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An Improved Approach to Relativistic Rotational Kinematics

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Abstract. The requirement of full compatibility between time–dilation and relative motion makes it meaningful to search for a direct theoretical way of mutually comparing just two clocks which are moving relative to each other. This goal can soundly be reached by introducing a "covariant" Cartesian projection of the space–time path of one clock onto the world–line of the other clock. Such a projection, besides reducing the given space displacement to a null length, also reduces (accordingly) the related time interval to its proper length, so as to set up a one–to–one correspondence (called "covariant" synchrony) between the "local" (proper) times that the two clocks have symmetrically been reading since their meeting. That gives a peculiar "local" physical content to the definition of four–velocity, which, in its turn, yields some far–reaching effects on rotational kinematics. A natural geometrical conception of a rotating frame can in fact be regained, in terms of a suitable "world" radial coordinate that may freely run to infinity (with no failure of relativistic consistency) and does not need to be cut off beyond the value c/ω (ω being the angular velocity of rotation). Hence a new kinematical model of a spinning disc directly proceeds, which allows a disc initially at rest, of whatever (original) radius, to be brought to rotate with an arbitrarily great, uniform angular velocity. What would ensure a border tangential speed $v < c$ is a shape–preserving, global (both longitudinal and transversal) Lorentz–like contraction that the disc should apparently undergo when seen rather spinning than being at rest in the Laboratory system. For $v \ll c$, the effect of such a contraction on the rotational red–shift would not exceed a size of order $\omega^4 R_0^4/c^4$, where R_0 is the (original) uncontracted distance between the source (placed at the centre of the disc) and the absorber (placed on the disc edge).

1 Introduction

Recently, Bailey *et al.*'s measurements of circularly rotating muon lifetime [1–3] have been the main subject of a renewed controversy [4,5] as regards the experimental evidence for the relativistic time–dilation formula

$$dt = dt_o(1 - v^2/c^2)^{-1/2}\,, \tag{1}$$

dt_o being a proper–time interval by a clock travelling at speed v. In the present paper, the intrinsic reliability of such measurements is not under discussion; it is argued, however, that a further insight into the question about time–dilation can still be gained from the theoretical viewpoint, with some intriguing far–reaching effects upon rotational kinematics.

As is well–known, the relativity principle strictly demands an *identical* flow rate of the proper time in every inertial system. This is not contradicted by the time–dilation effect, that actually involves an *asymmetrical* comparison of clocks in relative motion [6]: the clock which is seen lagging behind is always the one which is being compared with *different* clocks set *along* its path in the other inertial system. What we are going to show in Section **2** is that in full consistency with the relativity principle, also a *symmetrical* rate comparison for just *two* relatively moving clocks may be conceived in the Minkowskian space–time, though it has a purely theoretical significance. This rate comparison can be made in geometrical terms, by introducing a "covariant" Cartesian projection of the space–time path of one ('moving') clock onto the world–line of the other ('rest') clock (taking their meeting world–point as the initial event). Under such a projection, which is rigorously defined in Section **3**, the given world displacement of the 'moving' clock is turned into a *pure* time displacement (at one and the same spatial point) while keeping its length *unvaried*: the result is that not only the space displacement alone is reduced to a null length, but also the related time interval is reduced, accordingly, to its *proper* length. This (covariant) procedure really succeeds in setting up a *one–to–one correspondence* between the "local" (proper) times that the two clocks have been reading far from each other; which, of course, cannot be obtained merely under a (non–covariant) Cartesian projection in ordinary space. The correspondence so established – we shall call "covariant" synchrony – gives a peculiar physical meaning to the definition of four–velocity, that may now be viewed as being relative to a *single* rest observer and his "local" (proper) time. As is seen in Section **4**, such "local" view of the four–velocity becomes particularly significant within rotational kinematics: it enables one to recover a geometrically congruous (and still relativistically consistent) internal description of a rotating frame, in terms of an appropriate "world" radial coordinate that may *naturally run to infinity*, instead of needing a *cut off* for values greater than c/ω (ω being the angular velocity of rotation). The most immediate physical application, discussed in Section **5**, concerns a disc which is initially at rest and is then brought to spin with a uniform angular velocity. What should happen is that a disc with an *arbitrarily great* (original) radius can be brought to rotate at an *arbitrarily great* angular velocity: it should apparently undergo, with spinning, a *global* (both longitudinal and transversal) Lorentz–like contraction leaving its shape undistorted. Such an outcome does not affect the standard expectation for a radial length of the spinning disc that should look the same in the Laboratory system as well as in the frame rotating along with the disc.

