Zeitschrift: Helvetica Physica Acta
Band: 69 (1996)
Heft: 3

Artikel: Sphaleron on S^3
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DOI: https://doi.org/10.5169/seals-116940

Helv Phys Acta
Vol. 69 (1996)

0018-0238/96/030289-04$1.50+0.20/0

Sphaleron on S^3

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Abstract. An exactly solvable sphaleron model in $3+1$ spacetime dimensions is described

1 Introduction

The notion of *sphaleron* refers to the special type of static classical solution in a gauge field theory with periodic vacuum structure and broken scale invariance [1]. Specifically, sphaleron relates to the top of the potential barrier between distinct topological vacua, such that its energy determines the barrier height. Sphaleron can play the important role in the transition processes when the system interpolates between distinct topological sectors. Consider a thermal ensemble over one of the perturbative topological vacua. Such a system is metastable since the field modes are able to reach the neighboring topological sectors both via the quantum tunneling and due to the thermal overbarrier fluctuations. If temperature is high enough, the latter effect is dominant, in which case the sphaleron, 'sitting' on the top of the barrier, controls the transition rate. To evaluate the rate of such sphaleron-mediated thermal transitions, the Langer-Affleck formula is often used [2]:

$$\Gamma = -\frac{|\omega_-|}{\pi}\frac{\mathrm{Im}Z_1}{Z_0}, \tag{1.1}$$

which relates the probability of the decay of the unstable phase with the imaginary part of the free energy. Here Z_0 and Z_1 are the partition functions for the small fluctuations around the vacuum and the sphaleron, respectively. Since the sphaleron has one unstable mode whose eigenvalue $\omega_-^2 < 0$, the quantity Z_1 is purely imaginary. To compute Γ at the one-loop level is usually rather difficult. The problem becomes especially hard in the standard model case, where the sphaleron solution itself is known only numerically. That is why the other sphaleron models for which Γ can be evaluated exactly have been investigated [3], [4], however these models exist only in $1+1$ spacetime dimensions.

2 The sphaleron on S^3

To find an exactly solvable sphaleron model in $3+1$ dimensions [5], we consider the theory of a *pure* non-Abelian $SU(2)$ gauge field in the static Einstein universe $(M, \mathbf{g})$, where $M = R^1 \times S^3$, and the metric is

$$ds^2 = \mathbf{a}^2\{-d\eta^2 + d\xi^2 + sin^2\xi(d\vartheta^2 + sin^2\vartheta d\varphi^2)\}; \tag{2.1}$$

here $\mathbf{a}$ is a constant scale factor. Consider the following $SU(2)$-valued function on the S^3:

$$U = U(\xi, \vartheta, \varphi) = \exp\{-i\xi\ n^a\tau^a\}, \tag{2.2}$$

where $n^a = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ and τ^a are the Pauli matrices. This function defines the mapping $S^3 \to SU(2)$ with the unit winding number. Using U, we construct the following sequence of the static gauge field potentials:

$$A[h] = i\frac{1+h}{2}UdU^{-1}, \tag{2.3}$$

where the parameter $h \in [-1, 1]$. When $h = -1$ this field vanishes, whereas for $h = 1$ it is a pure gauge whose winding number is one, by construction. Thus fields (2.3) interpolate between the two distinct topological vacua, and the energy

$$E[h] = \int T_0^0\sqrt{^3\mathbf{g}}d^3x = \frac{3\pi^2}{g^2\mathbf{a}}(h^2-1)^2 \tag{2.4}$$

has the typical barrier shape — it vanishes at the vacuum values of h, $h = \pm 1$, and reaches its maximum in between, at $h = 0$; (g in (2.4) stands for the gauge coupling constant). The top of the barrier relates to the field configuration

$$A^{(sp)} = \frac{i}{2}\ UdU^{-1}, \tag{2.5}$$

which obeys the Yang-Mills equations and therefore can be naturally called sphaleron. It is worth noting that the sphaleron configuration consists of the gauge field alone. The violation of the scale invariance in this case is provided by the background curvature. Since the spacetime geometry is $SO(4)$-symmetric, the sphaleron inherits the same symmetries, such that, for instance, the energy-momentum tensor for the field (2.5) has the manifest $SO(4)$-symmetric structure.

3 The sphaleron transition rate

Our main task is to compute the transition rate (1.1) for the sphaleron solution (2.5). We pass to the imaginary time τ in the metric (2.1) and impose the periodicity condition, $\tau \in [0, \beta]$. Let us introduce $A_\mu^{\{j\}} = jA_\mu^{(sp)}$, which corresponds to the vacuum of the gauge field for $j = 0$ and to the sphaleron field for $j = 1$. Next we consider small fluctuations around the background gauge field: $A_\mu^{\{j\}} \rightarrow A_\mu^{\{j\}} + \phi_\mu$. Notice that we assume the spacetime metric (2.1) to be fixed and therefore do not take into account the gravitational degrees of freedom. The partition functions Z_j are then given by the Euclidean path integral over ϕ_μ. To compute the integral, we impose the background gauge condition and use the Faddeev-Popov procedure. The result is [5]:

$$Z_j = \exp(-S[A^{\{j\}}])\ \mathcal{N} \frac{Det'(\hat{M}_j^{FP}/\mu_0^2)}{\sqrt{Det'(\hat{M}_j/\mu_0^2)}}, \tag{3.1}$$

where S is the Euclidean action, the factor $\mathcal{N}$ is due to the zero and negative modes whereas Det' has all such modes omitted, μ_0 is an arbitrary normalization scale, and the fluctuation operators are

$$\hat{\mathbf{M}}_j\phi^\nu = -D_\sigma D^\sigma \phi^\nu + R^\nu_\sigma \phi^\sigma + 2i[F^\nu{}_\sigma, \phi^\sigma], \qquad \hat{\mathbf{M}}_j^{FP}\alpha = -D_\sigma D^\sigma \alpha. \tag{3.2}$$

Here $D_\mu = \nabla_\mu - i[A_\mu^{\{j\}},\ ]$ is the covariant derivative, R^ν_σ is the Ricci tensor for the geometry (2.1), $F^\nu{}_\sigma$ is the gauge field tensor for $A_\mu^{\{j\}}$, and α is a Lie algebra valued scalar field.

