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Extended AKS Theorem, the Moment Map and New Integrable Systems

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Abstract. Extended Version of AKS theorem is used in conjunction with co–cycle and moment map to derive some new integrable systems in loop algebra. In the first case we get equations like, three wave interaction, second order matrix nonlinear system, while in the case of moment map we obtain generalizations of dispersive water wave equation and MKdV system.

1 Introduction

The Adler–Kostant–Symes theorem [1] is one of the most important tools for the construction and investigation of the properties of nonlinear integrable systems. The AKS theorem gives a purely Lie theoretic setting to the study of such systems. Whether the underlying Lie algebra is finite or infinite dimensional, the theorem can always be used to construct the integrable equation. Its use provides a Lax pair and a Hamiltonian structure just by construction. Moreover, the complete integrability of the nonlinear dynamical systems [2] so constructed, is always guaranteed due to Liouville's theorem, which demands the existence of an infinite number of conserved quantities in involution.

On the other hand, moment maps [3] have also been used in relation to integrable systems. Efforts have been made to provide a systematic link between integrable systems, flows in loop algebras and integrable systems of partial differential equations through the use of moment maps. The construction of such maps allows us to apply the results of the AKS theorem to derive large class of integrable systems from the commutativity of the flows.

In the present paper, we follow a generalization of the AKS theorem due to Reyman et al [4] to derive some new integrable systems in loop algebra. Among these is the three wave interaction system which is of considerable interest in plasma physics [5], later in the paper, we utilize the moment map construction along with the extended AKS theorem to deduce some more integrable equations among which are the familiar mKdV and the generalized Nonlinear Schrödinger and dispersive water wave equations.

Let us outline the motivation for this paper. Although the AKS theorem has been around for quite some time, we felt that its application in obtaining new as well as interesting integrable systems is fairly nontrivial and therefore worth studying. Moreover, the use of the extended AKS theorem in the back drop of the moment map technique seems to be a pretty interesting formalism.

Our paper is organized as follows. In section 2, we introduce the extended AKS theorem in the context of loop algebra and use it in section 3 to derive some new integrable systems. In section 4 we introduce the moment map concept and study Poisson commutativity via the extended AKS theorem, Finally in section 5 we derive some coupled integrable PDEs by using the formulation of section 4.

2 The Loop algebra and the extended AKS Theorem

Let g be a finite dimensional Lie algebra which can be written as the vector space direct sum, $g = k \bigoplus n$, where k and n are not only subspaces but they are also subalgebras. Suppose we are adding a co–cycle with g, then $g^* = g \bigoplus \Re$. Further, suppose that we have a non degenerate, ad–invariant bilinear form $<,>$: $g^* X g^* \longrightarrow C$; by means of which we can identify the dual of g^*, i.e. $\tilde{g}^*$. From this bilinear form one can identify $k^* \sim n^\perp$ and $n^* \sim k^\perp$, where $< k^\perp, k >= 0 =< n^\perp k^\perp, n >$. Let $(x, a) \in g^*$ where $X \in g$ and $a \in \Re$. Then we have the commutator bracket,

$$[(x,a),(y,b)] = ([x,y], \int \text{tr } xy') \tag{1}$$

On g^*, the defining bilinear form will be,

$$< (x,a),(y,b) >= ab + \int \text{tr } xy \tag{2}$$

Let $\hat{g} = g^* \times C[\lambda, \lambda^{-1}]$ be the affine Lie algebra related to g^*, that is, the loop algebra of formal Laurent series in λ with coefficients in g^*.

The Lie bracket on $\hat{g}$ is defined by

$$[(\sum_{i=r}^{s} x_i\lambda^i, a(\lambda)), (\sum_{j=k}^{l} y_j\lambda^j, b(\lambda))] = \sum_{i=r}^{s}\sum_{j=k}^{l}[x_i, y_j]\lambda^{i+j}, Res(\text{tr } \int (\sum_i x_i\lambda^i)(\sum_j y_j'\lambda^j)) \tag{3}$$

With this preparation we can now state the basic idea of AKS Theorem. Let L be a co–adjoint orbit (which is actually the Lax operator under consideration) through an element of the Lie algebra, then the AKS theorem asserts,

$$\frac{\partial L}{\partial t} = [L, \pi_n(\nabla H)] + \partial_x(\pi_n(\nabla H)) \tag{4}$$

π_n denoting projection on the subalgebra n and ∇H are ad* invariant functions obtained as solution of the equation

$$[L, \nabla H] + \partial_x(\nabla H) = 0 \tag{5}$$

Let us now introduce the extended AKS theorem due to Reyman et al. [4]. An element Y of $n^* \sim k^\perp$ is called an (infinitesimal) character of n if

$$[X, Y]_{k^\perp} = 0\ \forall X\ \in\ n \tag{6}$$

Let I_Y be defined by $I_Y(\mu) = I(\mu + Y)$, $\mu \in g^*$.

