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Some Anisotropic Schrödinger Operators without Singular Spectrum

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Abstract.

We study Schrödinger operators with potentials not decaying at infinity. For these we prove a limiting absorption principle and the absence of singular spectrum. This is done by an abstract method, relying on the positivity of a commutator, related to the Kato-Putnam and Mourre methods.

0 Introduction

This article is devoted to the study of the spectral properties of some anisotropic Schrödinger operators $H = \Delta + V$ acting in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$.

Much is known on the spectrum when V is Δ-relatively compact. The essential spectrum is $[0, \infty)$ and, under some mild extra conditions (essentially on the large-distance behaviour of V), the singular continuous spectrum is absent and the point spectrum in any compact subset of $\mathbb{R} \setminus \{0\}$ is discrete. The case when V is not relatively compact is less understood. Detailed spectral informations are available when V has some special properties: repulsivity, quasi-periodicity, a N-body structure or monotony as in the Stark effect problem. We do not give details or references. Let us mention that — except in the quasi-periodic case — methods using the positivity of a commutator, as Mourre's approach [5] or the Kato-Putnam theory [4], [6] are fit.

Our intention is to show that some classes of V's which are not relatively compact can also be studied by means of the positivity of a suitable commutator. To give an idea, let us look at the operator $H = \Delta + V$. We have $i[H, A] = 2\Delta - \widetilde{V}$ where $A = \frac{1}{2}\{(P, Q) + (Q, P)\}$ is the generator of dilations in $\mathbb{R}^n$, P, Q are the momentum (resp. position) operators and $\widetilde{V} = (x, \nabla V)$. It is known that, for $N \geq 3$, $\Delta \geq \left(\frac{N-2}{2}\right)^2 |Q|^{-2}$; hence, if we require

$$|\widetilde{V}(x)| \leq \frac{C_N}{|x|^2}, \qquad \text{with } C_N < \frac{(N-2)^2}{2} \tag{0.1}$$

we get $i[H, A] > 0$. This seems to be a good starting point for studying the spectral properties of H in the anisotropic case, because (0.1) allows a rather general behaviour of V at infinity (roughly V must have radial limits, which may depend on the direction, and a $O(r^{-2})$ type of convergence towards them). But the existing commutator methods are not able to exploit the situation above in a suitable generality. First, in the Kato-Putnam theory, the weak positivity $i[H, A] > 0$ is enough, but one needs strong regularity properties for H and A. Secondly, suppose that one aims at a Mourre estimate for the interval J:

$$E_H(J) i[H, A] E_H(J) \geq a E_H(J) + K \tag{0.2}$$

where E_H is the spectral measure of H, a is a strictly positive number and K a compact operator. In our case $i[H, A]$ does not dominate a strictly positive constant (except in some very restricted circumstances). Hence the best we can do is to adopt a perturbative point of view and write

$$i[H, A] = 2H - (2V + \widetilde{V}).$$

One can try to get some positivity out of the first term and a compact operator out of the second by using suitable spectral projections $E_H(J)$. But this requires some compactness assumptions on V, which are not implied by (0.1) and which we would like to avoid. Therefore, one aims at a result relying on the condition $B \equiv i[H, A] > 0$, but with less regularity required on H and A. This will be done in the first section. The main result is Theorem 1.1. It is shown roughly that if $i[H, A]$ is positive, injective and H-bounded and the second commutator $[B, A]$ is not too singular (in a sense to be specified), then the spectrum of H is purely absolutely continuous. We also get some estimates on the behaviour of the resolvent of H close to the real axis (the limiting absorption principle and smoothness estimates). In Corollaries 1.1 and 1.2 we reformulate these as criteria for the existence and unitarity of the wave operators.

1 The Method of the Weakly Conjugate Operator

Let us consider a self-adjoint operator H in the Hilbert space $\mathscr{H}$ (with scalar product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$). We denote by $\mathscr{G}^2$ its domain and by $\mathscr{G}^1$ the form-domain. Endowed with the corresponding graph norms, they are also Hilbert spaces. By identifying $\mathscr{H}$ with its topological anti-dual (its adjoint) $\mathscr{H}^*$, one has the following continuous, dense embeddings:

$$\mathscr{G}^2 \subset \mathscr{G}^1 \subset \mathscr{H} \subset \mathscr{G}^{-1} \subset \mathscr{G}^{-2}$$

where, for example, $\mathscr{G}^{-1}$ is the adjoint of $\mathscr{G}^1$. Notice that H extends to a bounded operator: $\mathscr{G}^1 \to \mathscr{G}^{-1}$ (denoted by the same letter), which is symmetric with respect to the duality between $\mathscr{G}^1$ and $\mathscr{G}^{-1}$.

We shall study the spectrum of H by means of another self-adjoint operator A, which will be now introduced through its unitary group $\{W(t) = e^{itA} \mid t \in \mathbb{R}\}$. We assume that all operators $W(t)$ leave $\mathscr{G}^2$ invariant. It is a standard fact that W will induce C_0-groups in the spaces $\mathscr{G}^s$ $(s = \pm 1, \pm 2)$. They act in a coherent manner, hence no notational difference will be made between them, neither between their infinitesimal generators, but it is useful to distinguish between their domains by writing $D(A;\mathscr{K})$, where $\mathscr{K}$ stands for one of the spaces above.

