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# Multi-Black-Holes in 3D and 4D anti-de Sitter Spacetimes

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*Abstract.* The (single) black hole solutions of Bañados, Teitelboim and Zanelli (BTZ) in 2+1 dimensional anti-de Sitter space are generalized to an arbitrary number  $n$  of such black holes. The resulting multi-black-hole (MBH) spacetime is locally isometric to anti-de Sitter space, and globally it is obtained from the latter as a quotient space by means of suitable identifications. The MBH spacetime has  $n$  asymptotically anti-de Sitter exterior regions, each of which has the geometry of a single BTZ black hole. These exterior regions are separated by  $n$  horizons from a common interior region. This interior region can be described as a “closed” universe containing  $n$  black holes. Similar configurations in 3+1 dimensions, with horizons of toroidal and higher genus topologies, are also presented.

## 1 Introduction

In 3D Einstein's equations determine the full Riemann tensor. In vacuum (with only a cosmological constant  $\Lambda$ ) the spacetime has constant curvature, there are no local gravitational degrees of freedom. If one thinks of black holes as concentrations of curvature it is surprising that in 2+1 dimensions there are “black hole” solutions [1]. Their black hole properties derive solely from the global structure of the spacetime.

Because there are no local degrees of freedom it is reasonable that there should also be multi-black-hole solutions. In this contribution I sketch the construction of such solutions,

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and mention the corresponding spacetimes in 3+1 dimensions.

## 2 Initial values

The black hole construction is possible only if the constant curvature is *negative*,  $\Lambda = -1/\ell^2$ . The universal covering space of such spaces is (unwrapped) 3D anti-de Sitter (adS) space. Its wrapped version can be embedded in 4D flat space with signature  $--++$ ,

$$ds^2 = -dU^2 - dT^2 + dx^2 + dy^2 \quad (1)$$

as a hyperbolic surface,

$$-U^2 - T^2 + X^2 + Y^2 = -\ell^2. \quad (2)$$

When the induced metric on this surface is expressed in terms of the (non-spinning) BTZ [1] coordinates  $r, \phi, t$ ,

$$\frac{X}{T} = \sqrt{-\ell^2 + \frac{r^2}{M}} \frac{\cosh}{\sinh} \left( \frac{\sqrt{M}}{\ell} t \right), \quad \frac{U}{Y} = \frac{r}{\sqrt{M}} \frac{\cosh}{\sinh} (\sqrt{M}\phi) \quad (3)$$

it takes the form

$$ds^2 = \left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2}} + r^2 d\phi^2, \quad (4)$$

which is similar to that of the Schwarzschild black hole metric. It also has many similar global properties (as expressed, for example, in a Penrose diagram).

The surface  $t = \text{const}$  is totally geodesic, and therefore its intrinsic 2D geometry is also of constant, negative curvature. The universal covering space of such constant curvature spaces is 2D hyperbolic space,  $\mathbf{H}^2$ , which can be represented as the Poincaré disk (or as the Poincaré upper half space.) The disk can be obtained by *stereographic projection* [2] of the  $T = 0$  subspace of the surface of Eq (2) on the plane  $U = -\ell$  from a projection center at  $X = Y = 0, U = \ell$ .

The lines  $\phi = \text{const}$  are geodesics, and are therefore represented in the Poincaré model as circular arcs perpendicular to the boundary curve (circumference in the case of the disk model). We cut the  $t = 0$  initial state of the BTZ black hole along  $\phi = 0$  and  $\phi = \pi$ , obtaining two congruent strips in the Poincaré model, one of which is shown in Figure 1a. To reassemble the BTZ initial state we take two copies of this figure and join them together along the boundaries  $\phi = 0$  and  $\phi = \pi$ . We call this procedure of joining two copies of a region along geodesic boundaries “doubling” of the figure. (That the two copies fit together smoothly follows from the identity of the intrinsic geometry of the boundaries, and vanishing of their extrinsic curvature, since they are geodesics.) The region between any pair of non-intersecting geodesics on the Poincaré disk, when doubled, yields a BTZ black hole of finite mass parameter  $M$ . If the geodesics meet on the “limit circle” (at infinity),  $M$  vanishes; otherwise any such region can be brought in the symmetrical position of Fig. 1a by means of an isometry of the Poincaré disk. In that position it is obvious that the horizon occurs at

the minimal geodesic connecting the two boundary geodesics, and  $M$  is determined by the length of that geodesic compared to  $\ell$ . (Alternatively,  $M$  measures the angle that “infinity” subtends at the horizon.)

By the same method one can obtain multi-black-hole (MBH) initial states, with several asymptotically adS regions [3]. Figure 1b shows a strip which, when doubled, gives an initial state with three asymptotic regions, which we call a 3-black-hole state. (Figure 1c shows the same geometry in the upper half plane model.) By identifying two of the horizons, for example  $h_2$  and  $h_3$ , we get a “wormhole-type” geometry with only one asymptotic region. By identifying horizons by pairs in an originally 4-black-hole geometry we obtain a compact, negative curvature geometry.