Now, $[X, Y] = \sum_{i=-\infty}^{-l} \sum_{j=0}^{\infty} [a_i, b_j]\lambda^{i+j}$. For $[X, Y] = 0$ the commutator must contain only negative powers of λ. Hence, the only allowed value of j which satisfies (6) is $j = 0$. Then in the loop algebra case, we have $Y(\lambda) = Y$, $Y \in g$ is a character of n.

It is now easy to deduce that in case of the extended AKS theorem, the ad* – invariant ring of functions which are in involution will be determined from the condition,

$$[L', \nabla H(L')] + \partial_x(\nabla H(L')) = 0 \tag{7}$$

where

$$L' = L + Y \tag{8}$$

The Hamiltonian flow is given by an equation of the following form,

$$L_t = [L', \pi_n(\nabla H(L'))] + \partial_x(\pi_n(\nabla H(L'))) \tag{9}$$

In equations (4) and (9) the derivative is with respect to the spatial coordinate x.

3 Integrable Systems via Extended AKS Theorem

Before applying the extended AKS theorem to derive new integrable systems, it is necessary to calculate explicitly the co–adjoint orbit in $n^\perp$. We assume, without any loss of generality, that g is already in a faithful, linear representation.

Let $\delta = \lambda^{-3}M \in n^\perp \subseteq K^*$ and let Θ_δ be the co–adjoint orbit in k^* generated by $\phi \in \exp(k)$. Then a general element L in Θ_δ is given by [6,7],

$$\begin{aligned} L(\lambda) &= [\phi^{-1}\,\delta\,\phi]_- = [\exp(-(\lambda\xi_1 + \lambda^2\xi_2 + \ldots))\lambda^{-3}\,M\exp((\lambda\xi_1 + \lambda^2\xi_2 + \lambda^3\xi_3)]_- \\ &= \lambda^{-3}A + \lambda^{-2}B + \lambda^{-1}E \end{aligned} \tag{10}$$

Here A, B and E are functions determined by M, ξ_1, ξ_2, etc., $[\ ]_-$ denotes the projection into the sub algebra n.

In case of the extended AKS theorem, the orbit will be given by (8).

To determine the ad* invariant functions, we take

$$\nabla H = \sum_{k=0}^{\infty} \lambda^{k-3} k_{-3} \tag{11}$$

and use (7). Easy calculation shows that,

$$\pi_n(\nabla H) = \lambda^{-3}A + \lambda^{-2}B + \lambda^{-1}E \tag{12}$$

which is the second Lax operator for the integrable equation to be deduced below. Equation (9) leads to the following Hamiltonian flows,

$$A_t = [Y, A] + A_x, \; B_t = [Y, B] + B_x, \; E_t = [Y, E] + E_x \tag{13}$$

Equation (13) shows clearly how the extended AKS theorem gives rise to nonlinearity in the system. Setting $Y = 0$ would give a linear system of equations. Choosing $Y = A + B + E$, we arrive at the following system of equation,

$$A_t = [B + E, A] + A_x, \; B_t = [E + A, B] + B_x, \; E_t = [A + B, E] + E_x \tag{14}$$

Equations (14) are similar to those for the system of three wave interaction encountered in plasma physics.

For clarity we write out the first of equations (13) in non–matrix form. Choosing SL(2) as the underlying finite dimensional algebra and writing a general element X of this element in the form,

$$X = X_1\sigma_3 + X_2\sigma_+ + X_3\sigma_- \tag{15}$$

(σ_3, σ_+ and σ_- being Pauli matrices) we obtain the system of equations,

$$A_{1t} = A_{1x} + Y_2A_3 - Y_3A_2, \; A_{2t} = A_{2x} + 2(Y_1A_2 - Y_2A_1), \; A_{3t} = A_{3x} + 2(Y_3A_1 - Y_1A_3) \tag{16}$$

It is apparent that application of the usual AKS theorem ($Y_1 = Y_2 = Y_3 = 0$) would never give rise to the nonlinear system of equations (16).

As another example of our formalism, we take the orbit as

$$L' = \lambda^{-2}A + \lambda^{-1}B + Y \tag{17}$$

Taking

$$\nabla H = \sum_{k=0}^{\infty} \lambda^{k-1} h_{k-1} \tag{18}$$

and proceeding as before, we arrive at,

$$\pi_n \nabla H(L') = \lambda^{-1}A \tag{19}$$

which is the second Lax operator for this problem. Use of equation (9) leads to

$$A_t = [B, A],\ B_t = [Y, A] + A_x \tag{20}$$

If we take Y to be a consistent matrix and $A = W_t$ then equation (20) gives

$$W_{tt} = [[Y, W], W_t] + [W_x, W_t] \tag{21}$$

This second order nonlinear matrix system was considered by Calogero [8] earlier. Again it is clear that use of the usual AKS theorem would not lead to equation (21), We write out (21) in non–matrix form as follows:

$$W_{1tt} = 2y_1(W_2W_{3t} + W_3W_{2t}) - 2y_2W_1W_{3t} - 2y_3W_1W_{2t} + W_{2x}W_{3t} - W_{3x}W_{2t} \tag{22}$$

$$W_{2tt} = 2y_2(W_3W_{2t} + 2W_1W_{1t}) - 2y_3W_2W_{2t} - 4y_1W_2W_{1t} + 2(W_{1x}W_{2t} - W_{2x}W_{1t} \tag{23}$$

$$W_{3tt} = 2y_3(2W_1W_{1t} + W_2W_{3t}) - 4y_1W_3W_{1t} - 2y_2W_3W_{2t} + 2(W_{3x}W_{1t} - W_{1x}W_{3t}) \tag{24}$$