Definition 1.1 We will say that $H \in C^1(A;\mathscr{G}^2,\mathscr{H})$ if the mapping

$$\mathbb{R} \ni t \mapsto W(-t)HW(t) \in B(\mathscr{G}^2,\mathscr{H}) \tag{1.1}$$

is strongly C^1, i.e. for any $f \in \mathscr{G}^2$, $t \mapsto W(-t)HW(t)f \in \mathscr{H}$ is C^1 in norm-sense.

The strong derivative of the function (1.1) at $t = 0$ will be denoted by B. It belongs to $B(\mathscr{G}^2,\mathscr{H})$, the space of all linear, bounded operators: $\mathscr{G}^2 \to \mathscr{H}$. By duality and interpolation, we can think of B as a symmetric element of $B(\mathscr{G}^1,\mathscr{G}^{-1})$.

Definition 1.2 We say that A *is weakly conjugate to* H if $H \in C^1(A;\mathscr{G}^2,\mathscr{H})$ and $B > 0$ (i.e. $B \geq 0$ and the kernel of B is trivial).

Definition 1.3 (a) We denote by $\mathscr{B}$ the Hilbert completion of $\mathscr{G}^1$ for the norm $||f||_{\mathscr{B}} = \langle f, Bf\rangle^{1/2}$.

(b) We denote by $\mathscr{B}^*$ the Hilbert completion of $B\mathscr{G}^1$ for the norm $||g||_{\mathscr{B}^*} = \langle g, B^{-1}g\rangle^{1/2}$.

It is easy to see that B extends to a unitary operator: $\mathscr{B} \to \mathscr{B}^*$. $\mathscr{B}^*$ can be identified with the adjoint of $\mathscr{B}$ (this explains the notation). The duality form of the couple $(\mathscr{B}, \mathscr{B}^*)$ coincides with the scalar product of $\mathscr{H}$ on $\mathscr{G}^1 \times B\mathscr{G}^1$ and will therefore also be denoted by $\langle \cdot, \cdot \rangle$. In general, $\mathscr{H}$ is not comparable with $\mathscr{B}$ or $\mathscr{B}^*$. Note that we have dense embeddings $\mathscr{G}^1 \subset \mathscr{B}$ and $\mathscr{B}^* \subset \mathscr{G}^{-1}$. If the C_0-group W in $\mathscr{G}^{-1}$ leaves $\mathscr{B}^*$ invariant, we get C_0-groups respectively in $\mathscr{B}^*$ and in $\mathscr{B}$, also denoted by W. Hence, two new Hilbert spaces appear: $D(A; \mathscr{B}^*)$ and $D(A; \mathscr{B})$. Since $D(A; \mathscr{B}^*)$ will play a distinguish role, we use the short notation $\mathscr{A} \equiv D(A; \mathscr{B}^*)$. It carries the Hilbert norm

$$||f||_{\mathscr{A}} = \left(||f||^2_{\mathscr{B}^*} + ||Af||^2_{\mathscr{B}^*}\right)^{1/2} = \left(\langle f, B^{-1}f\rangle + \langle Af, B^{-1}Af\rangle\right)^{1/2}.$$

We finally assume $B \in C^1(A; \mathscr{B}, \mathscr{B}^*)$. As explained before, this means that the map

$$\mathbb{R} \ni t \mapsto W(-t)BW(t) \in B(\mathscr{B}, \mathscr{B}^*) \tag{1.2}$$

is strongly C^1. Equivalently the sesquilinear form

$$D(A; \mathscr{B}) \times D(A; \mathscr{B}) \ni (f, g) \mapsto i\langle f, BAg\rangle - i\langle Af, Bg\rangle \in \mathbb{C} \tag{1.3}$$

is continuous with respect to the topology of $\mathscr{B} \times \mathscr{B}$. A rough way to say this is $i[B, A] \in B(\mathscr{B}, \mathscr{B}^*)$, where the second commutator $i[B, A]$ is either the operator associated with (1.3) or the derivative at $t = 0$ in (1.2).

We may now state our main result:

Theorem 1.1 *Suppose that A is weakly conjugate to H and that $B \in C^1(A; \mathscr{B}, \mathscr{B}^*)$.*

(a) $|\langle f, (H - \lambda \mp i\mu)^{-1} f\rangle| \le C||f||^2_{\mathscr{A}}$, *with C independent of $\lambda \in \mathbb{R}$, $\mu > 0$ and $f \in \mathscr{A}$.*

(b) *Any operator $T \in B(\mathscr{A}^*, \mathscr{K})$ is H-smooth where $\mathscr{K}$ stands for an arbitrary Hilbert space.*

(c) *H has purely absolutely continuous spectrum.*

Remark that $(H - \lambda \mp i\mu)^{-1}$ belong to $B(\mathscr{G}^{-1}, \mathscr{G}^1) \subset B(\mathscr{B}^*, \mathscr{B}) \subset B(\mathscr{A}, \mathscr{A}^*)$, hence the uniform estimate (a) (the "limiting absorption principle") makes sense. The point is that it cannot be true in $B(\mathscr{G}^{-1}, \mathscr{G}^1)$ if λ belongs to the spectrum of H. The precise statement of the point (b) is that a closed operator T_0 in $\mathscr{H}$, taking values in $\mathscr{K}$, which extends to an element of $B(\mathscr{A}^*, \mathscr{K})$ is H-smooth in the usual sense (see [7]). This also makes sense, because if T_0 is H-smooth, its domain must contain the domain of H, which is dense in $\mathscr{A}^*$.