2 Time–Dilation Revisited: An Insight Gained into the Equivalence Property of Two Relatively Moving Clocks

The whole question about the real compatibility of the time–dilation effect with the relativity principle has been the subject of a very long controversy [7]. As particularly concerns the clock–rate'paradox' for two distinct inertial systems, a way out of contradiction is provided by the well–known standard argument [6] mentioned above. In short, the actual *asymmetrical* way of comparing relatively moving clocks is to be taken as just an essential condition for time–dilation to make its appearance (in spite of the relative motion).

One basic point emerging from these considerations is the following: if we could *directly* and *unambiguously* compare the rates of two *single* clocks in relative motion, without having recourse to any further clocks in the two systems, we should find them to be neither slower nor faster than each other and we should observe no time dilation at all. Were such a statement incorrect, the argument itself making the time–dilation effect fully compatible with the equivalence of inertial frames would be unavoidably contradicted. But what significance (other than the strict logical one) may be attributed to this statement?

Actually, given two clocks, C and C', belonging to the respective inertial frames $K(x,y,z)$ and $K'(x',y',z')$, a *direct* comparison of their own rates seems to be conceivable only in *geometrical* terms. Suppose C and C' to be placed at the origins of K and K', so that they may read the same time instant $t = t' = 0$ when the two origins are happened to coincide. Let E_o denote such initial event (geometrically represented by the common space–time origin of the K and K' systems). If K' is assumed to be moving (relative to K) in the direction of the x–axis, let E be a successive event marked by the instantaneous position $x = x_1$ (> 0) of the clock C' on that axis. Of course, the actual time instant $t = t_1$ associated with E in the K system cannot be read (remotely) by C, but can be determined (locally) by another clock (synchronized with C) set at the spatial point of coordinates $(x_1,0,0)$. If $t' = t'_1$ is the corresponding instant read by C', then, according to (1), the time interval $\Delta t = t_1$ and the proper–time interval $\Delta t_o = t'_1$ (both measured between E_o and E) should be such that

$$\Delta t = \gamma(V)\,\Delta t_o \tag{2}$$

$\gamma(V) = (1 - V^2/c^2)^{-1/2}$, where V is the K' speed relative to K. In view of that, the simplest attempt at getting in principle a *direct* comparison of the C and C' rates might be to project the space location $(x_1,0,0)$ of the event E onto the yz–plane (perpendicular to the direction of the C' motion): a correlated event $E_\perp$ is thus obtained, which lies on the world–line itself of clock C. The outcome is trivial, but by no means convincing. The time interval between E_o and $E_\perp$ (in the C system) would, of course, be still equal to Δt, and one should conclude that clock C' would be *slower* than C. Such a conclusion is indeed at variance with what should be expected for a really unambiguous and truly direct comparison of the C and C' rates. The questionable point is that a quite *opposite* outcome could equally be obtained, once allowing for the relativity requirement and applying the same procedure to the motion of C with respect to C'. Hence, this cannot be taken as being a sound method, since it would lead, on the whole, to a self–contradictory result: *the same clock might equally be claimed to be either slower or faster than the other clock.*

The intrinsic ambiguousness of the above method does not seem to exclude the possibility of really conceiving a correct procedure for the purpose in question. Strictly speaking, the projective operation we have taken into account is *not* a covariant one, as it is just defined in ordinary space and merely affects the spatial coordinates of the event E. The non–covariant character is especially pointed out by the fact that such a projection is apparently able to reduce Δt to a proper–time interval (at the C location) *without being able, however, to reduce its length accordingly.* This is at the origin of the questioned ambiguousness and makes it natural to think of some "covariant" sort of projection in space–time that may really exhibit such further essential requirement. Let then E' be that event – quite *symmetrical* to E – which is specified by the world coordinates $(t'=t_1, x'=x_1, 0, 0)$ in the K' frame, or by the world coordinates $(t=t'_1, 0,0,0)$ in the K frame; and let $E'_\perp$ be the correlated event which is obtained by simply projecting the spatial location of E' onto the $y'z'$–plane. We may argue that the improved projective method looked for should prescribe the "covariant" mapping

$$E\,(E') \longrightarrow E_\perp^{(c)} \equiv E'\,(E'^{(c)}_\perp \equiv E)\,. \tag{3}$$

A comparison between (3) and the ordinary mapping, $E\,(E') \longrightarrow E_\perp\,(E'_\perp)$, shows that they overlap merely in the spatial domain: the whole effect of (3) in K (K') also includes the *full* reduction of the time interval between E_o and E (E') to its *proper* length (equal to Δt_o). Prescription (3) is really able to define a *self–consistent* direct comparison of the C and C' rates: in strict accordance with their relative motion, clocks C and C' can unambiguously be claimed, via (3), to be neither slower nor faster than, but *"covariantly" synchronous* with, each other.

