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On the Completeness of Some Subsystems of q-Deformed Coherent States

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Abstract. The von Neumann type subsystems of q-deformed coherent states are considered. The completeness of such subsystems is proved.

Introduction

The systems of coherent states related to Lie groups introduced in [Pe 1972], play the important role in many branches of theoretical and mathematical physics and pure mathematics [CS 1985], [Pe 1986].

The basic feature of such systems is that they are overcomplete, i.e. contain subsystems, which are themselves complete. The most interesting of them are subsystems related to discrete subgroups of Lie groups, the first of which were considered by von Neumann [Ne 1929], [Ne 1932]. The completeness properties of such system were investigated in [BBGK 1971] and [Pe 1971].

In the last few years q-deformed coherent states were introduced and some their properties were investigated (see, for example, [AC 1976], [Bi 1989], [Ma 1989], [Ju 1991]). Note that these states are related to q-deformed Lie algebras [Dr 1985], [Dr 1986], [Ji 1985], [Ji 1986], [FRT 1991].

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We investigate in the present paper the completeness properties of q-deformed coherent states for simplest q-deformed Lie algebras, namely for $w_q(1), su_q(2)$ and $su_q(1,1)$. It appears that some important properties of such systems are changed essentially after q-deformation.

1 System of Standard Coherent States

In this section we recall the basic properties of the system of standard coherent states. For more details see books [CS 1985], [Pe 1986].

The basic quantities are the creation and annihilation operators a^+ and a and the unit operator I, which act in the Hilbert space $\mathcal{H}$ and generate the Heisenberg–Weyl algebra:

$$[a, a^+] = aa^+ - a^+a = I, \quad [a, I] = [a^+, I] = 0. \tag{1.1}$$

The standard orthonormal basis $\{|n\rangle\}$, $n = 0, 1, \ldots$, in $\mathcal{H}$ is defined by

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle, \tag{1.2}$$

where $|0\rangle$ is the vacuum vector satisfying the condition

$$a\,|0\rangle = 0. \tag{1.3}$$

The operators a and a^+ act as follows

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle, \quad a^+\,|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle. \tag{1.4}$$

Let us introduce the operators

$$E(\alpha) = \exp(\overline{\alpha}a), \quad E^+(\alpha) = \exp(\alpha a^+), \quad \alpha \in \mathbf{C}. \tag{1.5}$$

Then the standard system of coherent states that are non-normalized, may be defined by the formula

$$||\alpha\rangle = E^+(\alpha)\,|0\rangle, \tag{1.6}$$

or

$$||\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \tag{1.7}$$

It is easy to see that coherent states are eigenstates of the annihilation operator

$$a\,||\alpha\rangle = \alpha\,||\alpha\rangle, \quad \alpha \in \mathbf{C}, \tag{1.8}$$

and we can calculate the norm of such state

$$\langle\alpha||\alpha\rangle = \sum_{m,n=0}^{\infty} \frac{\overline{\alpha}^m \alpha^n}{\sqrt{m!\,n!}} \langle m|n\rangle = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\sqrt{n!}} = \exp\left(|\alpha|^2\right). \tag{1.9}$$

Hence the normalized state $|\alpha\rangle$ has the form

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)||\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.10}$$

The coherent states are not orthogonal to one another. The scalar product of two such states has the form

$$\langle\alpha||\beta\rangle = \exp\left(\bar{\alpha}\beta\right). \tag{1.11}$$

We also have the"resolution of the unity"

$$\frac{1}{\pi}\int d^2\,\alpha\,|\alpha\rangle\,\langle\alpha| = \sum_{n=0}^{\infty} |n\rangle\,\langle n| = I, \tag{1.12}$$

from which it follows that the system of coherent states is complete.

This gives us the possibility to expand an arbitrary state $|\psi\rangle$ on the states $|\alpha\rangle$

$$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha\, c(\alpha)\,|\alpha\rangle, \quad c(\alpha) = \langle\alpha|\psi\rangle. \tag{1.13}$$

Note that if a coherent state $|\beta\rangle$ is taken as $|\psi\rangle$, Eq. (1.13) defines a linear dependence between different coherent states. It follows that the system of coherent states is overcomplete, i.e. it contains subsystems that are complete.

Using (1.10) we obtain the following expression for $\langle\alpha|\psi\rangle$ in (1.13):

$$\langle\alpha|\psi\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\psi(\overline{\alpha}), \tag{1.14}$$

where

$$\psi(\alpha) = \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}}\alpha^n, \quad c_n = \langle n|\psi\rangle. \tag{1.15}$$

At the same time, the inequality $|c_n| = |\langle n|\psi\rangle| \leq 1$ means that $\psi(\alpha)$ is an entire function of the complex variable α for the normalizing state $|\psi >$. We also have $|\langle\alpha|\psi\rangle| \leq 1$ and therefore have a bound on the growth of $\psi(\alpha)$:

$$|\psi(\alpha)| \leq \exp\left(\frac{|\alpha|^2}{2}\right). \tag{1.16}$$

The normalization condition may now be written as

$$I = \frac{1}{\pi}\int d^2\alpha \exp\left(-|\alpha|^2\right)|\psi(\alpha)|^2 = \langle\psi|\psi\rangle = 1. \tag{1.17}$$

The expansion of an arbitrary state $|\psi\rangle$ with respect to coherent states takes the form

$$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\psi(\overline{\alpha})\,|\alpha\rangle. \tag{1.18}$$

Thus, we have established a one-to-one correspondence between the vectors $|\psi\rangle$ of the Hilbert space and the entire functions $\psi(\alpha)$, for which the integral (1.17) is finite. This correspondence is established by Eqs. (1.15) and (1.18).