Proof of Theorem 1.1. (b) and (c) follow from (a) in a standard way. We shall divide the proof of (a) into several easy steps. The same letter may denote different constants from line to line.

Lemma 1.1 *There exists $\varepsilon_0 > 0$ such that for all $\lambda \in \mathbb{R}$, $\mu > 0$ and $\varepsilon \in (0, \varepsilon_0)$, the operators*

$$H - \lambda \mp i\mu \mp i\varepsilon B : \mathscr{G}^2 \to \mathscr{H}$$

are isomorphisms.

Proof. As a consequence of the open mapping theorem, one needs only to show that the operators $H-\lambda\mp i\mu\mp i\varepsilon B$, which are elements of $B(\mathscr{G}^2,\mathscr{H})$, are bijections: $\mathscr{G}^2\to\mathscr{H}$. For $f\in\mathscr{G}^2$

$$\begin{aligned}||(H-\lambda\mp i\mu\mp i\varepsilon B)f||^2 &= \mu^2||f||^2+||(H-\lambda\mp i\varepsilon B)f||^2\\ &\quad +2\,\mathrm{Re}\langle (H-\lambda\mp i\varepsilon B)f,\mp i\mu f\rangle\\ &\geq \mu^2||f||^2+2\varepsilon\mu||f||^2_{\mathscr{B}}\\ &\geq \mu^2||f||^2.\end{aligned}$$

In particular, for $\mu\neq 0$ the operators $H-\lambda\mp i\mu\mp i\varepsilon B:\mathscr{G}^2\to\mathscr{H}$ are injective. But, setting $\varepsilon_0\equiv 1/||B||_{\mathscr{G}^2\to\mathscr{H}}$ and restricting to $\varepsilon\in(0,\varepsilon_0)$, it is easy to see that they are closed operators in $\mathscr{H}$ and adjoint to each other. This gives immediately the surjectivity and this finishes the proof.

(ii) Now, let us set

$$G_\varepsilon^\pm\equiv G_\varepsilon^\pm(\lambda,\mu)=(H-\lambda\mp i\mu\mp i\varepsilon B)^{-1}. \tag{1.4}$$

They are in $B(\mathscr{H},\mathscr{G}^2)$, hence in $B(\mathscr{G}^{-1},\mathscr{G}^1)$ and in $B(\mathscr{B}^*,\mathscr{B})$ too. One can easily show that

$$\langle f,G_\varepsilon^\pm g\rangle=\langle G_\varepsilon^\mp f,g\rangle\qquad \text{for all } f,g\in\mathscr{G}^{-1}. \tag{1.5}$$

This is why one sometimes uses the notations $G_\varepsilon^+=G_\varepsilon$ and $G_\varepsilon^-=G_\varepsilon^*$. In the next lemma we give the crucial a priori estimates satisfied by $G_\varepsilon^\pm$. Note that, for lack of a better positivity of B, one cannot avoid the new spaces $\mathscr{B}$ and $\mathscr{B}^*$.

Lemma 1.2 (a)

$$||G_\varepsilon^\pm f||_{\mathscr{B}}\leq\frac{1}{\sqrt{\varepsilon}}|\langle f,G_\varepsilon^\pm f\rangle|^{1/2}, \tag{1.6}$$

(b)

$$||G_\varepsilon^\pm f||_{\mathscr{B}}\leq\frac{1}{\varepsilon}||f||_{\mathscr{B}^*} \tag{1.7}$$

(c)

$$||G_\varepsilon^\pm||_{\mathscr{G}^{-1}\to\mathscr{G}^1}\leq\frac{c(\lambda)}{\mu}\,. \tag{1.8}$$

Here $\lambda\in\mathbb{R}$, $\mu\in(0,\infty)$ and $\varepsilon\in(0,\varepsilon_0)$; in (c) *the constant $c(\lambda)$ does not depend on μ nor ε.*

Proof. (a) We write $G_\varepsilon^+-G_\varepsilon^-=2i\mu G_\varepsilon^+G_\varepsilon^-+2i\varepsilon G_\varepsilon^+BG_\varepsilon^-$, which, combined with (1.4), gives for $f\in\mathscr{B}^*\subset\mathscr{G}^{-1}$:

$$\frac{1}{2i\varepsilon}\langle f,[G_\varepsilon^+-G_\varepsilon^-]f\rangle\geq\langle G_\varepsilon^-f,BG_\varepsilon^-f\rangle=||G_\varepsilon^-f||^2_{\mathscr{B}}$$

and this is stronger than one of inequalities in (1.5). The other one is obtained in the same way.

