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The Quantum Two-Body Problem in the presence of Curvature

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Abstract Two scalar particles undergoing a mutual interaction can be described by a pair of coupled Klein-Gordon equations. This approach takes into account the composite nature of the system (recoil effects) and allows for a covariant setting of symmetries and first integrals. Assuming that the form of these equations is explicitly given in the flat-spacetime limit, we discuss their coupling to an external gravitational field. Beside compatibility, we require preservation of the isometries. Writting down the wave equations in closed form is possible for some particular metrics; we give examples.

Twenty years ago it was reckognized that a system of two mutually interacting particles in flat spacetime can be described by a pair of coupled wave equations, say $2H_a^{(0)}\Psi = m_a^2\Psi \qquad a,b = 1,2$, for scalar bosons. $\Psi = \Psi(q_1, q_2)$. The squared-mass operators $2H_a^{(0)}$ reduce to p_1^2, p_2^2 for free particles [1].
The contact with quantum field theory is ensured either through the quasipotential approach or directly by invoking the Bethe-Salpeter (B-S) equation [2].

Generalization to quantum dynamics in curved spacetime is formally straightforward. In the presence of curvature it is natural to replace the D'Alembert box by the Laplace-Beltrami operator, which yields

$$2H_a\Psi \equiv (-g^{\mu\nu}(a)\nabla_{a\mu}\nabla_{a\nu} + V_a)\Psi = m_a^2\Psi \tag{1}$$

for $a = 1,2$. Here Ψ depends on two points in spacetime, $g^{\mu\nu}(a)$ and $\nabla_{a\mu}$ respectively represent the metric tensor and the covariant differentiation at point q_a.
For most practical purposes we can assume that $V_1 = V_2 = V$. But this term is required

to satisfy the condition $[H_1, H_2] = 0$ which ensures that the wave equations are mutually compatible. This condition is regarded as an equation for the explicit determination of V

$$[H_1 - H_2, V] = 0 \tag{2}$$

Insofar as mutual interaction corresponds to a perturbative field theory one should, in principle, derive eq (1) from a BS equation taking curvature into account. Several difficulties of quantum field theory in curved manifolds make this program rather problematic, so we follow a different approach. Let us assume that we know the mutual interaction potential $V^{(0)}$ when considering the system in flat spacetime. Say $V^{(0)} = f(Z, P^2, y \cdot P)$ in the notations of a previous work [3]. In order to carry out the coupling of this system to a curvature field, we use the compatibility condition (2) complemented with a few reasonable requirements expressing our concern for invariance and correct limits.
Therefore, in addition to compatibility we have proposed [4] this axiomatics:

(i) Hermiticity. Squared-mass operators in the left-hand-side of eq.(1) should be hermitian, that is *symmetric* with respect to the Hilbert space $L^2(\mathbf{R}^8) = L^2(\mathbf{R}^8, \sqrt{g(1)g(2)}d^4q_1\, d^4q_2)$.

(ii) Coupling Separability. Correct limits must be recovered when either mutual interaction or curvature vanishes.
In fact we suppose that the form of the mutual interaction in flat spacetime is explicitly known; in the absence of curvature, the mutual interaction would be represented by some operator $V^{(0)}$. In general V will differ from $V^{(0)}$ because relativistic interactions cannot be linearly composed.

(iii) Isometric Invariance. If the spacetime metric admits some isometry, two-body dynamics must be invariant under the isometry group.

Justifications and comments are in order.
Though $L^2(\mathbf{R}^8)$ has no direct physical interpretation, it is of importance for the consistancy of our operator formalism, notice that axiom (i) is obviously satisfied when V vanishes.
Axiom (ii) rises a conceptual problem because the *no-curvature limit* of a given spacetime is generally ill-defined. Nevertheless some particular spacetimes have a preferred flat background, which permits to consider the notion of an isolated system, free of (external) gravitational coupling. Needless to say, this ambiguity about the "turning off" of gravity is systematically ignored when the weak-field approximation is performed.
When the metric is invariant under some infinitesimal isometry, one-body dynamics reflects this symmetry. In contradistinction it is not sufficient that also $V^{(0)}$ be invariant, in order to automatically ensure invariance of the two-body motion.