It is clear that there is no triviality or overdeterminacy in the system of equations (22-24).

Furthermore, examples of new integrable systems may be constructed following the procedure outlined above.

4 Moment Maps

We first recapitulate the necessary definitions regarding moment maps. More details can be found in [3].

Let (M, ω) be a symplectic manifold, for $f \in C^\infty(M)$ the associated Hamiltonian vector field X_f is defined by

$$X_f \rfloor \omega = df \tag{25}$$

an the Poisson bracket in $C^\infty(M)$ is :

$$\{f, g\} = -X_f(g) \tag{26}$$

Then $C^\infty(M)$ may be regarded as a Lie algebra with respect to the Poisson–Lie bracket and denoting the Lie algebra of vector fields on the manifold M by $\chi(M)$, the map

$$\beta : C^\infty(M) \to \chi(M), \beta(f) = -X_f \tag{27}$$

is a homomorphism of Lie algebras.

Let $\phi : G \times M \to M$ be a smooth group action preserving ω and denote by $\hat{G}$ the Lie algebra of G. The infinitesimal $\hat{G}$ action is given by the homomorphism [9,10].

$$\sigma : \hat{G} \to \chi(M), \quad \sigma(\xi)(x) = -\frac{d}{dt}\phi(\exp(t\xi), x)\Big|_{t=0}, \quad \xi \in \hat{G},\ x \in M \tag{28}$$

The G action is called Hamiltonian if there exists a moment map:

$$J : M \to \hat{G}^*$$

such that the hamiltonian flow generated by $< j, \xi >$ coincides with $x \to \phi(\exp(t\xi), x)$, i.e.

$$X_{<J,\xi>} = \sigma(\xi) \tag{29}$$

Let $M_{N,r}$ be the space of complex $N \times r$ matrices, and let us identify $M_{N,r} \sim M^*_{N,r}$ via the pairing

$$(F, G) = \text{tr}\,(E^T G), F, G \in M_{N,r} \tag{30}$$

Let us consider several group actions on $M_{N,r} \times M_{N,r}$ which are hamiltonian with respect to the symplectic form

$$\omega = \text{tr}(dF \wedge dG^T) \tag{31}$$

For $n \leq N$ let

$$G^n_r = G\,L(r, C)x...X\,G\,L(r, C) \quad \text{(n times)} \tag{32}$$

be the direct product Lie group and

$$\hat{G}^n_r = gl(r, C) \bigoplus ... \bigoplus gl(r, C) \quad \text{(n times)} \tag{33}$$

be its Lie algebra.

Let K_1, K_2,..., K_n be positive integers with $K_i < r$ and $\sum_{i=1}^n K_i = N$. For $F \in M_{N,r}$, let F_i denote the $K_i \times r$ block whose j-th row is the $(K_1 + \cdots + K_{i-1} + j)$-th row of F; i.e. F has the block form

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_i \\ \vdots \\ F_n \end{pmatrix}$$

Now, let us define a Hamiltonian G^n_r action on $M_{N,r} \times M_{N,r}$ by

$$(g(F, G))_i = (F_i\, g_i^{-1}, G_i\, g_i^T), g = (g_1 ... g_n)\; G^n_r \tag{34}$$

Let $\sigma^n_r : G^n_r \to \chi(M_{N,r} \times M_{N.R})$ denotes the corresponding infinitesimal action. The associated moment map $J^n_r : M_{N,r} \times M_{N.R}(\hat{G}^n_r)^*$ is given by

$$J^n_r(F, G)(X_1, X_2 ... X_n) = -\sum_{j=1}^{n} \text{tr}\,(F_j\; X_j\; G_j^T) \tag{35}$$

Identifying $gl(r, C)^*$ with $gl(r, C)$ through the trace of matrix products, and hence $(\hat{G}^n_r)^*$ with $\hat{G}^n_r$, we obtain,

$$J^n_r(F, G) = -(G_1^T\, F_1, G_2^T\, F_2, ..., G_n^T\, F_n) \in \hat{G}^n_r \tag{36}$$