(b) follows from (a) and from $|\langle f, g\rangle| \leq ||f||_{\mathscr{B}^*}||g||_{\mathscr{B}}$ ($f \in \mathscr{B}^*$ and $g \in \mathscr{B}$).

(c) is straightforward.

(iii) We set $F_\varepsilon \equiv F_\varepsilon(\lambda, \mu; f) = \langle f, G_\varepsilon f\rangle$, where $\lambda \in \mathbb{R}, \mu > 0, \varepsilon \in (0, \varepsilon_0)$ and $f \in \mathscr{A}$. Our strategy is to differentiate with respect to ε, to use the a priori estimate (1.5) in order to get a differential inequality on $|F_\varepsilon|$ and integrate this. We will obtain (a) in the end by letting $\varepsilon \to 0$. In fact, a formal calculation gives easily

$$F'_\varepsilon = \langle G^*_\varepsilon f, Af\rangle - \langle Af, G_\varepsilon f\rangle - i\varepsilon\langle G^*_\varepsilon f, [B, A]G_\varepsilon f\rangle.$$

We give no details about the rigorous proof which can be easily supplied. By (1.5) we get

$$|F'_\varepsilon| \leq \frac{2}{\sqrt{\varepsilon}}||f||_{\mathscr{A}}|F_\varepsilon|^{1/2} + ||[B, A]||_{\mathscr{B}\to\mathscr{B}^*}|F_\varepsilon|. \tag{1.9}$$

(iv) By a version of Gronwall's lemma which is proven in [2], Appendix B and by (1.6) we conclude from (1.8) that the limit $F_0 \equiv \lim_{\varepsilon\to 0} F_\varepsilon$ exists and satisfies

$$|F_0| \leq C\{|F_{\varepsilon_0}| + ||f||^2_{\mathscr{A}}\} \leq C\left\{\frac{1}{\varepsilon_0}||f||^2_{\mathscr{B}^*} + ||f||^2_{\mathscr{A}}\right\} \leq C||f||^2_{\mathscr{A}}.$$

(v) To finish the proof we need only to show that F_0 is the right object, i.e. that $\langle f, G_\varepsilon(\lambda, \mu)f\rangle$ converges to $\langle f, (H - \lambda - i\mu)^{-1}f\rangle$ when $\varepsilon \to 0$. For this we write

$$|\langle f, G_\varepsilon(\lambda, \mu)f\rangle - \langle f, (H - \lambda - i\mu)^{-1}f\rangle| \leq ||G_\varepsilon(\lambda, \mu) - (H - \lambda - i\mu)^{-1}||_{\mathscr{B}^*\to\mathscr{B}}||f||^2_{\mathscr{B}^*}.$$

This goes to zero when $\varepsilon \to 0$ because of the second identity of the resolvent and of (1.7).

As consequences of Theorem 1.1 (b) and Theorems XII.24 and XIII.26 from [7], we can state the following scattering results:

Corollary 1.1 *For $j = 1, 2$, let H_j, A_j be self-adjoint operators in the Hilbert space $\mathscr{H}$. Assume that A_j is weakly conjugate to H_j and that $B_j \equiv i[H_j, A_j] \in C^1(A; \mathscr{B}_j, \mathscr{B}^*_j)$ (the objects $\mathscr{G}^2_j$, $\mathscr{B}_j$ and $\mathscr{A}_j$ are as explained before). Assume also that $H_1 - H_2 \in B(\mathscr{A}^*_1, \mathscr{A}_2)$ in the sense that there exists $H_{12} \in B(\mathscr{A}^*_1, \mathscr{A}_2)$ such that for $f_j \in \mathscr{G}^2_j$*

$$\langle H_1 f_1, f_2\rangle - \langle f_1, H_2 f_2\rangle = \langle H_{12} f_1, f_2\rangle$$

($\langle\cdot,\cdot\rangle$ denotes here different but coherent pairings). Then the wave operators

$$\Omega^\pm_{jk} \equiv \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_j}e^{-itH_k} \quad (j, k = 1, 2,\ j \neq k)$$

exist and are unitary. In particular, H_1 and H_2 are unitary equivalent.

Corollary 1.2 *Assume that A is weakly conjugate to H and that $B \in C^1(A; \mathscr{B}, \mathscr{B}^*)$. Let U be self-adjoint, such that $|U|^{1/2}$ extends to an element of $B(\mathscr{A}^*, \mathscr{H})$. Assume either that H is bounded below or that U is H-bounded in operator-sense. Then there is a constant $\Gamma > 0$ such that for any $\gamma \in (-\Gamma, \Gamma)$, $H_\gamma \equiv H + \gamma U$ is self-adjoint, purely absolutely continuous and unitary equivalent to H through the wave operators.*

The constant Γ is proportional to $\| \, |U|^{1/2} \|_{\mathscr{A}^* \to \mathscr{H}}$. For the self-adjointness of H_γ for small γ we use K.L.M.N. or Rellich's theorem. Note that $|U|^{1/2} \in B(\mathscr{G}^1, \mathscr{H})$, because $\mathscr{G}^1 \subset \mathscr{A}^*$. Neither of the two corollaries is stronger than the other. In the first, it is important that $\mathscr{G}_1^1 \neq \mathscr{G}_2^1$ and $\mathscr{A}_1 \neq \mathscr{A}_2$ are allowed. The second is fit to situations when the weak conjugation is easy to get only for one of the operators involved.