According to (3), the proper time which is being read by each of the two clocks C, C' may also be regarded as a "local" time connected with a far–away displacement of the other clock. This enables one to think of the four–velocity in more physical terms, beyond its pure geometrical significance in the Minkowskian space–time. By virtue of (3), the four–velocity of a moving object may in fact stand for a (covariant) velocity with respect to a *single* observer at rest and in terms of his "local" (proper) time far from the object. It will be seen in Section **4** that such an insight into the concept of four–velocity turns out to play an essential role for a deeper understanding of the metric properties of a rotating frame, as particularly concerns points which are not very near the rotation axis and whose tangential speed is not very small relative to the light speed.

3 A "Covariant" Cartesian Projection of a Time–like World Displacement in an Inertial Frame

Unlike the purely spatial mapping $E\,(E') \longrightarrow E_\perp\,(E'_\perp)$, the "covariant" mapping (3) in the K (K') frame really succeeds in defining a *one–to–one correspondence* between the proper–time instants that the clocks C and C' have *symmetrically* been reading since their meeting. This may happen, because the world displacement connecting E_o with E (E') is so projected by (3) as to keep its original length, $c\,\Delta t_o$, *unvaried.*

Such distinctive property tells us that (3) should be taken as a special case of a generalized kind of projection in space–time, we may call *"covariant" Cartesian projection.* The feature peculiar to it should just be the fact that unlike an ordinary Cartesian projection, it *preserves* the length of the whole world displacement involved.

Take, e.g., in the K system, an infinitesimal world displacement $(c\,dt, d\mathbf{r})$ carried out by a moving object starting from a given spatial point $\mathbf{r}$, and let

$$c^2dt^2 - d\mathbf{r}^2 = c^2dt_o^{\,2} \qquad (4)$$

be its squared length. Indeed, the "covariant" Cartesian projection of $(c\,dt, d\mathbf{r})$ onto a given q–axis $(q = x, y, z)$ can be conveniently represented by introducing one *ficticious* time coordinate, t_q, associated with that axis: we may thus say that $(c\,dt, d\mathbf{r})$ is mapped into a world displacement $(c dt_q, dq)$ *of squared length*

$$c^2dt_q^2 - dq^2 = c^2dt_o^2 \qquad (q = x, y, z)\,. \qquad (5)$$

So, if making reference to the $d\mathbf{r}$ direction, we may in particular speak of a "transverse" time component, $dt_\perp$, that *equals* the proper time dt_o spent by the object and stands just for the corresponding "local" (proper) time elapsed at the starting point $\mathbf{r}$:

$$dt_\perp = dt_o\,. \qquad (6)$$

Of course, we have still to do, consistently, with a total number of *four* freedom degrees in space–time, since the three time displacements dt_x, dt_y, dt_z are already determined by knowing the four quantities dx, dy, dz, dt_o. A formalism like this, characterized by a "three–dimensional" time–dilation like

$$dt_q = dt_o(1 - v_q^2/c^2)^{-1/2} \qquad (v_q = dq/dt_q;\, q = x, y, z)\,, \qquad (7)$$

is not a novelty in the literature [8,9], though herein it is being adopted *anew* and without departing from the *strict* framework of the (four–dimensional) Minkowskian space–time. According to the present approach, time is still a *scalar* (one–dimensional) quantity in space, which is left invariant under an *ordinary* Cartesian projection of the motion: each "component" dt_q that may be assigned to dt is but the result of a "covariant" Cartesian projection (onto the corresponding q–axis). Likewise, each quantity v_q in (7) should not be confused with an ordinary Cartesian component of three–velocity, rather standing for a "covariant" Cartesian component (in terms of dt_q) that obeys one *independent* composition rule of the type

$$v_q = (v'_q + V_q)\,(1 + v'_q V_q/c^2)^{-1} \qquad (q = x, y, z) \qquad (8)$$

(V_q being itself a "covariant" Cartesian component of the three–velocity of frame K' relative to frame K). This also means that the *same* ultrarelativistic limit $v_x = v_y = v_z = c$ covariantly holds for all v_q's, as can be seen by substituting $c^2dt_o^2 = 0$ in Eqs. (4),(5).