Let us now return to our loop algebra situation. We follow the notations of section 2. However, in contrast to the def. of the inner product introduced there, here we define the non degenerate ad–variant inner product by

$$< X(\lambda), Y(\lambda) >= \text{tr}\ ((X(\lambda)\ Y(\lambda))_0) = \text{res}(\lambda = 0)\ \text{tr}(\lambda^{-1}X(\lambda)Y(\lambda)) \tag{37}$$

where $X(\lambda)\ Y(\lambda))_0$ denotes the constant term in the formal series $X(\lambda)$, $Y(\lambda)$. Using (49), we have the identification

$$(\widetilde{gl(r)}^{+})^* \sim (\widetilde{gl(r)}^{-})^{\perp} = \widetilde{gl(r)}_0^{+} \tag{38}$$

Let us now fix n distinct complex numbers λ_1, λ_2, ...,λ_n. Since $X(\lambda) \in \widetilde{gl(r)}^{+}$ is a polynomial we can evaluate at $\lambda = \alpha_i$ to obtain $X(\alpha_i) \in gl(r, C)$. This gives a Lie algebra homomorphism $A : \widetilde{gl(r)}^{+} \to \hat{G}_r^n$ defined by

$$A(X(\lambda)) = (X(\alpha_1), X(\alpha_2), ..., X(\alpha_n)) \tag{39}$$

The kernel of this map is $a(\lambda)\widetilde{gl(r)}^{+}$, where $a(\lambda) = \prod_{i=1}^{n} (\lambda - \alpha_i)$. Hence we have the exact sequence of Lie algebra homomorphisms:

$$0 \to a(\lambda)\widetilde{gl(r)}^{+} \to (l)\ \widetilde{gl(r)}^{+} \to (A)\ \hat{G}_r^n \to 0$$

where l is inclusion. The dual sequence is thus

$$0 \to (\hat{G}_r^n)^* \to (A)^*\ (\widetilde{gl(r)}^{+})^* \to (l)^*\ (a(\lambda)\ \widetilde{gl(r)}^{+})^* \to 0 \tag{40}$$

If we identify $(\hat{G}_r^n)^*$ with $(\hat{G}_r^n)$ by using the trace component wise, and $(\widetilde{gl(r)}^{+})^*$ with $(\widetilde{gl(r)}^{-})_0$ we get

$$A^*\ (Y_1...Y_n) = \lambda \sum_{i=1}^{n} \frac{Y_1}{\lambda - \alpha_i} = \sum_{k=0}^{\infty}(\sum_{i=1}^{n} Y_i\alpha_i^k)\ \lambda^{-k} \tag{41}$$

Using (39) and (41) we arrive at the following expression for the moment map $\widetilde{J}_r$.

For $(F, G) \in M_{N,r} \times M_{N,r}$, we have

$$\widetilde{J}_r(F, G) = \sum_{i=1}^{n} \frac{\lambda\, G_i^T\, F_i}{\alpha_i - \lambda} \tag{42}$$

where we identify $(\widetilde{gl(r)}^{+})^*$ with $\widetilde{gl(r)}_0^{-}$

For further formulation, let us consider the loop algebra $\widetilde{sl(2, C)}$. Denote the column vectors of F by $\frac{1}{\sqrt{2}}(\vec{x}, \vec{y})$ where $\vec{x}$, $\vec{y} \in C^n$. Those of G are then $\frac{1}{\sqrt{2}}(-\vec{y}, \vec{x})$ and the symplectic form is given by $\omega = d\vec{x} \wedge d\vec{y}$. This follows from the fact that in the $\widetilde{sl(2, C)}$ case we can take $G = FY_1$ where $Y_1 = \sigma_+ - \sigma_-$. The moment map for the $\widetilde{sl(2, C)}$ action is

$$\widetilde{J}_2(\vec{x}, \vec{y}) = \frac{\lambda}{2}\ (Q_\lambda(\vec{x}, \vec{y})\sigma_3 + Q_\lambda(\vec{y}, \vec{y})\sigma_+ - Q_\lambda(\vec{x}, \vec{x})\sigma_- \tag{43}$$

where

$$Q_\lambda(\vec{\xi},\vec{\eta}) = \sum_{i=1}^{n} \frac{\xi_i\eta_i}{\lambda - \alpha_i} \tag{44}$$

and $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, $\vec{\eta} = (\eta_1, \eta_2, \cdots, \eta_n) \in \Re^n$ and we now have the expression for $\hat{\phi}_Y$ as,

$$\hat{\phi}_Y\,(X(\lambda)) = \hat{\phi}(X(\lambda) + \lambda Y) \tag{45}$$

where $Y \in \widetilde{sl(2,C)}$ and $\hat{\phi} \in I(\widetilde{sl(2)}^*)$. We shall now try to generate new integrable systems by using the concept of the moment map in conjunction with the extended AKS Theorem.