2 System of Coherent States for q-Deformed Heisenberg-Weyl Algebra

The generalization of the coherent states for q-deformed Heisenberg–Weyl algebra was given in the papers [AC 1976], [Bi 1989], [Ma 1989], [Ju 1991]. The corresponding formulae of the previous section should be modified.

Here the basic quantities as in the previous section are the creation and annihilation operators a^+ and a and unit operator I, which act in the Hilbert space $\mathcal{H}$ and satisfy the relations

$$[a, a^+] = aa^+ - qa^+a = I, \quad [a, I] = [a^+, I] = 0. \tag{2.1}$$

The orthonormal basis $|n\rangle$ in $\mathcal{H}$ is defined by

$$|n\rangle = \frac{(a^+)^n}{\sqrt{[n]!}}\,|0\rangle, \tag{2.2}$$

where

$$[n]! = [1]\cdot[2]\cdot\ldots\cdot[n], \quad [n] = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}, \tag{2.3}$$

and $|0\rangle$ is the vacuum vector satisfying the condition

$$a\,|0\rangle = 0. \tag{2.4}$$

The operators a and a^+ act here as

$$a\,|n\rangle = \sqrt{[n]}\,|n-1\rangle, \quad a^+\,|n\rangle = \sqrt{[n+1]}\,|n+1\rangle. \tag{2.5}$$

Let us introduce the operators

$$E(\alpha) = e_q(\overline{\alpha}a), \quad E^+(\alpha) = e_q(\alpha a^+), \quad \alpha \in \mathbf{C}, \tag{2.6}$$

where the function $e_q(x)$ is the generalization of the exponential function and is defined by the formula (see [Ex 1983] and [An 1986] for details)[2]

$$e_q(x) = \sum \frac{x^n}{[n]!}. \tag{2.7}$$

It is easy to see that this series converges at $|x| < R_q = (1-q)^{-1}$ (for all finite values of x at $|q| > 1$), and at $q \to 1$, $[n]! \to n!$. This function coincides with a standard exponent and satisfies the equation

$$\left(\frac{d}{dx}\right)_q e_q(x) = e_q(x), \tag{2.8}$$

where the q-derivative $(\frac{d}{dx})_q$ is defined by the formula

$$\left(\frac{d}{dx}\right)_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}, \tag{2.9}$$

so that $(\frac{d}{dx})_q \to (\frac{d}{dx})$ at the limit $q \to 1$.

By using (2.8) and (2.9) one can show that

$$e_q(x) = \frac{1}{\prod_{k=0}^{\infty}\Big(1 - q^k(1-q)x\Big)}. \tag{2.10}$$

So the $e_q(x)$ is the meromorphic function, which has no zeros and has simple poles at the points $x_k = q^{-k}/(1-q)$.

One can show [Ex 1983] that the inverse function $\Big(e_q(x)\Big)^{-1}$ (an entire function) is given by

$$\Big(e_q(x)\Big)^{-1} = e_{1/q}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}\, x^n}{[n]!} = \prod_{k=0}^{\infty}\Big(1 - q^k(1-q)x\Big). \tag{2.11}$$

Following [Bi 1989] and [Ma 1989] we now define the system of coherent states by the formula

$$||\alpha\rangle = E^+(\alpha)\,|0\rangle \tag{2.12}$$

or

$$||\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}\,|n\rangle. \tag{2.13}$$

It is easy to see that coherent states are eigenstates of the annihilation operator

$$a\,||\alpha\rangle = \alpha\,||\alpha\rangle, \quad \alpha \in \mathbf{C}, \tag{2.14}$$

[2] For simplicity, we restrict the consideration of the case $0 \le q \le 1$, used mainly in mathematical literature.

and we can calculate the norm of such states

$$\langle\alpha||\alpha\rangle = \sum_{m,n=0}^{\infty} \frac{\overline{\alpha}^m \alpha^n}{\sqrt{[m]!\,[n]!}} \langle m|n\rangle = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!} = e_q\left(|\alpha|^2\right). \tag{2.15}$$

Note that this series converges at

$$|\alpha|^2 < R_q^2 = (1-q)^{-1}. \tag{2.16}$$

Hence the normalized state $|\alpha\rangle$ has the form

$$|\alpha\rangle = \left(e_q(|\alpha|^2)\right)^{-1/2} ||\alpha\rangle = \left(e_q(|\alpha|^2)\right)^{-\frac{1}{2}} \sum \frac{\alpha^n}{\sqrt{[n]!}} |n\rangle, \quad |\alpha| < R_q. \tag{2.17}$$