In conclusion, we may state that the previously defined "covariant" synchrony of two clocks in relative motion can strictly be obtained by making *a "covariant" Cartesian projection of the space–time trajectory of one clock onto the world–line of the other clock.* Let us see now what happens on extending all that to the case of an infinitesimal world displacement performed by a clock in uniform circular motion around a clock at rest.

4 A Deeper View of the Kinematics of a Rotating Frame

The reason for utilizing a 'clock' in uniform circular motion to test time–dilation lies in the common (sensible) idea that we can truly simulate a uniform rectilinear motion by restricting ourselves to an *infinitesimal* displacement, $d\ell = R\, d\varphi$, which the 'clock' is performing along its circular path of radius R. This peculiar feature can be expressed in the following essential terms. Let C' denote the 'clock' under consideration. Suppose $d\tau'$ to be the proper time spent by C' in covering $d\ell$, and dt to be the corresponding actual time interval elapsed in the reference system at rest. Then,

$$dt/d\tau' = dt/dt_o = \gamma(v) \qquad (v = d\ell/dt)\,, \tag{9}$$

dt_o strictly being the proper time that C' would spend (in covering $d\ell$) if let go free along the tangent line.

The actual (constant) angular velocity of C' (with respect to the centre of the drawn circumference) can be defined as

$$\omega = d\varphi/dt \tag{10}$$

and is such that $v = \omega R$. Strictly speaking, time dt is *not* just a *proper* time in the rest frame, which may be read, i.e., by one (and the same) clock, say C, placed at the centre of the circumference: considering that C belongs as well to the rotating frame linked with C', we may regard dt as an actual proper time measured by C, only if we are *also* making reference to the latter frame (where C' is *at rest* relative to C). In principle, to define a "true" instantaneous angular velocity, one should, rather, make reference *but* for the rest frame and replace dt with a *truly proper* (virtual) time interval, say $d\tau$, suitably obtained from dt by *projecting* the C' world displacement onto the perpendicular line: one should take then, in place of ω, a quantity like

$$\omega_o = d\varphi/d\tau\,. \tag{11}$$

By definition, this is not, however, a directly measurable quantity, and the value of it may depend on the projective method adopted to define $d\tau$. Similarly, as strictly concerns the viewpoint of the clock C *alone* in the rest frame, one should introduce, together with (11), an instantaneous linear velocity like

$$v_o = d\ell/d\tau = \omega_o R\,. \tag{12}$$

Quantity v_o can equivalently be defined in a reversed manner, by exploiting the fact that C is belonging to the rotating frame as well: from the latter frame, during an actual proper time interval dt, the clock C sees C' moving in the rest system by a (virtual) spatial length $d\lambda = R'd\varphi$, where R' strictly stands for the radial coordinate of C' relative to C *in the rotating frame.* It may, therefore, also be put

$$v_o = d\lambda / dt = \omega R' , \tag{13}$$

whence,

$$\omega_o R = \omega R' . \tag{14}$$

All that has merely a formal significance, until the projective method fixing the value of $d\tau$ is really specified. If limiting ourselves to projecting the spatial component of the C' world displacement onto the perpendicular line, we trivially obtain

$$d\tau = dt , \tag{15}$$

so that

$$\omega_o = \omega \quad , \quad v_o = v . \tag{16}$$

This leads us, via (14), to *define* R' in line with the standard identity

$$R = R' . \tag{17}$$

If on the contrary, in view of the foregoing, we make a "covariant" Cartesian projection of the *whole* C' world displacement onto the perpendicular line, we non–trivially find that C' is "covariantly" synchronous with C:

$$d\tau = d\tau' = dt_o . \tag{18}$$

That implies, due to (9),

$$\omega_o = \gamma(v)\,\omega \quad , \quad v_o = \gamma(v)\,v \tag{19}$$

where v_o now looks just like the space component of the C' *four–velocity.* Hence, via (14), it is drawn that by a "radial coordinate" R' we can alternatively mean also a quantity (no longer coincident with R) *such that*

$$R = \gamma^{-1}(v)\,R' \tag{20}$$

or

$$R' = R_o \equiv \gamma(v)\,R . \tag{21}$$

The resulting non–uniqueness of the R' definition is clearly admissible on account of the non–Euclidean nature of the C' frame, but it is left to see what different meaning should be assigned to R' in the two cases (17) and (20).