To this end, let $N(\lambda) = \widetilde{J_2}(\vec{x},\vec{y})$ and define $N'(\lambda) = N(\lambda) + \lambda Y$, write

$$L'(\lambda) = \frac{a(\lambda)}{\lambda_n}\, N'(\lambda) = L(\lambda) + \frac{a(\lambda)}{\lambda^{n-1}}\, Y \tag{46}$$

where

$$L(\lambda) = \frac{a(\lambda)}{\lambda^n}\, N(\lambda) \tag{47}$$

We take,

$$L(\lambda) = L_0 + L_1\lambda^{-1} + L_2\lambda^{-2} + ... + L_{n-1}\lambda^{n-1} \tag{48}$$

$$\lambda\frac{a(\lambda)}{\lambda^n} = \lambda + a_1 + \lambda^{-1}a_2 + ... + \lambda^{-(n-1)}a_n \tag{49}$$

Consider the ad* invariant functions on $\widetilde{SL(2,C)}^*$ given by

$$\phi_k(X,(\lambda)) = \frac{1}{2}\,\mathrm{tr}\,(\frac{a(\lambda)\lambda^k}{\lambda^n}\,(X(\lambda)^2) \tag{50}$$

and let t_k denote the time parameter for the Hamiltonian flow of $\hat{\phi}_k$. Then we have the equation of motion ,as

$$\frac{d}{dt_k}N(\lambda) = [(\frac{a(\lambda)\lambda^k}{\lambda^n}\, N'(\lambda))_+, N'(\lambda)] \tag{51}$$

$$\frac{d}{dt_k}(\frac{\lambda^n}{a(\lambda)}\, L(\lambda)) = [(\lambda^k L'(\lambda))_+\,, \frac{\lambda^n}{a(\lambda)}\, L'(\lambda)] \tag{52}$$

$$\frac{d}{dt_k}L(\lambda) = [(\lambda^k\, L'(\lambda))_+,\, L'(\lambda)] \tag{53}$$

In particular,

$$L'(\lambda) = \lambda Y + (L_0 + a_1Y) + (L_1 + a_2Y)\,\lambda^{-1} + ... + (L_{n-1} + a_nY)\,\lambda^{-(n-1)} \tag{54}$$

It may be noted that the Lax equation (53) is constructed with the help of the moment map - $N(\lambda) = \tilde{J}_2$

Having completely described the formulation of the extended AKS Theorem in the context of the moment map, we can now use this procedure to generate new integrable systems. This will be discussed in next section. Obviously, we can take $Y = 0$ and thereby return to the AKS Theorem and construct integrable systems also from there. Further, we can use automorphisms of the Lie algebra to generate integrable systems starting from the AKS Theorem. All these will be done in the next section.

5 Some Integrable Systems using Moment Map.

Example I

Let us take $k = 0$ and $k = 1$ successively in (53). Then we obtain the first few nontrivial equations as follows

$$\frac{dL_0}{dt_0} = [Y,\ L_1] \tag{55}$$

$$\frac{dL_1}{dt_0} = [L_0,\ L_1] + [Y,\ L_2 + a_1 L_1 - a_2 L_0] \tag{56}$$

and

$$\frac{dL_0}{dt_1} = [Y,\ L_2] \tag{57}$$

An invariant of the flows is $\frac{1}{2}\text{tr}\,(L'(\lambda))^2$ which when taken as zero and $L'(\lambda)$ substituted reduces to,

$$\text{tr}\,[YL_1 + L_1Y + 2a_2Y^2 + L_0^2 + a_1^2Y^2 + a_1(L_0Y + YL_0)] = 0 \tag{58}$$

Taking

$$Y = Y_1\sigma_3 + Y_2\sigma_+ + Y_3\sigma_-,\, L_0 = a\sigma_3 + b\sigma_+ + c\sigma_-,\, L_1 = s\sigma_3 + v\sigma_+ + u\sigma_-,\, L_2 = s\sigma_3 + v\sigma_+ + u\sigma_- \tag{59}$$

we obtain, from above, the following sets of equations

$$a_x = Y_2u - Y_3v\ ,\ b_x = 2(Y_1v - Y_2s)\ ,\ c_x = (Y_3s - ;Y_1u) \tag{60}$$

$$s_x = bv - cv + (Y_2U - Y_3V) + a_1(Y_2u - Y_3v) - a_2(Y_2c - Y_3b) \tag{61}$$

$$v_x = 2(av - bs) + 2(Y_1v - Y_2s) + 2a_1(Y_1v\ ; - Y_2s) - 2a_2(Y_1b - Y_2a) \tag{62}$$

$$U_x = 2(cs - au) + 2(Y_3s - Y_1u) + 2a_1(Y_3s - Y_1u) - 2a_2(Y_3a - Y_1c) \tag{63}$$

and

$$a_t = Y_2\,U - Y_3\,V\ ,\ b_t = 2(Y_1\,V - Y_2s)\ ,\ c_t = 2(Y_3s - Y_1\,U) \tag{64}$$

where we have taken $t_0 \equiv x$ and $t_1 \equiv t$.

$$(a^2 + bc) + 2sY_1 + (vY_3 + uY_2) + (2a_2 + a_1^2)\,(Y_1^2\ : + Y_2Y_3) + a_1\,(2a_1Y_1 + bY_3 + cY_2) = 0 \tag{65}$$