2 Schrödinger Operators

Let us consider an euclidean space X, i.e. a finite-dimensional, real vector space equipped with a scalar product $(\cdot\,,\cdot)$. The corresponding norm is $|\cdot|$. For each subspace Y we denote by $\mathscr{H}(Y)$ the Hilbert space $L^2(Y; dy)$ and by $\mathscr{H}^s(Y)$ the usual Sobolev space of order $s \in \mathbb{R}$ associated to Y. $\mathscr{H}(X)$ will be identified freely with $\mathscr{H}(Y) \otimes \mathscr{H}(Y^\perp)$ and we write $\langle \cdot, \cdot \rangle$ for the scalar product in any $\mathscr{H}(Y)$. By Q^Y we denote the usual multiplication operator by the free variable in $\mathscr{H}(Y)$. $P^Y = -i\nabla^Y$ will be the corresponding momentum. The index X will be usually dropped.

We intend to study Schrödinger operators $H = \Delta + V(Q)$, where $V(Q)$ is the multiplication by a real Borel function V defined on X and Δ is the Laplace-Beltrami operator assigned to X, with the convention $\Delta = |P|^2$ (it is positive). We also set $\Delta^Y = |P^Y|^2$; it acts in $\mathscr{H}(Y)$, with domain $\mathscr{H}^2(Y)$. $\Delta_Y = \Delta^Y \otimes 1$ is an operator in $\mathscr{H}(X)$ defined in $\mathscr{H}^2(Y) \otimes \mathscr{H}(Y^\perp)$.

Let us also set $A^Y = \frac{1}{2}\{(P^Y, Q^Y) + (Q^Y, P^Y)\}$, the generator of dilations in Y. As a weakly conjugate operator we shall try $A_Y = A^Y \otimes 1$, for some suitable Y. It generates in $\mathscr{H}(X)$ the unitary group $W_Y(\cdot) = W^Y(\cdot) \otimes 1$, where W^Y is the dilation group in $\mathscr{H}(Y)$. Namely

$$[W_Y(t)f](x) = e^{\frac{n_Y}{2}t} f(e^t x^Y, x^Z) \quad \text{for all } t \in \mathbb{R},\ x \in X \text{ and } f \in \mathscr{H}(X).$$

Here n_Y is the dimension of Y, $Z = Y^\perp$ and (x^Y, x^Z) is the decomposition of x with respect to the splitting $X = Y \oplus Z$. Let us advocate the use of A_Y. We have

$$B \equiv i[H, A_Y] = 2\Delta_Y - (D^Y V)(Q), \tag{2.1}$$

where the notation $D^Y V = (x^Y, \nabla^Y V)$ will be systematically used. Suppose that V is *Y-homogeneous of degree* 0, i.e.

$$V(\lambda x^Y, x^Z) = V(x^Y, x^Z) \text{ for all } \lambda > 0, (x^Y, x^Z) \in X. \tag{2.2}$$

Then $D^Y V = 0$ and $B = 2\Delta_Y > 0$. In fact, under some mild conditions (we may suppose for simplicity that V is bounded), Theorem 1.1 may be applied to show that H has no singular spectrum and this is not trivial if $Y \neq X$.

The discussion above suggests the introduction of *the homogeneous Sobolev space of order* 1 *on* Y, denoted by $\overset{\circ}{\mathscr{H}}{}^1(Y)$, which is the completion of $\mathscr{D}(Y) = C_0^\infty(Y)$ in the norm $||f||_{\overset{\circ}{\mathscr{H}}{}^1(Y)} = \langle f, \Delta^Y f\rangle^{1/2} = ||\,|P^Y|f\,||$. It is a Hilbert space and $\mathscr{H}^1(Y) \subset \overset{\circ}{\mathscr{H}}{}^1(Y)$. $\overset{\circ}{\mathscr{H}}{}^1(Y)$ is not comparable with $\mathscr{H}(Y)$. Its adjoint may be identified with $\overset{\circ}{\mathscr{H}}{}^{-1}(Y)$, the homogeneous Sobolev space of order -1, defined by the norm $||g||_{\overset{\circ}{\mathscr{H}}{}^{-1}(Y)} = ||\,|P^Y|^{-1} g\,||$ (here we may start with g belonging to the Fourier transform of $C_0^\infty(Y \setminus \{0\})$; if $n_Y \geq 3$, one might simply take g in $\mathscr{D}(Y)$, since $|\cdot|^{-1}$ will be in L^2_{loc}. We also set $\overset{\circ}{\mathscr{H}}{}^{\pm 1}_Y = \overset{\circ}{\mathscr{H}}{}^{\pm 1}(Y) \otimes \mathscr{H}(Z)$, with the scalar products $\langle f, g\rangle_{\overset{\circ}{\mathscr{H}}{}^{\pm 1}_Y} = \langle f, \Delta_Y^{\pm 1} g\rangle$. Of course, these two Hilbert spaces stay in duality in a natural way.