The unusual link (21) makes it non–trivial to rewrite formula $v = \omega R$ just in terms of $R' = R_o$, so as to obtain

$$v = \gamma^{-1}(v)\,\omega R_o . \tag{22}$$

This equation , if solved relative to v, yields

$$v = \omega R_o (1 + \omega^2 R_o^2/c^2)^{-1/2} , \tag{23}$$

$$\gamma^{-1}(v) = \gamma^{-1}(\omega R) = (1 + \omega^2 R_o^2/c^2)^{-1/2} \equiv \delta(\omega R_o)\,. \tag{24}$$

Formula (23) consistently binds v to be *less* than c for *any* (arbitrarily large) value of R_o and *any* fixed value of ω: looking at (23), it can immediately be checked that $v \to c$ as $R_o \to \infty$. Such a property becomes particularly significant on defining a rotating frame. Let $R' = R_o$, z', φ' be the cylindrical coordinates of a rotating frame, with the new coordinate $R' = R_o$ replacing the ordinary radial coordinate $R' = R$, and let the z'–axis be coincident with the rotation axis. By virtue of (23) we can thus recover the *natural* geometric definition of *a radial coordinate R' freely running to infinity* (rather than being *cut off* for values exceeding the usual maximum admissible value $R'_{max} = c/\omega$). From the viewpoint of the inertial system at rest, such a frame is still seen rotating *as a whole* with an actual angular velocity ω, and still appears with a *finite* radial extension up to a distance $R_{max} = c/\omega$ from the rotation axis. Evidently, the invariance property of the world interval being understood, an *unusual* reading of the ordinary space–time metric of a rotating frame must be involved. The "new" metric will be formally reducible to the usual one, as soon as it is rewritten in terms of the ordinary radial coordinate $R' = R$: it can then be expressed by a squared world interval of the form

$$ds^2 = \delta^2(\omega R_o)\, c^2 dt^2 - \delta^6(\omega R_o)\, dR_o^2 - 2\delta^2(\omega R_o)\, \omega^2 R_o\, d\varphi' dt - \delta^2(\omega R_o) R_o^2\, d\varphi'^2 - dz'^2, \tag{25}$$

where $\delta(\omega R_o)$, as given by (24), is such that $R = \delta(\omega R_o) R_o$ and $dR = \delta^3(\omega R_o)\, dR_o$. The spatial metric related to (25) reads

$$d\ell'^2 = \delta^6(\omega R_o)\, dR_o^2 + R_o^2\, d\varphi'^2 + dz'^2; \tag{26}$$

and a comparison with the corresponding usual metric form,

$$d\ell'^2 = dR^2 + \gamma^2(\omega R) R^2 d\varphi'^2 + dz'^2\,, \tag{27}$$

points out the following *distinctive* meaning for R_o as opposed to R: *The coordinate R_o should stand for the Euclidean radius proper to a circumference which is drawn in the rotating frame at an actual radial distance R from the rotation axis.* So R_o, unlike R, does *not* give the actual radial length of such a circle in the rotating frame, but rather gives the corresponding radial length that the same circle would exhibit if the metric were Euclidean. In this sense, R_o may just be taken as the spatial analogue of the world time t (measured at the rotation axis) and may then be referred to as a "world" radial coordinate.