Instead of pursuing the general case, which will be too complicated we shall now consider two special cases as follows :

Case 1: $Y_2 = Y_3 = 0$, $Y_1 =$ const. whence we obtain $a_x = 0a_t$ whereby $a = 0$ (by choice)

$$b_x = 2Y_1\,v\ ,\ c_x = -\,2Y_1\,u \tag{66}$$

$$b_t = 2Y_1\,V\ ,\ c_t = -\,2Y_1\,U \tag{67}$$

$$s_x = bu - cv, v_x = -2bs + 2Y_1V + 2a_1Y - 1v - 2a_2Y_1b, u_x = 2cs - 2Y_1U - 2a_1Y_1u + 2a_2Y_1c, \quad (68)$$

$$b\,c \,+\, 2\,s\,Y_1 \,+\, (2\,a_2 \,+\, a_1^2)\,Y_1^2 \,=\, 0 \quad (69)$$

From the above equations, we finally obtain the flows

$$b_t \,=\, \frac{1}{2Y_1}\,(b_{xx} \,-\, 2b^2\,c \,-\, 2a_1^2\,Y_1^2\,b \,-\, a_1\,b_x \quad (70)$$

$$c_t \,=\, \frac{1}{2Y_1}\,(c_{xx} \,-\, 2b\,c^2 \,-\, 2a_1^2\,Y_1^2\,c \,-\, a_1\,c_x \quad (71)$$

which is very similar to this NLS eqn.

Case 2: $Y_1 = 0$, Y_2, Y_3 constants. Proceeding as in case 1, we obtain the flows,

$$a_t \,=\, -\,\frac{1}{2Y_2}\,(b_{xx} \,-\, 2\,a^2\,b) \,-\, \frac{Y_3}{Y_2^2}\,b^3 \,+\, a_1^2\,Y_3\,b \,-\, a_1\,a_x \quad (72)$$

$$b_t \,=\, -\,\frac{1}{2Y_3}\,(a_{xx} \,-\, 2\,a^3) \,-\, \frac{a\,b^2}{Y_2} \,+\, a_1^2\,Y_2\,a \,-\, a_1\,b_x \quad (73)$$

This is a generalization of the Dispersive Water Wave Equations (10).

Example II

For this example, we turn to the ordinary AKS Theorem and use it in conjunction with the moment map construction i.e setting $Y = 0$ and $k = 0$ and 2 successively we obtain the following nontrivial equations,

$$\frac{dL_0}{dx} = 0\,,\ \frac{dL_1}{dx} = [L_0,\, L_2]\,,\ \frac{dL_2}{dx} = [L_0,\, L_3] + [L_1,\, L_2]\,,\ \frac{dL_3}{dx} = [L_0,\, L_4] + [L_1,\, L_3] \quad (74)$$

$$\frac{dL_0}{dt} = 0\,,\ \frac{dL_1}{dt} = [L_0,\, L_4] \quad (75)$$

Since L_0 is conserved, we take $L_0 = \frac{1}{2}(\sigma_3 + \sigma_+ + \sigma_-)$. Further, we take

$$L_1 = s\,\sigma_3 + v\,\sigma_+ + u\,\sigma_-\,,\ L_2 = S\,\sigma_3 + V\,\sigma_+ + U\,\sigma_-\,,\ L_3 = S\,\sigma_3 + V_1\,\sigma_+ + U_1\,\sigma_-\,, \quad (76)$$

The equations turn out to be:

$$s_x \,=\, \frac{1}{2}(U \,-\, V)\,,\ v_x \,=\, V \,-\, S\,,\ u_x \,=\, S \,-\, U \quad (77)$$

$$S_x = \frac{1}{2}(U_1 - V_1) + vU - uV\,,\ U_x = S_1 - U_1 + 2(uS - sU), V_x = V_1 - S_1 + 2(sV - vS) \quad (78)$$

$$s_t =: S - 1x - (v\,U_1 - uV_1)\,,\ u_t = U_{1x} - 2(uS_1 - sU_1)\,,\ v_t = V_{1x} - 2(sV_1 - vS_1) \quad (79)$$

We obtain the following connecting equations.

$$U = -\frac{1}{4} V_x - \frac{3}{4} u_x - \frac{1}{2}(s^2 + \, : u\, v)\,,\ V = \frac{3}{4} v_x - \frac{1}{4} u_x - \frac{1}{2}(s^2 + u\, v) \tag{80}$$

with

$$V + U = -2\, s \tag{81}$$

We also have the invariant conditions

$$S + \frac{1}{2}(U + V) + (s^2 + uv) = 0 \tag{82}$$

$$S_1 + \frac{1}{2}(U_1 + V_1) + 2\, s\, s + u\, V + v\, U = 0 \tag{83}$$

which lead to relations,

$$U_1 = \frac{1}{2} u_{xx} + \frac{1}{2}(u_x\, v - u\, v_x) - \frac{3}{2} u^2\, v - \frac{1}{4} u\, v^2 - \frac{1}{4} u^3 \tag{84}$$

