^{il} \partial \mathcal{P}^{jk} / \partial X^l + \text{cycl}(i, j, k) = 0$ . Up to appropriate dilaton-dependent prefactors,  $A_i$  collects zweibein, spin-connection, and  $a$ , while, in the torsion free case, the  $X^i$  comprise two Lagrange multipliers enforcing torsion zero, the dilaton field  $\Phi$ , and momenta associated to the YM-connection [2]. The identity satisfied by  $\mathcal{P}^{ij}$  turns out to equip the manifold  $\mathcal{N}$  spanned by the  $X^i$  with a *Poisson bracket*  $\{X^i, X^j\}_{\mathcal{N}} := \mathcal{P}^{ij}(X)$ . This is the reason for calling the two-dimensional topological field theories given by (0.2) Poisson- $\sigma$ -models. The analysis of  $(1+1)$ -gravity-Yang-Mills systems is greatly facilitated by the reformulation.<sup>2</sup>

In particular, using the formalism provided by Poisson- $\sigma$ -models (cf. Part I of [2] or [3]), the classical field equations can be brought into an extremely simple form locally. They give rise to a  $(1 + \text{rank}(\text{gauge group}))$ -parameter family of gauge inequivalent solutions, labelled by a mass parameter and Yang-Mills charges associated to the Casimirs of the gauge group. The corresponding local expression for the metric depends on the mass parameter  $M$  and the charge  $q$  of the quadratic Casimir only and can be brought into the form

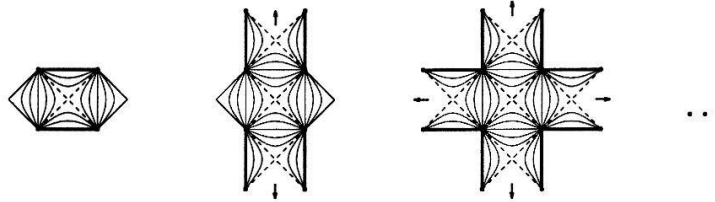
$$g = 2drdv + h_{M,q}(r)dv^2 , \quad (0.3)$$

where the family of functions  $h_{M,q}$ , labelled by  $M$  and  $q$ , is determined by the potentials  $U, V, W, K$ , characterizing the model (cf. Part I of [2]); e.g. for  $U \equiv \Phi$ ,  $W \equiv K \equiv 0$  one has  $h = -\int V(r)dr + M$ .

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<sup>2</sup>The situation closely resembles  $(2+1)$ -dimensional Einstein gravity, which can be reformulated as an  $ISO(2,1)$ -Chern-Simons theory; however, now there is a whole family of such gravity theories and the reformulation as a topological field theory is of the form of an ordinary non-abelian gauge theory only in exceptional cases.

In general the local expression (0.3) for the metric does not represent a complete space-time. To find the global solutions we have to construct maximal extensions. Restricting ourselves in a first step to (simply connected) universal covering solutions, this may be done using the building block principle developed in Part II of [2]. Typically, this leads to black-hole configurations such as depicted below. Any solution with non-trivial first homotopy can then



**Figure:** Possible Penrose diagrams for an increasing number of zeros of  $h$  in (0.3) .

be obtained in a second step by dividing out some discrete subgroup of the isometry group from the universal covering. (Note that the isometries of the metric extend to symmetries of the entire solution). Restricting our attention to orientable and time-orientable manifolds, the isometry group takes the simple form of a *direct* product  $\mathbb{R} \times \mathcal{F}(k)$ , where  $\mathbb{R}$  corresponds to transformations generated by the Killing field  $\partial/\partial v$  and  $\mathcal{F}(k)$  is a *free* group of rank  $k$  with

$$k = \# \text{simple zeros of } h + 2 \# \text{multiple zeros of } h - 1. \quad (0.4)$$

For a given universal covering the diffeomorphism inequivalent maximally extended space-time solutions are in one-to-one correspondence with the conjugacy classes of the subgroups of this isometry group. For the above group they can be determined explicitly and, in particular, for  $k > 1$  smooth solutions on two-surfaces of arbitrary genus with an arbitrary nonzero number of punctures can be obtained! The full space of gauge inequivalent solutions of (0.1) on such a two-surface has dimension  $(\text{rank}(\pi_1(\mathcal{M})) + 1)$  times  $(\text{rank}(\text{gauge group}) + 1)$ . For more details cf. Part III of [2].

