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Einstein-Yang-Mills equations on cosmological space-times [1]

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Abstract. The equations of Einstein-Yang-Mills fields admitting as symmetry the full isometry group of space-times of the type $\mathbb{R} \times \Sigma$ where Σ is a homogeneous Riemannian manifold are derived and the method to classify possible equations for the larger compact gauge groups is briefly discussed.

1 Introduction

The combined system of Einstein's gravitational and the Yang-Mills field equations has lead to many interesting problems and results over the last few years. Both the gravitational and the Yang-Mills fields are perhaps the most fundamental in modern physics and it makes sense to study their interaction in various settings and from different points of view. While the results obtained for static spherically symmetric space-times are perhaps the most interesting also EYM cosmologies have soon been considered both in the classical and quantized versions. (See, for example, [1],[2].) Although classical solutions of spatially homogeneous gravitational fields coupled to pure Yang-Mills fields do not seem to represent what is presently regarded as realistic universes they are still likely to exhibit features that would be expected when also other non gauge fields are present.

This report is a brief account of investigations about how to set up systematically, rather than by an ansatz, the equations of classical cosmological models with the topology $\mathbb{R} \times \Sigma$

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where Σ is a homogeneous Riemannian manifold and where the Yang-Mills connection is invariant under the full isometry group of the space-time.

It is convenient to first consider the EYM equations on a space-time with a transitive isometry group. Let (M, g) be a connected pseudo-Riemannian manifold with its Levi-Civita connection, and K its isometry group. Its (left) action, $\bar{\psi} : K \times M \to M : (a, x) \mapsto \bar{\psi}_a x$ is transitive and effective on M. Fixing a point $x_0 \in M$ (to be called the *origin*) the isotropy subgroup K_0 of K is defined by $K_0 := \{a \in K \mid \bar{\psi}_a x_0 = x_0\}$. For practical calculations we introduce a basis $\{\mathbf{e}_\alpha | \alpha = 1 \ldots m\}$ of the Lie algebra $\mathfrak{k}$ such that the corresponding generators $\{\mathbf{e}_a | a = 1 \ldots n\}$ span the tangent space $T_{x_0}M$ at x_0 while the $\{\mathbf{e}_\Gamma | \Gamma = n + 1 \ldots m\}$ span the Lie subalgebra $\mathfrak{k}_o$. Thus, if the structure constants are introduced by $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = c^\lambda_{\alpha\beta}\mathbf{e}_\lambda$, then $c^a_{\Gamma\Delta} = 0$.

The infinitesimal generators $\bar{\mathbf{e}}_a$ on M corresponding to $\mathbf{e}_a$ form a frame field in a neighborhood of x_0 and a pseudo-Riemannian metric $g_{ab}\bar{\theta}^a \otimes \bar{\theta}^b$ on M, where $\{\bar{\theta}^a\}$ is the frame dual to $\{\bar{\mathbf{e}}_a\}$, is fixed by giving the g_{ab} at x_0 subject to

$$g_{r(a}c^r_{b)\Gamma} = 0 \tag{1}$$

because of the ad_{K_0}-invariance.

We now assume that the full isometry group K of space-time acts on the principal bundle P by PB-automorphisms $\tilde{\psi}_a$ and leaves the YM-connection on P invariant. Then the equivalence classes of such $\tilde{\psi}$-invariant principal bundles P over M are in one-to-one correspondence with conjugacy classes of homomorphisms $\lambda : K_0 \to G$. Here λ and $\tilde{\psi}$ are related by $\tilde{\psi}_a(u_0) = R_{\lambda(a)}u_0 \quad \forall\, a \in K_0$ where u_0 is any fixed element of $\pi^{-1}(x_0)$. Moreover, Wang's theorem states that (for fixed λ) the set of $\tilde{\psi}$-invariant connections $\tilde{\omega}$ on P is in one-to-one correspondence with the set of linear maps $\Lambda : \mathfrak{k} \to \mathfrak{g}$ that agree with λ on $\mathfrak{k}_o$ and are equivariant with respect to ad on $\mathfrak{k}$. It is thus possible to compute all derivatives of the gauge fields explicitly and to formulate the Yang-Mills-Einstein equations on a homogeneous space-time in purely algebraic form. Let σ be a local section of P and introduce the local gauge potential A and the gauge field F by $A = \sigma^*\tilde{\omega}$ and $F = \sigma^*\tilde{\Omega}$. Then it can be shown that $\mathcal{L}_{\bar{X}}F = -[\Lambda(X), F] - [A(\bar{X}), F]$ where $\bar{X}$ is the generator on M. This allows us to express the frame components of the YM-field F and the field equations in terms of Λ and the structure constants only (without involving the gauge, but only at the 'origin' x_0),