The coherent states are not orthogonal to one another. The scalar product of two such states has the form

$$\langle\alpha||\beta\rangle = e_q(\overline{\alpha}\beta), \tag{2.18}$$

We also have the "resolution of the unity"

$$\frac{1}{\pi}\int_{D_q} d_q^2\alpha\,|\alpha\rangle\,\langle\alpha| = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{R_q^2} d_q(r^2)\,|\alpha\rangle\,\langle\alpha| = \sum_{n=0}^{\infty} |n\rangle\,\langle n| = I, \tag{2.19}$$

$$\alpha = re^{i\theta}, \quad D_q = \{\alpha : |\alpha| < R_q\}.$$

which follows from the formula

$$\int_0^{x_1} \left(e_q(x)\right)^{-1} x^n\, d_q x = [n]!, \tag{2.20}$$

where $x_1 = (1-q)^{-1}$ is the first zero of the entire function $\left(e_q(x)\right)^{-1}$ and the integral $\int_0^1 f(x)\,d_q x$ is the so-called Jackson integral [Ex 1983]:

$$\int_0^a f(x)\,d_q x = a(1-q)\sum_{k=0}^{\infty} q^k f(q^k a). \tag{2.21}$$

Note that from the"resolution of unity" (2.19) it follows that the system of coherent states is complete. This gives us the possibility to expand an arbitrary state $|\psi\rangle$ on the states $|\alpha\rangle$

$$|\psi\rangle = \frac{1}{\pi}\int d^2\,\alpha\,\langle\alpha|\psi\rangle\,|\alpha\rangle. \tag{2.22}$$

If a coherent state $|\beta\rangle$ is taken as $|\psi\rangle$, (2.22) defines a linear dependence between different coherent states. Therefore, the system of coherent states is overcomplete, i.e. contains subsystems, which are complete.

By using (2.17) we obtain an equation for $\langle\alpha|\psi\rangle$

$$\langle\alpha|\psi\rangle = \left(e_q(|\alpha|^2)\right)^{-1/2} \psi(\overline{\alpha}), \tag{2.23}$$

where

$$\psi(\alpha) = \sum \frac{c_n}{\sqrt{[n]!}}\, \alpha^n, \quad c_n = \langle n|\psi\rangle. \tag{2.24}$$

At the same time, the inequality $|c_n| = |\langle n|\psi\rangle| \leq 1$ means that $\psi(\alpha)$ for the normalizing state $|\psi\rangle$ is an analytical function of the complex variable α in the disc $D_q = \{\alpha||\alpha| < R_q\}$. We also have $|\langle\alpha|\psi\rangle| \leq 1$, and therefore a bound on the growth of $\psi(\alpha)$:

$$|\psi(\alpha)| \leq \left(e_q(|\alpha|^2)\right)^{\frac{1}{2}}. \tag{2.25}$$

We can now rewrite the normalization condition as

$$I = \frac{1}{\pi} \int_{D_q} d_q^2\alpha \left(e_q(|\alpha|^2)\right)^{-1} |\psi(\alpha)|^2 = \langle\psi|\psi\rangle = 1. \tag{2.26}$$

The expansion of an arbitrary state $|\psi\rangle$ with respect to coherent states takes the form

$$|\psi\rangle = \frac{1}{\pi} \int_{D_q} d_q^2\alpha \left(e_q(|\alpha|^2)\right)^{-1/2} \psi(\overline{\alpha})\, |\alpha\rangle. \tag{2.27}$$

Thus, we have established a one-to-one correspondence between the vectors $|\psi\rangle$ of the Hilbert space and the functions $\psi(\alpha)$ analytical in D_q, for which the integral (2.26) is finite. This correspondence is established by (2.23) and (2.27).

3 Completeness of Subsystems of q-Deformed Coherent States

As it was shown in the foregoing section, the system of q-deformed coherent states

$$\{|\alpha\rangle : \alpha \in D_q\}, \quad D_q = \{\alpha : |\alpha| \leq (1-q)^{-1/2}\} \tag{3.1}$$

is overcomplete, and hence there exist subsystems of coherent states that are complete ones. We describe these subsystems in this section.

Let us take some set of points $\{\alpha_k\}$ in the disc D_q and take the corresponding subsystem of coherent states $\{|\alpha_k\rangle\}$. Then if there exists a vector $|\psi\rangle$ of the Hilbert space $\mathcal{H}$, which is orthogonal to all states $\{|\alpha_k\rangle\}$:

$$\langle\alpha_k|\psi\rangle = 0, \tag{3.2}$$

then the system $\{|\alpha_k\rangle\}$ is incomplete. It is complete if such a vector does not exist.

We may reformulate this criterion in terms of the function

$$\psi(\alpha) = \langle\psi||\alpha\rangle = \sum_n \langle\psi|n\rangle \frac{\alpha^n}{\sqrt{[n]!}}, \tag{3.3}$$

which is analytic inside D_q, and is equal to zero at the points α_k

$$\psi(\alpha_k) = 0. \tag{3.4}$$

If such a function has a finite norm

$$||\psi||^2 = \int_{D_q} |\psi(\alpha)|^2 \left(e_q(|\alpha|^2)\right)^{-1} d_q^2\alpha < \infty, \tag{3.5}$$

then the system $\{|\alpha_k\rangle\}$ is incomplete. But if any such function has infinite norm $||\psi|| = \infty$, then such a system is complete.

Note that the function $\psi(\alpha)$, having the finite norm, should satisfy the condition

$$|\psi(\alpha)|^2 \left(e_q(|\alpha|^2)\right)^{-1} \leq |\psi(\alpha)|^2 \left(1 - |\alpha|^2(1-q)\right) \leq C, \quad \alpha \in D_q. \tag{3.6}$$

It follows from this condition that

$$\lim_{|\alpha|^2 \to (1-q)^{-1}} |\psi(\alpha)| \left(1 - |\alpha|^2(1-q)\right)^{\frac{1}{2}} < \infty. \tag{3.7}$$

We give the simple example of the complete subsystem of coherent states.