In the same way, only by changing P^Y into Q^Y, we define $\overset{\circ}{\mathscr{H}}_{\pm 1}(Y)$, the homogeneous Lebesgue spaces with weight of order ± 1, as well as the spaces $\overset{\circ}{\mathscr{H}}_{\pm 1, Y} = \overset{\circ}{\mathscr{H}}_{\pm 1}(Y) \otimes \mathscr{H}(Z)$.

We shall rely heavily on the classical inequality

$$\Delta^Y \geq \left(\frac{n_Y - 2}{2}\right)^2 |Q^Y|^{-2}, \tag{2.3}$$

valid on $\mathscr{D}(Y)$ if $n_Y \geq 3$. Obviously, it extends on $\mathscr{H}^1(Y)$. As an immediate consequence, we have

$$\begin{aligned} &\overset{\circ}{\mathscr{H}}{}^1(Y) \subset \overset{\circ}{\mathscr{H}}_{-1}(Y), && \overset{\circ}{\mathscr{H}}_1(Y) \subset \overset{\circ}{\mathscr{H}}{}^{-1}(Y), \\ &\overset{\circ}{\mathscr{H}}{}^1_Y \subset \overset{\circ}{\mathscr{H}}_{-1,Y}, && \overset{\circ}{\mathscr{H}}_{1,Y} \subset \overset{\circ}{\mathscr{H}}{}^{-1}_Y. \end{aligned}$$

It is possible now to give the main result of this section.

Theorem 2.1 *Assume that $V(Q)$ is Δ-bounded with subunitary relative bound and that there is a subspace $Y \subset X$, with dimension $n_Y \geq 3$ such that*

(i) $(D^Y V)(Q) \in B(\mathscr{H}^2(X), \mathscr{H}(X))$,

(ii) $|(D^Y V)(x)| \leq \frac{C_Y}{|x^Y|^2}$, *where* $C_Y \leq \frac{(n_Y - 2)^2}{2}$, (2.4)

(iii) $|(D^Y D^Y V)(x)| \leq \frac{C}{|x^Y|^2}$. (2.5)

Then

(a) *H has purely absolutely continuous spectrum,*

(b) $|\langle f, (H - \lambda \mp i\mu)^{-1} f\rangle| \leq C ||f||^2_{\overset{\circ}{\mathscr{H}}_{1,Y}}$ *for all $\lambda \in \mathbb{R}$, $\mu > 0$ and $f \in \overset{\circ}{\mathscr{H}}_{1,Y}$,*

(c) *Any closed, densely defined operator in* $\mathscr{H}(X)$ *which extends to an element of* $B(\overset{\circ}{\mathscr{H}}_{-1,Y}(X), \mathscr{H}(X))$ *is* H*-smooth.*

Proof. The proof consists in verifying (straightforwardly) the assumptions of Theorem 1.1 mainly by making use of (2.3). We only note that the choices $\mathscr{B} = \overset{\circ}{\mathscr{H}}{}_Y^{1}$ and $\mathscr{B}^* = \overset{\circ}{\mathscr{H}}{}_Y^{-1}$ are possible (with equivalent norms) and that $\overset{\circ}{\mathscr{H}}_{1,Y}$ is obviously embedded continuously in $D(A_Y; \overset{\circ}{\mathscr{H}}{}_Y^{-1})$. We shall not make explicit the assumption that $V(Q)$ is Δ-small, neither the assumption (i); since our main purpose is to master potentials which behave anisotropically at infinity, there is not a great loss to suppose that V and $D^Y V$ are bounded.

Remark 2.1 It is easy to see that there is no monotony in Y, neither in the hypothesis nor in the conclusion. It is an important fact that one is allowed to take $Y \neq X$. For example, if V is Y-homogeneous of degree 0 (see (2.2)), it seems that the absence of the singular spectrum is known only for $Y = X$ (case treated by I. Herbst in [3] in great detail; he emphasizes the asymptotical property of e^{-itH}, obtaining refined informations). The examples covered by setting $Y \neq X$ have unexpected generality. Choose for example $V : X \to \mathbb{R}$ a function which is C^2 outside the origin, whose derivatives of order ≤ 2 do not grow at infinity and which depends only on $(x^Y|x^Y|^{-1}, x^Z)$. This is is already quite anisotropic inside Y. With respect to the variables in Z it is "arbitrary"; in particular, there is no need of radial limits. Even the factorizable case

$$V(x) = V^Y\left(\frac{x^Y}{|x^Y|}\right) V^Z(x^Z) \tag{2.6}$$

is remarkably wild. But if V does not depend on x^Y, the situation becomes trivial; the operator $H = \Delta^Y \otimes 1 + 1 \otimes (\Delta^Z + V^Z(Q^Z))$ is, of course, purely absolutely continuous. It might happen, however, that the global resolvent estimates be new. The smoothness condition imposed on V is there for simplicity and can be relaxed. In (2.6) for example, V^Z may be any L^∞ function.