$$F_{ab} = [\Lambda_a, \Lambda_b] - c^r_{ab}\Lambda_r - c^\Sigma_{ab}\lambda_\Sigma, \tag{2}$$

$$[\Lambda^r, F_{ra}] + \Gamma^s_{ar}F_s{}^r + F_{a\ell}\Gamma^\ell_{rs}g^{rs} = 0, \tag{3}$$

$$R_{ab} = \kappa T_{ab} = \langle F_{ar}, F_b{}^r\rangle - \tfrac{1}{4}\langle F_{rs}, F^{rs}\rangle g_{ab} \tag{4}$$

where $\langle , \rangle$ represents a biinvariant scalar product on the Lie algebra $\mathfrak{g}$.

Now let (M, g) now be an $n+1$-dimensional space-time manifold with an isometry group K whose orbits are n-dimensional spacelike hypersurfaces so that $M = \Sigma \times \mathbb{R}$ with K acting transitively on Σ and K_0 the isotropy subgroup at $x_0 \in \Sigma$. We choose to describe the metric by a coordinate time t and a frame field $\{\mathbf{e}_a\}$ of Killing vector fields on Σ, $g = -dt \otimes dt + g_{ab}\bar{\theta}^a \otimes \bar{\theta}^b$. Assume also that the $\bar{\mathbf{e}}_\Gamma$ $(\Gamma = n + 1 \ldots m)$ vanish at a point $x_0 \in \Sigma$. It then follows that the Σ_t-coordinate components of the frame vectors $\bar{\mathbf{e}}_a$ do not depend on the time t so that $[\partial_t, \bar{\mathbf{e}}_\alpha] = 0 \; \forall \; \alpha = 1 \ldots m$.

The further analysis is analogous to the one above and leads to the field equations

$$[E^r, \Lambda_r] - c^r_{rs} E^s = 0, \tag{5}$$

$$\dot{E}_a + [A_0, E_a] + K^r_r E_a - 2K^r_a E_r - [B_{ar}, \Lambda^r] + B_a{}^s c^r_{rs} - \tfrac{1}{2} g_{ar} c^r_{pq} B^{pq} = 0. \tag{6}$$

$$K^r_a c^s_{rs} + K^r_s c^s_{ra} = \kappa \epsilon_a{}^{rs} <E_r, B_s>, \tag{7}$$

$$\overset{\Sigma}{R} + (K^r_r)^2 - K^{rs} K_{rs} = \kappa(<E_r, E^r> + <B_r, B^r>), \tag{8}$$

$$\dot{K}_{ab} - 2K_{ar} K^r_b + K^r_r K_{ab} + \overset{\Sigma}{R}_{ab} = \kappa T_{ab}. \tag{9}$$

where $K_{ab} = \frac{1}{2}\dot{g}_{ab}$ is the extrinsic curvature of the hypersurfaces and a dot denotes the time derivative, $B_a = \frac{1}{2}\epsilon_a{}^{rs}\left([\Lambda_r, \Lambda_s] - c^t_{rs}\Lambda_t - c^\Sigma_{rs}\lambda_\Sigma\right)$, and $E_a = \partial_t \Lambda_a + [A_0, \Lambda_a]$.

The gauge can be chosen such that $A_0 = 0$. Then, after a basis of the symmetry Lie algebra $\mathfrak{k}$ and the homomorphism $\lambda : K_0 \to G$ are chosen and a point $x_0 \in \Sigma$ is fixed, we have as dynamical variables the functions $g_{ab}(t)$, subject to (1), and the $\mathfrak{g}$-valued functions $\Lambda_a(t)$, subject to

$$[\Lambda_a, \lambda_\Gamma] + c^r_{a\Gamma}\Lambda_r = -c^\Sigma_{a\Gamma}\lambda_\Sigma. \tag{10}$$