Let the set $\{\alpha_k\}$ have a limit point inside the disc D_q. The function that is analytic inside D_q and equal to zero at points α_k should be equal to zero identically. Hence this system of coherent states is complete.

For the future, it is convenient to introduce the new variable

$$\zeta = (1-q)^{1/2}\,\alpha. \tag{3.8}$$

So we may now consider the set of functions analytical inside the unit disc $D = \{\zeta\colon |\zeta| < 1\}$.

The characteristic property of $\psi(\zeta)$, related to the complete set $\{\zeta_k\}$, is that it has sufficiently many zeros inside $D_r = \{\zeta\colon |\zeta| < r\}$ and hence it sufficiently quickly grows at $|\zeta| \to 1$. So we may use some theorems from the theory of functions analytical inside the unit disk.

Let us give the theorem [Le 1964] that relates the growth of such function analytic in a disc with the distribution of its zeros.

Let $M(r)$ be the maximum modulus of $f(\zeta)$ on the circle $C_r = \{\zeta \colon |\zeta| = r\}$:

$$M_1(r) = \left[\frac{1}{2\pi}\int |f(re^{i\theta})|^2\, d\theta\right]^{\frac{1}{2}}, \tag{3.9}$$

and $n(r)$ be the number of zeros of $f(\zeta)$ in the disc $D = \{\zeta \colon |\zeta| < r\}$. We assume that the limit

$$\nu = \underline{\lim}_{r\to 1}\,(1-r^2)\,n(r) \tag{3.10}$$

exists and that $\nu \neq 0$. The numbers

$$\tau = \overline{\lim}_{r\to 1}\left[\ln M(r)/\ln\frac{1}{1-r^2}\right], \tag{3.11}$$

$$\tau_1 = \overline{\lim}_{r\to 1}\left[\ln M_1(r)/\ln\frac{1}{1-r^2}\right] \tag{3.12}$$

characterize the growth of $f(\zeta)$ at $|\zeta| \to 1$, and we call τ and τ_1 the generalized types of function $f(\zeta)$.

Note first of all the following relation between ν, τ and τ_1:

Theorem 3.1. *When $\nu > 0$, the following inequalities are true*

$$\tau \geq \frac{\nu}{2}, \quad \tau_1 \geq \frac{\nu}{2}. \tag{3.13}$$

Proof. Dividing $f(z)$ by az^n, if necessary, we obtain $\tilde{f}(z)$ with $\tilde{f}(0) = 1$. We use the Jensen formula

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|\tilde{f}(re^{i\theta})|\, d\theta = \int_0^r \frac{n(t)}{t}\, dt, \tag{3.14}$$

from which

$$\ln M(r) \geq \int_0^r \frac{n(t)}{t} dt. \tag{3.15}$$

On the another hand, from the generalized inequality between the arithmetic and geometric means

$$\ln M_1(r) = \frac{1}{2}\ln\left[\frac{1}{2\pi}\int_0^{2\pi} |\tilde{f}(re^{i\theta})|^2\, d\theta\right] \geq \frac{1}{2\pi}\int_0^{2\pi} \ln|\tilde{f}(re^{i\theta})|\, d\theta = \int_0^r \frac{n(t)}{t}\, dt. \tag{3.16}$$

It follows from the definitions of ν, τ and τ_1, that whatever the numbers $\varepsilon > 0$, $\varepsilon_1 > 0$ and $\delta > 0$, there exists the such number $r_0 < 1$ that

$$\ln M(r) \leq (\tau + \varepsilon)\ln\frac{1}{1-r^2}, \quad \ln M_1(r) \leq (\tau_1 + \varepsilon_1)\ln\frac{1}{1-r^2}, \tag{3.17}$$

$$n(r) \geq \frac{\nu - \delta}{1-r^2}\, r^2,$$

when $r > r_1$. We may now rewrite (3.15) and (3.16) as

$$(\tau+\varepsilon)\ \ln\frac{1}{1-r^2} \geq \ln M(r) \geq \int_0^{r_0}\frac{n(t)}{t}\,dt + \frac{\nu-\delta}{2}\left[\ln\frac{1}{1-r^2} - \ln\frac{1}{1-r_0^2}\right], \tag{3.18}$$

$$(\tau_1+\varepsilon_1)\ \ln\frac{1}{1-r^2} \geq \ln M_1(r) \geq \int_0^{r_0}\frac{n(t)}{t}\,dt + \frac{\nu-\delta}{2}\left[\ln\frac{1}{1-r^2} - \ln\frac{1}{1-r_0^2}\right]. \tag{3.19}$$

Considering the limit $r \to 1$ in (3.18) and (3.19) with $\nu > 0$, we arrive at (3.13).

The criterion of completeness of the subsystem of q-deformed coherent states follows from this theorem and from inequality (3.6).

Theorem 3.2. *The system of q-deformed coherent states $\{|\alpha_k\rangle\}$ is complete, if the limit*

$$\nu = \underline{\lim}_{r\to 1}\,(1-r^2)\,n(r) \tag{3.20}$$

exists and if $\nu > 1$. Here $n(r)$ is the number of points α_k inside the disc $D_r = \{\zeta\colon |\zeta| < r\}$.