Assuming that the mutual interaction potential describing the system in flat spacetime is given, it remains generally impossible to determine the explicit form of the equations of motion for the same system imbedded in a curvature field. A similar situation already arises in the framework of special relativity if we first consider an isolated two-body system an then we attempt to carry out its coupling to some external field. It was fortunately pointed out by J.Bijtebier [5] that, if the external potential is stationary in a *strong* sense, the wave equation can be written in closed form, at the price of a suitable transformation which amounts to a change of representation. In this special case the problem of coupling to an external field is finally solved by an Ansatz. Turning back to the two-body problem in curved spacetime it was reasonable to expect that some stationary metrics allow for explicit determination of the wave equations. Indeed if we introduce an auxiliary flat metric $\overline{g}$, curvature can be viewed as an external field. In this *bimetric approach* which is reminiscent of the Rosen-Papapetrou formalism, coupling to curvature turns out to be equivalent to a problem of coupling with external potentials in the framework of the flat metric. In order to implement axiom iii) the flat background must be in some sense compatible with the isometries of $g_{\mu\nu}$. But isometrically admissible backgrounds are generally such that $\Gamma = (g/\overline{g})^{\frac{1}{4}}$ is not a constant.

In exceptional cases where $g/\overline{g} = 1$ one may define, for the single-particle problem, an effective gravitational potential as the difference between $K = \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $\frac{1}{2}p^2 = \frac{1}{2}\overline{g}^{\mu\nu}p_\mu p_\nu$ (except for $g_{\mu\nu}$, we rise indices with the flat metric).

But in general it is convenient to replace K by $\widetilde{K} = \Gamma K \Gamma^{-1}$ which is flat-hermitian and correspond to a transformed Klein-Gordon equation $2\widetilde{K}\widetilde{\Psi} = m^2\widetilde{\Psi}$. According to this modification, we introduce an effective one-body potential F through the formula $\widetilde{K} = \dfrac{1}{2}p^2 + F$.

Naturally the transformation resulting from Γ extends to the two-body sector. Each particle separated gets coupled to curvature by $F_1 = F(q_1, p_1)$ and $F_2 = F(q_2, p_2)$ respectively; one considers the transformed equations $2\widetilde{H}_a\widetilde{\Psi} = m_a^2\widetilde{\Psi}$ where $\widetilde{\Psi} = \Gamma_{12}\Psi$, $\Gamma_{12} = \Gamma(1)\,\Gamma(2)$. Now all the operators are flat-hermitian, which is a transformed form of axiom i) [4].

Provided $g_{\mu\nu}$ is strongly static (in the sense of Ref.4), we perform a further transformation and obtain coupled wave equations in closed form, in terms of a new wave function $\widetilde{\Psi}'$. The Ansatz [5] consists in writting that $\widetilde{V}' = f(-\mathbf{z}^2P^2 - (\mathbf{z}\cdot\mathbf{P})^2, P^2, y^{(0)}P^{(0)})$. One combination of the wave equations allows for eliminating the relative time. The other one stems from the formula [4]

$$\widetilde{H}_1' + \widetilde{H}_2' = \frac{P^2}{4} + y^2 + F_1 + F_2 - 2T\frac{y_0}{P_0} + \frac{T^2}{P_0^2} + \widetilde{V}'$$

where $y = (1/2)(p_1 - p_2)$, $\qquad T = -\mathbf{y}\cdot\mathbf{P} + F_1 - F_2$.

Strong time translation invariance arises when the metric is static orthogonal with the additional property that $g_{00} = 1$. In order to have a preferred flat background we shall consider the following cases (notice that in both cases the isometries of $g_{\mu\nu}$ are also displacements of the flat metric $d\overline{s}^2 = (dq^0)^2 - (d\mathbf{q})^2$. We define the contravariant h, not necessarily small, by $g^{\mu\nu} = \overline{g}^{\mu\nu} + h^{\mu\nu}$).

Case a)

$$ds^2 = (dq^0)^2 - R^2(r)d\mathbf{q}^2$$

with R a nonvanishing function of $\mathbf{q}^2$ such that $R(0) = R(\infty) = 1$. Case a) exhibits spherical symmetry but has $\Gamma \neq 1$, which entails a complicated expression of F. For a weak field h, we can set $R^2 = 1 + G$ and get *at first order* $h^{\mu\nu} = diag(0, G, G, G)$, thus up to $O(G^2)$

$$F = -\frac{3}{4}(p^2G + Gp^2) + (1 - \frac{3}{4}G)p_\mu h^{\mu\nu}p_\nu(1 - \frac{3}{4}G) + \frac{3}{2}(1 - \frac{3}{4}G)p_\mu G g^{\mu\nu}p_\nu(1 - \frac{3}{4}G)$$

Case b)

$$ds^2 = (dq^0)^2 - S^2[(dq^1)^2 + (dq^2)^2] - S^{-4}(dq^3)^2$$

where the nonvanishing function S depends only on the q^3 coordinate and $S(0) = S(\infty) = 1$. This line element is invariant by displacements in the (q^1, q^2) plane. Moreover it satisfies $\Gamma = 1$ hence $F = p_\mu h^{\mu\nu} p_\nu$ and $\widetilde{\Psi} = \Psi$, etc. This is encouraging for detailed calculations involving $(H'_1 + H'_2)\Psi'$ along the lines of refs. 3-5.

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