To quantize the model (0.2) we choose a Hamiltonian Dirac approach. This restricts  $\mathcal{M}$  to be of the form  $\Sigma \times \mathbb{R}$  and we choose  $\Sigma = S^1$ . As (0.2) is in first order form already, the Hamiltonian structure may be read off easily: Denoting coordinates on  $S^1 \times \mathbb{R}$  by  $(r, t)$ , the  $r$ -components of the one-form valued  $A_i$ -variables serve as the momenta for the fields  $X^i(r)$ , while their  $t$ -components give rise to the set of first class constraints  $G^i(r) \equiv X^i(r)' + \mathcal{P}^{ij} A_{jr} \approx 0$ . On the quantum level the latter are imposed as operator conditions selecting the physical wave functions.

The solution to the quantum constraints of the field theory is intimately related to the geometry of the target space, recognized above as the one of a ‘Poisson manifold’  $\mathcal{N}$ . A Poisson structure  $\mathcal{P}$  on a manifold  $\mathcal{N}$  naturally induces a foliation of the manifold into symplectic leaves, each of which carries a symplectic structure  $\Omega$ .<sup>3</sup> These symplectic submanifolds may be regarded as classical mechanical systems. If their second homology is trivial,

<sup>3</sup>The matrix  $\mathcal{P}^{ij}$  determined from (0.1) is degenerate; only by restriction to one of the above-mentioned submanifolds, the integral surfaces of the vector fields  $\mathcal{P}^{ij} \partial/\partial X^j$ ,  $\mathcal{P}^{ij}$  becomes non-degenerate with inverse

then the mechanical systems can be quantized. If the symplectic leaves have non-trivial second homology, then the quantization is possible only if the symplectic form is integral, i.e.  $\int_{\sigma} \Omega = 2\pi n \hbar$ ,  $\forall \sigma \in H_2(\text{leaf})$ . This condition will select a discrete subset from the set of symplectic leaves with non-trivial second homology.

One finds the following connection between the quantum theory associated to the Poisson- $\sigma$ -model and the quantum mechanical systems defined on the symplectic leaves of  $\mathcal{N}$  [3]: Physical quantum wave functions of the Poisson- $\sigma$ -model correspond to the quantizable symplectic leaves of  $\mathcal{N}$ . If the first homotopy of the respective symplectic leaf is trivial, then there is precisely one quantum state corresponding to it. Otherwise the quantum states corresponding to a particular symplectic leaf are labelled by ‘winding numbers’  $l_1 \dots l_m$ , where  $m = \text{rank } \pi_1(\text{leaf})$ .

Applying these results to the gravity-Yang-Mills systems, the quantum states can effectively be written as wave functions depending on

- a continuous parameter  $M$  (the mass parameter characterizing the classical solutions)
- a set of quantum numbers  $j$  labelling the irreducible representations of the gauge group
- possibly further integer parameters  $l_1 \dots l_m$ , if the symplectic leaf corresponding to the value of  $M$  has non-trivial first homotopy.

The result presented above holds for Minkowskian signature of the space-time topology. In the case of Euclidean space-time the symplectic leaves will have a different topology. For some models (e.g. deSitter gravity) the mass spectrum becomes purely discrete, for others it remains continuous (e.g. spherically reduced gravity), while in general the spectrum of the mass operator  $M$  will be a mixture of both of these scenarios.

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$\Omega$ . The symplectic leaves may be characterized by constant values of functions on  $\mathcal{N}$  which may be identified with the mass parameter  $M$  and the YM-charges characterizing the local classical solutions.