In order to construct the examples of complete subsystems it is useful to consider the unit disc D as a Lobachevsky plane with standard measure

$$d\mu(\zeta) = \frac{d^2\zeta}{(1-|\zeta|^2)^2},$$

on which the group $G = SU(1,1)/\mathbb{Z}_2$ acts transitively. The simplest subsystems $\{|\alpha_k\rangle\}$ are related to the discrete subgroups Γ of the group G.

Let $\Gamma = \{\gamma_n\}$ and α_0 be any point of D.

Definition 3.3. *The set of states $\{|\alpha_k\rangle\}$, where $\alpha_k = \gamma_k\cdot\alpha_0$, is called the subsystem of coherent states related to subgroup Γ.*

Theorem 3.4. *The system of q-deformed coherent states related to the discrete subgroup Γ of the group $G = SU(1,1)/\mathbb{Z}_2$ is incomplete if the area S_Γ of the fundamental domain $\Gamma\backslash D$ is infinite.*

Proof. In this case one may show (see for example [Le 1964]) that there exists a function $f(\zeta)$, which is analytic and bounded in D, that has zeros at the points ζ_k. For such a function the norm defined by Eq. (3.5) is finite and hence this system of coherent states is incomplete.

Theorem 3.5. *Let the system of q-deformed coherent states $\{|\alpha_k\rangle\}$ be related to the discrete subgroup Γ, such that the area S_Γ of the fundamental domain $\Gamma\backslash D$ is finite and $S_\Gamma < \pi$. Then the system $\{|\alpha_k\rangle\}$ is complete.*

Proof. Let us remind that non-Euclidian area of the disc of radius r is equal to $S(r) = \pi r^2/(1-r^2)$. In this case, it follows from the condition $S_\Gamma < \pi$ that $\nu = \pi/S_\Gamma > 1$. Hence the norm of any analytic function, which has the zeros in the points $\{\alpha_k\}$, is infinite, and the system $\{|\alpha_k\rangle\}$ is complete.

Let us try to list the discrete subgroups Γ for which $S_\Gamma < \pi$. To this end, we need the information from the theory of discrete subgroups Γ of the group $SU(1,1)/\mathbb{Z}_2$, which we take from [Le 1964]. Let us restrict ourselves to consideration of groups with finite area of the fundamental domain $\Gamma\backslash D$. It is known that in this case the fundamental domain has the form of a polygon with an even number of sides $2n$. These sides being divided into pairs, are equivalent with respect to the action of transformations of the group Γ. The vertices of the polygon are joined in the cycles of vertices, which are equivalent to one another. With this, the sum of the angles of the polygon at the vertices of a given cycle equals to $2\pi/l$, where l is either a positive integer or ∞. If $l = 1$, the cycle is called random. If $l = \infty$, the vertices of the cycle lie on the boundary of the domain D, and the cycle is called parabolic, while in all the other cases, the cycle is called elliptic and l is called the order of the cycle. Let c be the number of cycles. By identifying equivalent sides and vertices, we obtain a Riemann surface. The genus p of this surface may be found by the formula

$$2\pi = 1 + n - c. \tag{3.21}$$

We call the set of numbers $(p, c;\ l_1,\ l_2, \ldots,\ l_c)$ the signature of the group Γ. We would like to mention that the area of the fundamental domain S_Γ is completely determined by the signature of the group and, for our choice of invariant measure $d\mu(\zeta) = (1 - |\zeta|^2)^{-2} d\xi d\eta$, is given by

$$S_\Gamma = \pi \left[p - 1 + \frac{1}{2} \sum_{j=1}^{c} \left(1 - \frac{1}{l_j}\right)\right]. \tag{3.22}$$

From (3.22) it is easy to see that the value of S_Γ cannot be arbitrarily close to zero. It may be shown [Si 1945] that the minimal value of $S_\Gamma = \frac{\pi}{84}$ corresponds to the group Γ with the signature (0, 3; 2, 3, 7). If the fundamental domain is not compact, i.e. the group Γ contains parabolic elements, then $S_\Gamma \geq \frac{\pi}{12}$; $S_\Gamma = \frac{\pi}{12}$ correspond to the modular group $\Gamma = (0, 3;\ 2, 3, \infty)$. It is known also that when $p \geq 2$, the signature of Γ may be arbitrary. For $p = 1$ the condition $c \geq 1$ should be satisfied, and for $p = 0$ we should have either $c \geq 5$, or $c = 4$ and $\sum l_j^{-1} < 2$, or $c = 3$ and $\sum l_j^{-1} < 1$.

We are interested here in the case

$$\frac{S_\Gamma}{\pi} = \left[p - 1 + \frac{1}{2} \sum_{j=1}^{c} \left(1 - \frac{1}{l_j}\right)\right] < 1. \tag{3.23}$$

As it will shown below, the number of such cases is finite.