Remark 2.2 Let us make some comments on the dimension n_Y. The trouble is that (2.3) is not true for $n_Y = 1$ or $n_Y = 2$. Generally, there is no way out; if $V \in C_0^\infty(\mathbb{R}^N)$ with $N = 1, 2$ and V is negative, then $H = \Delta + \varepsilon V$ has bound states for any $\varepsilon > 0$. But there are also good particular cases. For instance, if V is Y-homogeneous of degree 0, the use of (2.3) is avoided and n_Y may take any value.

Remark 2.3 It is obvious that one may replace (2.4) by some repulsivity condition. This does not give a very attractive result for $Y = X$, because of the extra assumptions needed. Using the Kato-Putnam theory and a more intricate A, R. Lavine obtained a better result (see for example Theorem XIII.29 in [7]). However, the present approach has also some pleasant features: one needs no tedious calculations and "the repulsivity inside a proper subspace" is enough.

Remark 2.4 We now make some considerations on the behaviour of V at infinity allowed in Theorem 2.1. We add a mild regularity assumption: for almost every $x^Z \in Z$, V is C^1 in the radial variable outside a compact set. Specifically, suppose that there is a negligible set $M \subset Z$ such that, for any $x^Z \in Z \setminus M$, there is a $r^{x^Z} > 0$ such that the map $(r^{x^Z}, \infty) \ni r \mapsto V(r \cdot \omega^Y, x^Z) \in \mathbb{R}$ is C^1 for all $\omega^Y \in S^Y$, where S^Y is the unit sphere in Y. From (2.4) we infer by means of "an integration of the derivative procedure" that the radial limits

$$V(\infty \cdot \omega^Y, x^Z) = \lim_{r \to \infty} V(r \cdot \omega^Y, x^Z)$$

exist and one has the following estimate on the convergence rate

$$|V(\infty \cdot \omega^Y, x^Z) - V(r \cdot \omega^Y, x^Z)| \leq \frac{C_Y}{2} r^{-2}. \tag{2.7}$$

Hence, roughly:

(i) V must have radial limits in the directions included in Y,

(ii) V must have a r^{-2}-convergence (with a small constant) to those limits,

(iii) $(\partial_r V)(r\omega^Y, x^Z) = O(r^{-3})$ when $r \to \infty$ (with a small constant),

(iv) $(\partial_r^2 V)(r\omega^Y, x^Z) = O(r^{-4})$ when $r \to \infty$.

There is no extra restriction on the Z-behaviour. The strength of Theorem 2.1 is shown even by the special case $V(x) = V^Y(x^Y) \cdot V^Z(x^Z)$. V^Z may be any L^∞ function and (2.4) (for example) reads now:

$$|(x^Y, (\nabla^Y V^Y)(x^Y))| \leq \frac{C_Y}{||V^Z||_{L^\infty}} \cdot |x^Y|^{-2}.$$

Up to our knowledge, neither of the two simple situations (a) $Y = X$, (b) $V^Y(x^Y) \to 0$ when $|x^Y| \to \infty$, with $Y \neq X$ was known before.

Remark 2.5 Let us work in the representation $\mathscr{H}(X) = \mathscr{H}(Y; \mathscr{H}(Z))$. One justifies easily the identification of $\overset{\circ}{\mathscr{H}}_{-1}(Y) \otimes \mathscr{H}(Z)$ with $\overset{\circ}{\mathscr{H}}_{-1}(Y; \mathscr{H}(Z))$, the completion of $C_0^\infty(Y; \mathscr{H}(Z))$ (the smooth functions: $Y \to \mathscr{H}(Z)$ having compact support) under the norm

$$||f||_{\overset{\circ}{\mathscr{H}}_{-1}(Y;\mathscr{H}(Z))} = \left[\int_Y ||f(y)||^2_{\mathscr{H}(Z)} \frac{dy}{|y|^2} \right]^{1/2}.$$

Taking into account this and the point of (c) of Theorem 2.1, we see that any measurable function $F: Y \to B(\mathscr{H}(Z))$ which satisfies $||F(y)||_{B(\mathscr{H}(Z))} \leq C|y|^{-1}$ defines in an obvious way a smooth operator.

We particularize now Corollary 1.1 for the case of Schrödinger operators.

Corollary 2.1 *Let $V_1, V_2 : X \to \mathbb{R}$ two Borel functions such that the corresponding multiplication operators in $\mathscr{H}(X)$ are Δ-bounded with bounds < 1. For $j = 1, 2$, let us set $H_j = \Delta + V_j(Q)$ and assume that there is a subspace Y_j of X, of dimension $n_j \geq 3$ and a constant $C_j < \frac{(n_j-2)^2}{2}$ such that*

$$|(D^{Y_j} V_j)(x)| \leq \frac{C_j}{|x^{Y_j}|^2}, \qquad D^Y V_j \in B(\mathscr{H}^2(X), \mathscr{H}(X)), \tag{2.8}$$

$$|(D^{Y_j} D^{Y_j} V_j)(x)| \leq \frac{C_j}{|x^{Y_j}|^2}, \tag{2.9}$$

$$|V_1(x) - V_2(x)| \leq \frac{C}{|x^{Y_1}| \cdot |x^{Y_2}|} \tag{2.10}$$

Then the wave operators

$$\Omega^{\pm}(H_2, H_1) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_2} \cdot e^{-itH_1}$$

exist and are complete. In particular, H_1 and H_2 are unitary equivalent.