$$V_1 = \frac{1}{2} v_{xx} + \frac{1}{2}(u_{x\, v} - u\, v_x) - \frac{3}{2} u\, v^2 - \frac{1}{4} u^2\, v - \frac{1}{4} v^3 \tag{85}$$

and finally to the following coupled mkdV equations in u and v,

$$4\, u_t = 2\, u_{xxx} - 3\, v^2\, u_x - 3\, u^2\, u_x - 18 u\, u_x\, v\,,\ 4\, v_t = 2\, v_{xxx} - 3\, u^2\, v_x - 3\, v^2\, v_x - 18 u\, v\, v_x \tag{86}$$

Example III

For this example, consider the moment map

$$\tilde{J}_2\,(F,\, G) = \frac{\lambda}{2} \sum_{i=1}^{n} \frac{G_i^T\, F_i}{\alpha_i - \lambda} \tag{87}$$

where, for the $sL(2, C)$ algebra under consideration, we have $F_i = \frac{1}{\sqrt{2}}(\vec{x}, \vec{y})$ and $G_i = \frac{1}{\sqrt{2}}(-\vec{y}, \vec{x})$ since $\mathrm{tr}(G_i^T F_i) = 0$. Under the automorphism [9]

$$\widetilde{J_2'}\,(F,\, G) = H\,(\lambda)\, \widetilde{J_2}\,(F,\, G)\, H\,(\lambda)^{-1}\,, \quad \lambda \to \lambda' = g(\lambda) = q\lambda \tag{88}$$

where

$$H(\lambda) = \sigma_3 + \lambda(q - 1)\sigma_+$$

and we take $q^2 = 1$, and finally take $q = -1$. The moment map becomes,

$$\hat{J}_2\,(F,\, G) = \frac{\lambda}{2} \sum_{i=1}^{n} \frac{G_i^T\, F_i}{\alpha_i - \lambda} + \frac{q\lambda}{2} \sum_{i=1}^{n} \frac{H(\lambda)\, G_i^T\, F_i\, H(\lambda)^{-1}}{\alpha_i - q\lambda} \tag{89}$$

Under further action with $H(\lambda)$ and $\lambda' = q\lambda$ we obtain

$$H(\lambda)\, \hat{J}_2\,(F,\, G)\, H(\lambda)^{-1} = \hat{J}_2\,(F,\, G) \tag{90}$$

which explicitly proves the automorphism in question. Written out in full, the moment map becomes,

$$\hat{J}_2\,(F,\,G) \;=\; \frac{\lambda}{2}\,\sum_{i=1}^{n}[A\,\sigma_3 \;+\; B\,\sigma_+ \;-\; D\,\sigma_-] \tag{91}$$

where,

$$A = \frac{\lambda^2 X_i^2+\lambda(X_iY_i-\lambda_iX_i^2)}{\lambda^2-\alpha_i^2}$$
$$B = \frac{(-2\lambda^3X_i^2+\alpha_iY_i^2+2\lambda\alpha_iX_iY_i+2\lambda^2(\alpha_iX_i^2-X_iY_i))}{\lambda^2-\alpha_i^2}$$
$$D = \frac{\alpha_iX_i^2}{\lambda^2-\alpha_i^2}$$

Now choose $N(\lambda) = \hat{J}_2(F,G)$ and

$$L(\lambda) \;=\; \lambda^2\,L^{(2)} \;+\; \lambda\,L^{(1)} \;+\; \lambda^{-(2n\,-\,1)}\,L_{2n\,-\,1} \tag{92}$$

and the Lax form of the equation

$$\frac{d}{dt_k}\,L(\lambda) \;=\; [(\lambda^{2k}\,L(\lambda))_+\,,L(\lambda)] \tag{93}$$

where the invariants are given by,

$$\phi_k(x(\lambda)) \;=\; \frac{1}{2}\,\mathrm{tr}\,((\frac{\lambda^{2k}\,a(\lambda)\,x(\lambda)^2}{\lambda^{2n}})_0) \tag{94}$$

and t_k denotes the time parameter for the hamiltonian flow of ϕ_k.

$$L^{(2)} \;=\; -\,2\,V^{(2)}\,\sigma_+\,,\; L^{(1)} \;=\; V^{(2)}\,\sigma_3 \;+\; V^{(1)}\,\sigma_+\,,\; L_0 \;=\; -\,\frac{V^{(1)}}{2}\,\sigma_3 \tag{95}$$

$$L_1 = u_1\sigma_3+V_1\sigma_++\omega_1\sigma_-,\, L_2 = u_2\sigma_3+V_2\sigma_+,\, L_3 = u_3\sigma_3+V_3\sigma_++\omega_3\sigma_3,\, L_4 = u_4\sigma_3+V_4\sigma_+ \tag{96}$$