Let us consider separately the different cases:

I. Let $p \geq 2$, then (3.23) cannot be satisfied.

II. Let $p = 1$, then (3.23) takes the form

$$\sum_{j=1}^{c} \frac{1}{l_j} > c - 2, \quad c \geq 1; \quad n = c + 1. \tag{3.24}$$

Hence here we may have:

a)

$$c = 1; \quad l_1 = 2, 3, \ldots, \infty, \tag{3.25}$$

b)

$$c = 2; \quad l_1 = 2, 3, \ldots, \infty, \quad l_2 = 2, 3, \ldots, \infty, \tag{3.26}$$

except of the case $l_1 = \infty$, $l_2 = \infty$,

c)

$$c = 3; \quad \sum_{j=1}^{3} \frac{1}{l_j} > 1, \tag{3.27}$$

c_1)

$$c = 3; \quad l_1 = 2,\ l_2 = 2,\ l_3 = 2, 3, \ldots, < \infty, \tag{3.28}$$

c_2)

$$c = 3; \quad (l_1, l_2, l_3) = (2,3,3),\ (2,3,4),\ (2,3,5). \tag{3.29}$$

III. Let

a)

$$p = 0, \quad \sum_{j=1}^{c} \frac{1}{l_j} > c - 4, \tag{3.30}$$

a_1)

$$c = 5, \quad \sum_{1}^{5} \frac{1}{l_j} > 1, \tag{3.31}$$

a_2)

$$c = 6, \quad \sum_{1}^{6} \frac{1}{l_j} > 2, \tag{3.32}$$

a_3)

$$c = 7, \quad \sum_{1}^{7} \frac{1}{l_j} > 3, \tag{3.33}$$

a_4)

$$c = 8, \quad \sum_{1}^{8} \frac{1}{l_j} > 4. \tag{3.34}$$

This case and also the case $c > \infty$ are impossible.

b)

$$c = 4, \quad \sum_{1}^{4} \frac{1}{l_j} > 0, \quad \sum_{j=1}^{4} \frac{1}{l_j} < 2, \tag{3.35}$$

c)

$$c = 3, \quad \sum_{1}^{3} \frac{1}{l_j} > -1, \quad \sum_{j=1}^{3} \frac{1}{l_j} < 1. \tag{3.36}$$

So, as a function having zeros at the points $\zeta_n = \gamma_n \cdot \zeta_0$, we may take the automorphic form related to discrete subgroup $\Gamma = \{\gamma_n\}$. If the fundamental domain $\Gamma\backslash D$ has finite area, we may take it as polygon with finite number of sides which are segments of geodesics. Vertices of a polygon lying on the boundary of disc are called parabolic vertices. We denote $\mathcal{P}$ the set of parabolic vertices, and $D^+ = D \bigcup \mathcal{P}$. Now we are ready to give the definition of the automorphic form.

Definition. *An automorphic form of weight m (m is integer) is a function $f_m(z)$ that is analytic in D, satisfies the functional equation*

$$f_m(\zeta \cdot \gamma_n) = (\beta_n z + \overline{\alpha}_n)^{2m} f_m(z), \quad \gamma_n = \begin{pmatrix} \alpha_n & \beta_n \\ \overline{\beta}_n & \overline{\alpha}_n \end{pmatrix} \in \Gamma$$

and is regular in D^+ (this means that at each parabolic vertex ζ_p of the domain $\Gamma\backslash D$, there should exist $\lim(z - z_p)^{2m} f_m(z)$ at $z \to z_p$, in the interior of domain $\Gamma \backslash D$). An automorphic form $f_m(z)$ is called parabolic if $f_m(z)$ vanishes at all parabolic vertices.

The set of automorphic forms of weight m builds a finite-dimensional vector space. We denote $d_m(\Gamma)$ ($d_m^+(\Gamma)$, correspondently) the dimension of the space of automorphic forms (the space of parabolic forms, correspondently). Let $m_0(m_0^+)$ be the least m for which $d_m(\Gamma) \geq 2$ ($d_m^+(\Gamma) \geq 2$, correspondently). It is known (see for example [Le 1964]) that if $\Gamma\backslash D$ is compact, then $d_m(\Gamma) = d_m^+(\Gamma)$, $m_0 = m_0^+$, and any automorphic form may be considered as parabolic one.

The dimension of the space of automorphic forms of weight m is given by

$$d_m(\Gamma) = \begin{cases} 0, & \text{for } m < 0, \\ 1, & \text{for } m = 0, \\ g_1, & \text{for } m = 1, \\ (2m-1)(p-1) + \sum_{j=1}^{c} \left[m\left(1 - \frac{1}{l_j}\right)\right], & \text{for } m \geq 2. \end{cases} \tag{3.37}$$

Here, p is the genus of the fundamental domain, $[m]$ is the integer part of the number m, and $g_1 \geq p$ is the number of holomorphic differentials on the Riemann surface $\Gamma\backslash D$.

With this, the number of zeros of function $f_m(z)$ in the interior of fundamental domain is given by the Poincaré formula [Po 1882] (written here in a somewhat different form)

$$N = 2mS_\Gamma/\pi. \tag{3.38}$$

It should be mention that if there are elliptic and parabolic vertices, this number need not be integer.

Further, from a comparison of (3.22) and (3.37) we found that

$$N \geq d_m + p - 1, \tag{3.39}$$

with the equality sign holding only in case when the numbers m/l_i are integers, including zero.

In what follows we shall be interested in automorphic forms for which $d_m(\Gamma) \geq 2$. We denote by m_0 the minimal weight of such forms. We consider now the values which m_0 may take.

I. If $p \geq 2$ then $m_0 = 1$, as it is evident from formula (3.37).

II. Let $p = 1$ and let c_2 be the number of parabolic cycles. Then, if

- **a)** $c_2 \geq 2$, then $m_0 = 1$,
- **b)** $c_2 = 1$, then $m_0 = 2$,
- **c)** $c_2 = 0$ and $\Gamma = (1, 1;\ 2)$, then $m_0 = 4$,
- **d)** $c_2 = 0$ and $\Gamma = (1, 1;\ l)$, $l > 3$, then $m_0 = 3$,
- **e)** $c_2 = 0, c \geq 2$, then $m_0 = 2$.