To find spectra of these operators, we introduce the 1-form basis $\{\omega^0, \omega^a\}$ on the spacetime manifold, where $\omega^0 = d\tau$, and ω^a are the left invariant 1-forms on S^3. It is convenient to expand the fluctuations as $\phi = (\phi_p^0\omega^0 + \phi_p^a\omega^a)\tau^p/2$. Let e_a be the left-invariant vector fields dual to ω^a, such that $\mathbf{L}_a = \frac{i}{2}e_a$ are the $SO(4)$ angular momentum operators. We introduce also spin and isospin operators as follows: $\mathbf{S}_a\phi_p^b = \frac{1}{i}\varepsilon_{abc}\phi_p^c$ and $\mathbf{T}_p\phi_r^a = \frac{1}{i}\varepsilon_{prs}\phi_s^a$. As a result, the fluctuation operators (3.2) can be expressed entirely in terms of the operators $\mathbf{L}_a$, $\mathbf{S}_a$ and $\mathbf{T}_p$, such that the spectra can be explicitly obtained by the purely algebraic methods [5]. All of the eigenvalues are positive except for the following ones: the sphaleron fluctuation operator $\hat{\mathbf{M}}_1$ has one negative mode, whereas the vacuum operators $\hat{\mathbf{M}}_0$ and $\hat{\mathbf{M}}_0^{FP}$ have three zero modes each. It is worth noting that, since the sphaleron field configuration is $SO(4)$ invariant, the sphaleron itself does not have zero modes at all (in the background gauge imposed).

The next step is to compute the products of the eigenvalues to evaluate the determinants in (3.1). For this, zeta function regularization scheme has been used. Omitting all technical details given in [5], the resulting expression for the transition rate can be represented in the following form:

$$\Gamma = \frac{1}{8\sqrt{2}\pi^2 \sin(\beta/\sqrt{2})} \exp\left\{-\frac{3\pi^2}{g^2(\mathbf{a})}\beta - \mathcal{E}_0\beta - \beta(F_1 - F_0)\right\}. \tag{3.3}$$

In this expression, the prefactor in the right hand side is the overall contribution of zero and negative modes. $3\pi^2\beta/g^2(\mathbf{a})$ is the Euclidean action of the sphaleron, where the gauge coupling constant receives the quantum correction due to the scaling behavior of the functional determinants:

$$\frac{1}{g^2(\mathbf{a})} = \frac{1}{g^2(\mathbf{a}_0)} - \frac{11}{12\pi^2}\ln\left(\frac{\mathbf{a}}{\mathbf{a}_0}\right). \tag{3.4}$$

Here we have replaced g by $g(\mathbf{a}_0)$, where $\mathbf{a}_0 = 1/\mu_0$. This expression agrees with the renormalization group flow, such that it does not depend on the scale $\mathbf{a}_0$ if $g(\mathbf{a}_0)$ is chosen to obey the Gell-Mann-Low equation. To fix the scale, we assume that the value of $g(\mathbf{a}_0)$ is determined by the physical temperature, $T(\mathbf{a}_0) = 1/\beta\mathbf{a}_0$, and use the QCD data:

$$T(\mathbf{a}_0) = 100 \text{ GeV}, \qquad \frac{g^2(\mathbf{a}_0)}{4\pi} = 0.12. \tag{3.5}$$

One can assume that the weak coupling region extends up to some $\mathbf{a}_{max} \sim 10 \div 100\mathbf{a}_0$. The next term in (3.3), $\mathcal{E}_0$, is the contribution of the zero field oscillations, that is, the Casimir energy. This quantity can be computed exactly [5], the numerical value is $\mathcal{E}_0 = -1.084$. The contribution of the thermal degrees of freedom in (3.3) is

$$\beta(F_1 - F_0) = 4\ln(1 - e^{-\beta}) + 2\sum_{\sigma=0,1,2}\sum_{n=3}^{\infty}(n^2 - \sigma^2)\ln(1 - e^{-\beta\sqrt{n^2+\sigma^2-3}}) -$$

$$- 6\sum_{n=2}^{\infty}(n^2 - 1)\ln(1 - e^{-\beta n}). \tag{3.6}$$

Altogether Eqs.(3.3)-(3.6) provide the desired solution of the one-loop sphaleron transition problem. The numerical curves of $\Gamma(\beta)$ evaluated according to these formulas for several values of $\mathbf{a}$ are presented in [5]. This solution makes sense under the following assumptions:

$$\mathbf{a} \leq \mathbf{a}_{max}, \qquad \frac{1}{\sqrt{2}\pi} < \frac{1}{\beta} \ll \frac{3\pi^2}{g^2(\mathbf{a})}. \tag{3.7}$$

The first condition is the the weak coupling requirement. When the scale factor $\mathbf{a}$ is too large, the running coupling constant (3.4) becomes big (confinement phase), and the effects of the strong coupling can completely change the semiclassical picture. That is why our solution can be trusted only for the small values of the size of $\mathbf{a}$. The other condition in (3.7) requires that the thermal fluctuations are small compared to the classical sphaleron energy, such that the perturbation theory is valid.

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