5 An Undistorted Contracted Model of a Uniformly Rotating Disc

What has been shown in the previous Section enables one to gain an insight into the kinematics of a uniformly rotating disc. Consider a disc which is initially at rest in the Laboratory system and is then brought to rotate (around its centre) with an actual uniform angular frequency ω. From locally applying Special Relativity we may draw the conclusion that when the disc is seen spinning, its edge should exhibit a Lorentz–contracted total length, as compared with the proper total length of it in the frame rotating along with the disc: the factor of contraction should just be the one, $\gamma^{-1}(\omega R)$, predicted for each infinitesimal rectilinear element of the disc edge, R strictly being the radius that the disc turns out to have (in the Laboratory system) *when rotating*. Likewise we may conclude that the radial distance R must also be coincident with the actual proper distance of the edge of the spinning disc from the rotation axis. In principle, this does *not* strictly mean that R should further be identical with the *original* radius of the disc (still at rest in the Laboratory system); as well as it cannot *a priori* be said that a Lorentz contraction of the disc edge should netly occur even relative to the *original* proper length of it (before spinning). Two *different* readings can in fact be provided according to whether the spatial metric relevant to the rotating disc has to look like (27) or (26). If such a metric is still to be Euclidean *along the radial direction*, as in (27), then R must stand as well for the original radius of the disc; if, on the contrary, the Euclidean character of the metric is still to be maintained *along the disc edge*, as in (26), then the length of the disc edge must *truly* appear to be Lorentz–contracted when the disc is seen rather spinning than being at rest in the Laboratory system. Two such metric varieties turn out to work *opposite*. As for (27), the appearance of the non–Euclidean metric factor $\gamma^2(\omega R)$ *along the disc edge* should imply an actual *dilatation* of the proper length of the edge, so that *no* net Lorentz contraction could be observed in the Laboratory system (relative the original length of the disc edge at rest) [10]. As for (26), which is *no more* Euclidean along the radial direction, it does *not* bind the actual radius R of the spinning disc to coincide with the *original* radius of the disc at rest, while it strictly tells us that *the proper length of the disc edge should be always the same, no matter whether the disc is at rest or uniformly rotating*. Let us take then (26) as the "true" metric for the reference system which is rotating along with the disc. Denoting by R_o the related radial coordinate to be assigned to any given point of the disc edge, we should therefore have, as concerns the radial coordinate R of the same point in the Laboratory system,

$$R = \gamma^{-1}(\omega R)\, R_o = \delta(\omega R_o)\, R_o = R_o(1 + \omega^2 R_o^2/c^2)^{-1/2} . \tag{28}$$

Note, by the way, that a formula like this should similarly apply to an inner point of the disc. On the other hand, we already know that R_o stands for the *Euclidean* radius fitting the actual proper length of the circumference drawn by the edge of the disc. So, after all, considering that the proper length in question should not depend upon whether the disc is spinning or not, we are led to conclude that *it is rather R_o, than R, to be identified with the original radius of the disc at rest in the Laboratory system*. Hence, when the disc is brought to rotate with a uniform angular velocity ω, its edge should exhibit a net Lorentz–contracted length $2\pi R_o \gamma^{-1}(\omega R)$, and its apparent new radius R as given by (28) should just be that one for a circle with an edge of length $2\pi R_o \gamma^{-1}(\omega R)$! This means that the (longitudinal) Lorentz contraction is really expected to affect a disc rather spinning than being at rest, but without causing any sort of distortions in its shape: *A uniformly spinning disc should apparently undergo a global (both longitudinal and transversal) contraction preserving its natural (undistorted) shape*. Such a model of a rotating disc is indeed the direct physical counterpart of the proposed model of a rotating frame with a radial coordinate *freely running to infinity*: by virtue of (23), a disc of *whatever* original radius R_o might actually be brought to rotate at *whatever* uniform angular velocity ω.

Let us look now at the experimental viewpoint. Of course, measurements like those of circularly rotating muon lifetime [1–3] are not so conceived as to be a suitable test for the new model of a rotating frame and its physical consequences: the radius R of the muon circular path is (like ω) a fixed parameter in the Laboratory system, and the metric (25) still predicts a dilated muon lifetime $\tau = \delta^{-1}(\omega R_o)\tau_o = \gamma(\omega R)\tau_o$ ($\tau_o \approx 2.2 \times 10^{-6}$ sec). Quite different, on the contrary, may be the case of an experiment aimed at measuring the rotational red–shift with the help of the Mössbauer effect [11]. In outline, let an absorber be circularly rotating around a source placed at the centre of the described circumference. Both the source and the absorber are supposed to be assembled within a cylindrical apparatus which is rigidly spinning at an angular velocity ω. Due to the circular motion, the (proper) resonant absorption frequency ν_o is expected to be red–shifted according to the standard formula, equivalent to (9),

$$\nu/\nu_o = \gamma^{-1}(v) \tag{29}$$

(v being the actual tangential speed of the absorber). Let the lower resonant absorption frequency ν be written in terms of ωR_o (rather than of v), where R_o is the original radius of the cylinder at rest (as well as the original proper distance between source and absorber). In the standard approach, one clearly has $v = \omega R_o$ and

$$\nu(\omega R_o) = \nu_o \gamma^{-1}(\omega R_o) \, . \tag{30}$$

In the new approach, on the contrary, Eq. (23) should hold, and one should have then, as can be obtained by substituting (24) in (29),

$$\nu(\omega R_o) = \nu_o(1 + \omega^2 R_o^2/c^2)^{-1/2} = \nu_o \gamma^{-1}(\omega R) \, , \tag{31}$$