III. If $p = 0$ then

$$m_0 \geq m_1 = \frac{\pi}{2S_\Gamma} = \left[\sum_{i=1}^{c}\left(1 - \frac{1}{l_i}\right) - 2\right]^{-1} = \left[c - 2 - \sum_{i=1}^{c}\frac{1}{l_i}\right]^{-1}. \tag{3.40}$$

Let us introduce the notation:

$$N_0 = 2m_0\, S_\Gamma/\pi.$$

It follows from (3.39) that $N_0 \geq p + 1$.Therefore, N_0 can be equal to one only in the case of $p = 0$.

With this, $d_{m_0} = 2$, and the value of m_0 may be determined from (3.38):

$$m_0 = \frac{\pi}{2S_\Gamma} = \left[\sum_{i=1}^{c}\left(1 - \frac{1}{l_i}\right) - 2\right]^{-1}. \tag{3.41}$$

Let l be the least common multiple of the numbers l_j which are not infinite. Then, (3.41) can be written in the form:

$$m_0 = l\left[\sum_i\left(l - \frac{l}{l_i}\right) - 2l\right]^{-1},$$

from which it follows that $m_0 \leq 1$. However, m_0 must be divisible by those of l_i which are not infinite. Therefore, they must coincide with l, i.e., $m_0 = l$.

Thus we have

Proposition. If group Γ of signature $(0, c;\ l_1 \ldots l_{c_1}, \infty, \ldots \infty)$ admits automorphic form $f_{m_0}(z)$ with one zero in the fundamental domain, then m_0 is the least common multiple of the numbers $l_1, l_2, \ldots, l_{c_1}$ and, moreover, must satisfy the condition

$$m_0(c - 2) - \sum_{i=1}^{c_1}\frac{m_0}{l_i} = 1. \tag{3.42}$$

It is not difficult to show that (3.42) has a solution only for $c = 3, 4, 5$, and that the number of solutions of this equation is finite. There are 21 discrete subgroups Γ corresponding to them. All of them are listed in Table I.

TABLE I

m_0	Γ
1	$(0{,}3;\infty,\infty,\infty)$
2	$(0{,}3;2{,}\infty,\infty), (0{,}4;2{,}2{,}2{,}\infty), (0{,}5;\ 2{,}2{,}2{,}2{,}2)$
3	$(0{,}3;3{,}3{,}\infty)$
4	$(0{,}3;4{,}4{,}4), (0{,}3;2{,}4{,}\infty), (0{,}4;2{,}2{,}2{,}4)$
6	$(0{,}3;2{,}3{,}\infty), (0{,}3;3{,}3{,}6), (0{,}3;2{,}6{,}6),\ (0{,}4;2{,}2{,}2{,}3)$
8	$(0{,}3;2{,}4{,}8)$
10	$(0{,}3;2{,}5{,}5)$
12	$(0{,}3;3{,}3{,}4), (0{,}3;2{,}3{,}12), (0{,}3;2{,}4{,}6)$
18	$(0{,}3;2{,}3{,}9)$
20	$(0{,}3;2{,}4{,}5)$
24	$(0{,}3;2{,}3{,}8)$
42	$(0{,}3;2{,}3{,}7)$

4 Case of Quantum Algebra $su_q(2)$

In this section we consider the basic properties of the system of q-coherent states (see [AC 1976], [Bi 1989], [Ma 1989], [Ju 1991] for other details).

The basic quantities here are the operators $J_\pm$ and J_0, which act in the Hilbert space $\mathcal{H}$ of finite dimension $2j+1$ (j is half-integer, $2j+1$ is a positive integer) with the basis

$$|j,\mu\rangle, \quad \mu = -j, -j+1, \ldots, j, \tag{4.1}$$

or

$$|n\rangle, \quad n = j + \mu, \quad n = 0, 1, \ldots, 2j. \tag{4.2}$$

The operators $J_\pm$ and J_0 act as follows

$$J_\pm\,|j,\mu\rangle = \sqrt{[j \mp \mu]\,[j \pm \mu + 1]}\,|j, \mu \pm 1\rangle, \tag{4.3}$$

$$J_0\,|j,\mu\rangle = \mu\,|j,\mu\rangle, \tag{4.4}$$

or

$$J_+\,|n\rangle = \sqrt{[n+1]\,[2j-n]}\,|n+1\rangle, \tag{4.5}$$

$$J_-\,|n\rangle = \sqrt{[n]\,[2j-n+1]}\,|n-1\rangle, \tag{4.6}$$

$$J_0\,|n\rangle = (n-j)\,|n\rangle. \tag{4.7}$$

Here $[n]$ is the Gauss symbol [Ga 1808]

$$[j,n] = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}, \quad [n]! = [1]\,[2]\ldots[n]. \tag{4.8}$$

>From (4.5) it is not difficult to obtain

$$|n\rangle = \sqrt{\frac{[2j-n]!}{[n]!\,[2j]!}}\,(J_+)^n\,|0\rangle. \tag{4.9}$$