Now let us set $k = 1$ and 2 successively in (93). Then we find $L^{(2)}$ is conserved and hence we may set it equal to zero. In turn, this implies that $L^{(1)}$ is conserved and therefore may be set equal to zero. Finally, we find that L_0 is conserved and may also be set to zero. identifying $t_1 \equiv x$ and $t_2 \equiv t$, we obtain the following nontrivial equations:

$$\frac{dL_1}{dx} \;=\; 0\,,\; \frac{dL_2}{dx} \;=\; [L_1, L_3]\,,\; \frac{dL_3}{dx} \;=\; [L_1, L_4] + [L_2, L_3]\,,\; \frac{dL_4}{dx} \;=\; [L_1, L_5] + [L_2, L_4] \tag{97}$$

and

$$\frac{dL_1}{dt} \;=\; 0\,,\; \frac{dL_2}{dt} \;=\; [L_1,\,L_5] \tag{98}$$

from which we obtain,

$$\frac{dL_2}{dt} \;=\; \frac{dL_4}{dx} \;-\; [L_2,\,L_4] \tag{99}$$

Since L_1 is conserved set $L_1 = \alpha\;\sigma_3 + \beta\;\sigma_+ + \gamma\;\sigma_-$ and for simplicity take $\alpha = \beta = \gamma = 1$. The explicit equations we get are the following:

$$u_{2x} \;=\; \omega_3 \;-\; v_3\,,\; v_{2x} \;=\; 2\,(v_3 \;-\; u_3)\,,\; u_3 \;=\; \omega_3 \tag{100}$$

$$u_{3x} = -v_4 + v_2\omega_3\,,\ \omega_{3x} = 2u_4 - 2u_2\omega_3\,,\ v_{3x} = 2(v_4 - u_4) + 2(u_2 v_3 - v_2 u_3) \quad (101)$$

and the invariant

$$\mathrm{tr}\,[\lambda^4\,(L(\lambda)^2)_0] = \mathrm{tr}\,[L_1\,L_3 + L - 3\,L_1 + {L^2}_2] = constant \quad (102)$$

From these relations, it is easy to obtain

$$v_2 = -2u_2, -3 = \omega_3 = -\frac{1}{4}u_2^2 - \frac{1}{8}v_{2x}, v_3 = -\frac{1}{4}u_2^2 + \frac{3}{8}v_{2x}, v_4 = -2u_4 = -\frac{1}{4}u_{2xx} + \frac{1}{2}u_2^3 \quad (103)$$

Hence, we get the flow [12] as:

$$8\,u_{2t} = u_{2xxx} - 6\,u_2^2\,u_{2x} \quad (104)$$

which is nothing but the mKdV equation.

6 Conclusion

In this paper, we have obtained some integrable systems via the extended AKS Theorem and the moment map technique. It is interesting to note that some new non linear equations along with those already known are deduced. The moment map formalism helps to reduce the number of dependent variables. An immediate extension is the application of the method to super-Lie algebra which will give rise to supersymmetric integrable system. The bi-Hamiltonian structure and other properties of these equation will be the subject of our future investigation. One of the authors (I.M.) is grateful to CSIR (Govt. of India) for a Fellowship.

References

[1] M. J. Adler, Invent. Maths. 50, 219 (1979)
B. Kostant, Adv. Math. 34, 195 (1979)
W.W. Symes, Invent. math. 59, 13 (1980)

[2] I. Marshall, Physics Letters A 127, 1, 19 (1988)

[3] Abraham and Marsden, *Foundations of mechanics 2nd ed.*
(Reading MA Benjamin/Cummings, 1978 chap 4.
Olshanetsky and Perelomov, Phys. rep. 71, 313 (1981)
Guillemin and Sternberg, Sympletic techniques in phys. C.U.P (1984)
A. Weinstein, Lectures on symplectic manifolds. CBMS conference series, 29 Providence RI : AM. Math. Soc.(1977)
M. R. Adams, J. Harnad and E. Previato, Commun. Math Phys., 117, 451 (1988)

[4] A. G. Reyman, M. A. Semenov-Tian-Shansky, I. B. Frenkel, Sov. Math. Doklady 20, 811 (1979)

[5] E. G. Harris, *Lectures in Advances in Plasma Physics Vol 3-Edited by A. Simon and W. Thompson* (Interscience Pub. N.Y, 1989)

[6] H. Flaschka, A. C. Newell and T. Ratiu, Physica 9D, 300 (1983)

[7] Guillemin and Sternberg, Lett. Math. Phys. 7, 113 (1983)
A. G. Reyman, M. A. Semenov-Tian-Shansky, Invent. Maths. 54, 81 (1979), 63 423 (1981)

[8] F. Calogero, A. Degasparis *Solitons* ed. by R. K. Bullough and P. J. Caudrey Berlin Springer) (1980), p.301

[9] M. A. Semonov-Tian-Shansky, Funct. Anal. Appl. 259 (1983)

[10] B. G. Konopelchenko, Phys. Lett. 108B,1, 26 (1980)