R being the *contracted* radius, given by (28), which is to be expected for the rotating cylinder. Formulas (30) and (31) are actually coincident *up to order* $\omega^2 R_o^2/c^2$. Experimental evidence for (30) is just to order $\omega^2 R_o^2/c^2$ [11], thus being *unable* to distinguish (30) from (31). For such a purpose to be achieved, a measurement of the resonance frequency shift to an accuracy at least of order $\omega^4 R_o^4/c^4$ is required. If we take the expansion of $\nu(\omega R_o)$ in either case and neglect higher–order terms, we get

$$\nu(\omega R_o) = \nu_o(1 - \omega^2 R_o^2/2c^2 - \omega^4 R_o^4/8c^4) \tag{32}$$

and

$$\nu(\omega R_o) = \nu_o(1 - \omega^2 R_o^2/2c^2 + 3\omega^4 R_o^4/8c^4) \tag{33}$$

respectively; and subtracting (32) from (33) gives

$$\nu_o(\omega^4 R_o^4/2c^4) \, . \tag{34}$$

Of course, a measurement to such an accuracy would be extremely difficult, but it could, perhaps simplier, be replaced by a direct search for the (lower–order) effect (28) concerning the radius of the rotating cylinder.

6 Concluding Remarks

It has first been shown, as an essential preliminary statement, that a *symmetric* theoretical comparison of the rates of just *two* clocks in relative motion is soundly conceivable in the framework of the Minkowskian geometry. This can be done by introducing a "covariant" Cartesian projection of the successive space–time positions of one clock onto the world–line of the other clock (starting from their meeting world–point). Such a projection has not only the ordinary effect of reducing the given space displacement to a null length, but also the complementary effect of accordingly reducing the related time interval to its *proper* length. As a result, a "covariant" synchrony may be defined for two relatively moving clocks, in line with the relativity–principle requirement of an *identical* flow rate of the proper time in every inertial system. The same unambiguous outcome cannot clearly be obtained under a mere Cartesian projection in ordinary space, because this would be a non–covariant operation that does not affect time instants and is unable to set up a *true* comparison of the proper–time rates of the two clocks.

It has further been pointed out that the new concept of "covariant" synchrony for two clocks in relative motion can actually give a peculiar "local" significance to the four–velocity of one clock *with respect to* the other clock, just in terms of the "local" (proper) time which the latter is reading (far away from the former). Such an argument has been extended to the limiting case of an infinitesimal world displacement performed by a clock in uniform circular motion around a clock at rest, and some fundamental conclusions about rotational kinematics have been drawn. What can essentially be gained is a more orthodox (and still relativistically consistent) geometrical definition of a rotating frame, in terms of a suitable "world" radial coordinate that may *naturally run to infinity*, with no need for values greater than c/ω to be ruled out. The new radial coordinate, R_o, differs from the standard one, R, by the following: it is identically equal to the *Euclidean* radius, $R\gamma(\omega R)$, of a circumference of proper length $2\pi R\gamma(\omega R)$ which is described in the rotating frame at an actual radial distance R from the rotation axis. A "new" metric should accordingly be assigned to a rotating frame, which can be obtained by just recasting the usual metric in terms of R_o. The most immediate physical application concerns the kinematics of a uniformly spinning disc (with presumable far–reaching effects on the physics of rotating black–holes). The result is that a disc of *whatever* (original) radius R_o might be brought to spin with an *arbitrarily great* uniform angular velocity ω: its shape should not undergo any distortion with spinning, but should appear to be *globally contracted* by a scale factor $\gamma^{-1}(\omega R)$, where $\omega R = \omega R_o(1 + \omega^2 R_o^2/c^2)^{-1/2}$ and R is the *new* radius that the disc should exhibit when it is seen rotating with an angular velocity ω.

The experimental viewpoint has been taken into account too, and it has been seen that the rotational red–shift as measured to an accuracy of order $\omega^2 R_o^2/c^2$ is unable yet to test this model of a uniformly spinning disc, an accuracy at least of order $\omega^4 R_o^4/c^4$ being required.

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