Only a time-independent basis transformation in $\mathfrak{k}$ by automorphisms leaving $\mathfrak{k}_o$ invariant can now be used to possibly eliminate some variables. The algebraic problem of finding the possible homomorphisms λ amounts to classifying the $\mathfrak{k}_o$ subalgebras of $\mathfrak{g}$ and is solved, at least for semisimple $\mathfrak{k}_o$ and $\mathfrak{g}$, in [3]. Since the isotropy group K_0 is a subgroup of $SO(n)$ and thus compact so that the homogeneous space is reductive the system (10) is homogeneous and Λ can be regarded as an intertwining operator between two linear representations of K_0. For fourdimensional space-times the isotropy group can only be either $SO(3)$ or $U(1)$ (or trivial). Also the g_{ab} are arbitrary, subject to (1). But not all choices need lead to nonisometric space-times. One can reduce the number of free parameters by bringing g_{ab} into a canonical form using basis transformations by automorphisms of K that leave the subgroup K_0 invariant.

The best known cosmological models are the isotropic ones. In this case the K_0 is $SO(3)$ or $SU(2)$ and Σ must be of constant curvature k and its isometry group K therefore $SO(4)$, $E(3)$ or $SO(3,1)$, respectively, depending on whether k is positive, zero or negative. The Lie algebra has thus a basis $\{e_i, f_i\}$ $(i = 1 \ldots 3)$ with commutators

$$[e_i, e_j] = k\epsilon_{ij}{}^r f_r, \quad [e_i, f_j] = \epsilon_{ij}{}^r e_r \quad \text{and} \quad [f_i, f_j] = \epsilon_{ij}{}^r f_r. \tag{11}$$

where the f_i span the Lie algebra of the isotropy group, we can choose k to be ± 1 or 0 and the $\epsilon_{ij}{}^r$ in this section now refers to the Euclidean metric in $\mathbb{R}^3$.

The geometry of these isotropic models is then already determined, namely it is the one of the well known Friedman-Robertson-Walker space-times and the metric can be given by $g = R(\tau)^2(-d\tau^2 + \delta_{ab}\bar{\theta}^a \otimes \bar{\theta}^b)$ where τ is a conformal time coordinate. In view of the fact that $T^\lambda_\lambda = 0$ the time evolution of the metric is completely determined and one only needs to formulate the equations for the Yang-Mills field. If $\Lambda_i := \Lambda(e_i)$ and $\lambda_i := \lambda(f_i)$ we have from (10) the homogeneous linear system

$$[\lambda_i, \Lambda_j] = \epsilon_{ij}{}^r \Lambda_r \tag{12}$$

for the $\Lambda_i(\tau)$. One solution is $\Lambda_i(\tau) = \lambda_i =: \Lambda_i^0$, say. If $\{\Lambda_i^K, K = 0, \ldots, r-1\}$ is a basis of the solution space then we can express Λ_i in terms of r time-dependent 'amplitudes', $\Lambda_i = \sum_{K=0}^r \Phi_K(\tau)\Lambda_i^K$. The YM equations then become

$$L^{KL}\Phi_K'\Phi_L = 0 \quad \text{and} \quad \Phi_K'' - 2k\Phi_K + \tfrac{1}{2}\gamma_K^{LM}\gamma_M^{PQ}\Phi_L\Phi_P\Phi_Q = 0, \tag{13}$$

and the YM field components are

$$E_i = R^{-1}\Phi_K'\Lambda_i^K \quad \text{and} \quad B_i = R^{-1}\left(\tfrac{1}{2}\gamma_M^{KL}\Phi_K\Phi_L - k\delta_M^0\right)\Lambda_i^M. \tag{14}$$

Here the quantities Λ_i^K and the L^{KL}, which form a skew symmetric matrix of elements of $\mathfrak{g}$, and the scalars γ_M^{KL} depend only on the Lie algebra $\mathfrak{g}$ and the homomorphism $\lambda : \mathfrak{su}(2) \to \mathfrak{g}$. To compute them systematically one needs the structure theory of the Lie algebra $\mathfrak{g}$. In special cases, as for $G = SU(n)$ or $SO(n)$ they can be obtained more directly from the well known unitary or orthogonal representations of $\mathfrak{su}(2)$. For homomorphisms that correspond to irreducible unitary representations one finds that there is only one amplitude but for other λ's there can be several.

For axisymmetric homogeneous cosmological models where K is fourdimensional and $K_0 = U(1)$ the possible homomorphisms $\Lambda : \mathfrak{k}_o \to \mathfrak{g}$ are easier to classify. This is done for arbitrary compact G in, for example, in [4] in the context of spherically symmetric spacetimes. Wang's equations are slightly different in the cosmological case but can be analyzed in an analogous way.

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