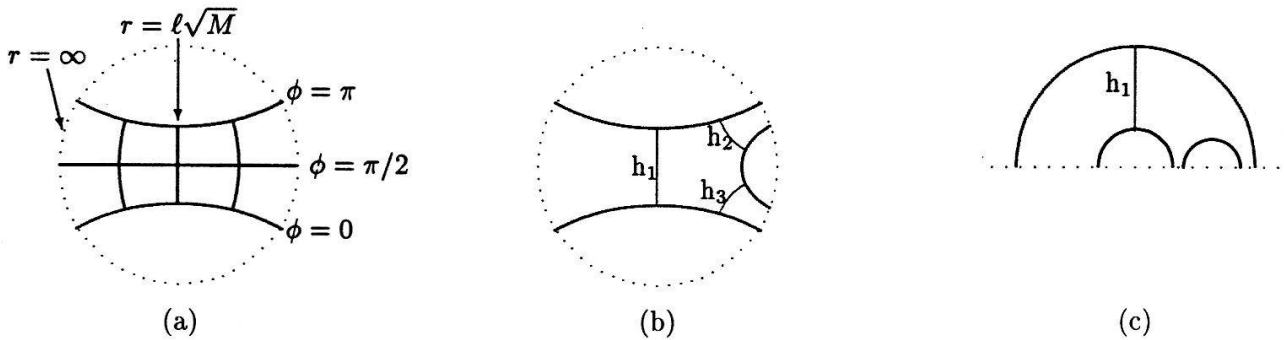


Fig. 1. Construction of MBH initial values, shown in the Poincaré model of  $\mathbf{H}^2$ . In this model geodesics are arcs of circles perpendicular to the boundary (dotted), which represents infinity.

(a) In the disk model the “single” BTZ black hole’s initial state is the region in which some BTZ coordinate lines are shown (thicker lines). This region is to be doubled, as described in the text. The horizon is the vertical geodesic segment at  $r = \ell\sqrt{M}$ .

(b) Initial state of a “3-black-hole” configuration, with three asymptotically adS regions reaching out to infinity. Each of these exterior regions is separated from the common interior region by a horizon  $h$  (thinner arcs), the shortest geodesic segment between a pair of non-intersecting geodesic boundaries (thicker arcs). The length  $\pi\ell\sqrt{M_i}$  of the horizon  $h_i$  measures the mass  $M_i$  associated with the  $i^{\text{th}}$  exterior. (In this figure the lengths of the three horizons, as determined by the Poincaré metric, and hence the three masses, are actually equal).

(c) The same geometry as in (b) in the upper half-plane model. Only one of the horizons is shown.

### 3 Time Development

The left half of Fig. 1b is identical to that of Fig. 1a, hence the time development in its domain of dependence will also be the same. But that domain is the region outside the spacetime horizon corresponding to  $h_1$ ; thus to an observer in an exterior region outside of one of the horizons  $h_i$  the future spacetime geometry is indistinguishable from that of a single BTZ black hole. Because the past of such observers includes more than the exterior’s past domain of dependence, they can be aware of the difference from a single BTZ black hole, for example via light from the “white hole” singularity.

The region between the horizons of a MBH initial state has finite area; except for the openings at the horizons it is a closed universe model. Because it is locally homogeneous, its time development in time-orthonormal coordinates will amount simply to an overall scale change, described for example by the Raychaudhuri equation [2]. The scale factor decreases to zero in the finite proper time  $\pi\ell/2$ , at which time a non-Hausdorff singularity, similar to that of the BTZ black hole [4], is reached.

## 4 Analogous Configurations in 3+1 Dimensions

The BTZ idea can be generalized to 4D spacetimes in various ways [5]. The following metrics, found in collaboration with Dr. J. Louko and obtained by analytic continuation and re-scaling of the Schwarzschild-de Sitter metric [6], satisfy the vacuum Einstein equations with cosmological constant  $\Lambda$ :

$$ds^2 = -F dt^2 + \frac{dr^2}{F} + r^2 \begin{pmatrix} d\theta^2 + \cosh^2 \sqrt{M} \theta d\phi^2 \\ d\theta^2 + \theta^2 d\phi^2 \end{pmatrix} \quad \text{with } F = \begin{cases} -M - \frac{2m}{r} - \Lambda r^2 & (M \neq 0) \\ -\frac{2m}{r} - \Lambda r^2 & (M = 0). \end{cases}$$

When  $\Lambda = -1/\ell^2$  is negative this has the appropriate signature at large  $r$  and becomes the adS metric there, and when  $m = 0$  and  $\theta = 0$  it becomes the BTZ metric (4). However, to complete the analogy, the surfaces of constant  $r$  and  $t$  should be compactified. In the case  $M = 0$  these 2D surfaces are flat, so a torus compactification is possible. If  $M \neq 0$  these surfaces have constant negative curvature, and can also be compactified, for example as described at the end of Section 2. When the Schwarzschild-type mass parameter  $m$  vanishes, joining two or more of these in their anti-de Sitter regions yields MBH geometries. Simple joining is not possible when  $m \neq 0$  because the curvature is then no longer constant. Instead, the single black hole with  $m \neq 0$  has a curvature singularity at  $r = 0$ , like the Schwarzschild geometry.

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