Proof. The proof consists in checking that the assumptions of Corollary 1.1 are fulfilled with $A_j = A_{Y_j}$. Note that (2.10) says precisely that $V_1(Q) - V_2(Q)$, defined at least as a continuous form on $\mathscr{G}^1 = \mathscr{H}^1(X)$, i.e. as a bounded operator: $\mathscr{H}^1(X) \to \mathscr{H}^{-1}(X)$, extends to an element of $B(\overset{\circ}{\mathscr{H}}_{-1,Y_1}, \overset{\circ}{\mathscr{H}}_{1,Y_2})$.

Remark 2.6 Let us take a look at the case $Y_1 = Y_2 = Y$, taking into account Remark 2.4. V_1 and V_2 will be supposed regular in the sense described there. Obviously, in order to satisfy (2.10) when (2.8) is true, it is enough to ask that $V_1(\infty \cdot \omega^Y, x^Z) = V_2(\infty \cdot \omega^Y, x^Z)$ for all $\omega^Y \in S^Y$ and almost all $x^Z \in Z$. The unitary equivalence criterion we get can be roughly described as follows: Take two potentials satisfying (i),...,(iv) in Remark 2.4. Assume that they have the same radial limits in the directions included in Y. Then they define unitarily equivalent Schrödinger operators. This seems to be interesting in both the particular cases "$Y = X$" and "$Y \neq X$ and $V_j(\infty \cdot \omega^Y, x^Z) = 0$". To illustrate the second situation, let us take $V_j(x) = \gamma_j V^Y(x^Y) \cdot V^Z_{(j)}(x^Z)$. Then

$$|V_1(x) - V_2(x)| = |V^Y(x^Y)| \cdot |\gamma_1 V^Z_{(1)}(x^Z) - \gamma_2 V^Z_{(2)}(x^Z)|$$

and it suffices, for example, to require that V^Y be a symbol of order -2, with small enough positive constants γ_1 and γ_2, and $V^Z_{(1)}, V^Z_{(2)}$ bounded, but otherwise arbitrary.

Remark 2.7 Let us set $Y_{12} = Y_1 + Y_2$. It follows at once from (2.10) and (2.7) (written alternatively for Y_1 and Y_2) that the limits $V_j(\infty \cdot \omega^{Y_{12}}, x^{Y_{12}^\perp}) = \lim_{r\to\infty} V_j(r \cdot \omega^{Y_{12}}, x^{Y_{12}^\perp})$ exist for any $\omega^{Y_{12}} \in S^{Y_{12}}, x^{Y_{12}^\perp} \in Y_{12}^\perp$, $j = 1, 2$ and are equal (we assumed some regularity on the dependence of V_j of the variable $x^{Y_{12}}$).

Corollary 1.2 is even more interesting in our context, because it extends the result on the absence of the singular spectrum to some potentials satisfying less than what was needed in Theorem 2.1.

Corollary 2.2 *Let $H = \Delta + V(Q)$, where V satisfies the hypotheses imposed in Theorem 2.1. Let $U : X \to \mathbb{R}$ a Borel function such that*

$$|U(x)| \leq \frac{C}{|x^Y|^2} \tag{2.11}$$

and set $H_\gamma = H + \gamma U(Q)$. There exists $\Gamma > 0$ such that for all $\gamma_1, \gamma_2 \in (-\Gamma, \Gamma)$, H_{γ_1} and H_{γ_2} are unitary equivalent through the corresponding wave operators. All H_γ's are purely absolutely continuous.

Remark 2.7 shows how to control the case when U is, more generally, a suitable function from Y to the self-adjoint elements of $B(\mathscr{H}(Z))$.

By means of Corollary 2.2 we cover a large class of perturbations with no condition on the derivatives. Let us take for example $Y = X$ and $V : X \to \mathbb{R}$ homogeneous of degree 0. Its radial limits are $V(\infty \cdot \omega) \equiv V(\omega)$ ($\omega \in S^X$). If one superposes a potential which obeys (2.11), there follows a result which is worth mentioning:

Corollary 2.3 *Let $W : X \to \mathbb{R}$ be a function which is smooth (outside a compact set). Assume that it has radial limits $W(\omega) = \lim_{r\to\infty} W(r \cdot \omega)$ for any $\omega \in S^X$ such that*

$$|W(\omega) - W(r \cdot \omega)| \leq \frac{\gamma}{r^2} \text{ for all } r > 0,\ \omega \in S^X.$$

Then, for γ small enough, $H = \Delta + W(Q)$ is a closed form on $\mathscr{H}^1(X)$ and it has no singular spectrum.

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