>From (4.5)–(4.7) it follows that the operators $J_\pm$ and J_0 satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0], \tag{4.10}$$

where the operator $[2J_0]$ is defined by the formula

$$[2J_0]\,|n\rangle = \lambda_n\,|n\rangle, \tag{4.11}$$

$$\lambda_n = \Big([j+\mu] - [j-\mu]\Big) = \Big([n] - [2j-n]\Big) = \begin{cases} q^{j-\mu}\,[2\mu], & \mu \geq 0; \\ -q^{j+\mu}\,[-2\mu], & \mu \leq 0. \end{cases} \tag{4.12}$$

Now following [Bi 1989] and [Ma 1989] we define the system of q-deformed coherent states by the formula

$$||z\rangle = e_q(zJ_+)\,|0\rangle. \tag{4.13}$$

>From (4.9) we have

$$||z\rangle = \sum_{n=0}^{2j} \sqrt{\frac{[2j]!}{[n]!\,[2j-n]!}}\,z^n\,|n\rangle \tag{4.14}$$

and we may calculate the norm of this state

$$\langle z||z\rangle = G_{2j}(|z|^2) = \sum_{n=0}^{2j} \frac{[2j]!}{[n]!\,[2j-n]!}\,|z|^2 = [1+|z|^2]^{(2j)}. \tag{4.15}$$

Here $G_{2j}(x)$ is a certain polynomial of degree $2j$. Let us give the simplest examples:

$$G_0 = 1, \quad G_1 = 1 + x, \quad G_2 = 1 + [2]x + x^2 = 1 + (1+q)x + x^2;$$

$$G_3 = 1 + [3]x + [3]x^2 + x^3 = (1+x)\Big(1 + ([3]-1)x + x^2\Big), \ldots \tag{4.16}$$

Note that these polynomials were first considered by Gauss [Ga 1808] and investigated in more detail in the paper by Szegö [Sz 1926]. Here we note the following important properties of these polynomials:

i) Their roots are located on the circle of unit radius and $x = 1$ is not a root.

ii) The relation of these polynomials to theta–functions [Sz 1926]. Namely the functions

$$\Phi_0 = 1, \ldots, \quad \Phi_n = \frac{(-1)^n q^{n/2}}{\sqrt{(1-q)(1-q^2)\ldots(1-q^n)}} G_n\left(-q^{-1/2}z\right)$$

are orthogonal on the unit circle $\{z\colon z = e^{i\theta}\}$ with the weight function $f(\theta)$, which coincides with theta–function

$$\int_0^{2\pi} \overline{\Phi}_j(\theta)\, \Phi_k(\theta)\, f(\theta)\, d\theta = 0, \quad j \neq k,$$

$$f(\theta) = \sum_{n=-\infty}^{\infty} q^{n^2/2} e^{in\theta} = \sum_{n=-\infty}^{\infty} q^{n^2/2} \cos n\theta = \left|D(e^{i\theta})\right|^2,$$

$$D(z) = \prod_{n=1}^{\infty} \sqrt{1-q^n}\left(1 + q^{(2n-1)/2}z\right).$$

Note also that we have

$$G_n\,(q^{1/2}) = \prod_{\nu=1}^{n} (1 + q^{\nu/2}),$$

the expression for the generating function for $G_n(x)$,

$$\sum_{n=0}^{\infty} \frac{G_n(x)}{(1-q)(1-q^2)\ldots(1-q^n)} t^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^n t)(1-q^n t x)},$$

and the recurrence formulae

$$G_{n+1}(x) = (1+x)\, G_n(x) - (1-q^n)\, x G_{n-1}(x);$$

$$G_n(qx) - (1-q^n)\, G_{n-1}(qx) = q^n G_n(x).$$

Let us denote the roots as $\zeta_1, \ldots, \zeta_{2j}$. Then $|\zeta_k| = 1$ and $\overline{\zeta}_k$ is also the root as ζ_k. So

$$G_n(x) = \prod_{j=1}^{n} (x - \zeta_j).$$

The normalized coherent states now take the form

$$|z\rangle = \left(G_{2j}(|z|^2)\right)^{-1/2} \sum_{n=0}^{2j} \sqrt{\frac{[2j]!}{[n]!\,[2j-n]!}}\, z^n\, |n\rangle, \tag{4.17}$$

and the scalar product of two such states is

$$\langle w|z\rangle = \frac{G_{2j}(\overline{w}z)}{\left(G_{2j}(|z|^2)\, G_{2j}(|w|^2)\right)^{1/2}}. \tag{4.18}$$

So for the fixed coherent state $|z\rangle$ there are $2j$ coherent states $|w_k\rangle$, $k = 1, \ldots, 2j$, which are orthogonal to state $|z\rangle$. Here

$$w_k = (\bar{z})^{-1}\bar{\zeta}_k. \tag{4.19}$$

As for the standard system of coherent states, for q-coherent states we also have the resolution of unity

$$\int ||z\rangle\,\langle z||\, d_q\mu(z) = I, \tag{4.20}$$

$$d_q\mu(z) = \frac{[2j+1]}{2\pi}\Big(G_{2j+2}(|z|^2)\Big)^{-1} d_q(|z|^2)\,d\theta, \quad z = |z|e^{i\theta}. \tag{4.21}$$

To prove this, let us consider the integral

$$I_{n,l} = \int_0^\infty x^n \Big(G_l(x)\Big)^{-1} d_q x. \tag{4.22}$$

Then after an integration by parts [Ex 1983] we have

$$I_{n,l} = \frac{q^{-n}\,[n]}{[l-1]}\int_0^\infty x^{n-1}\Big(G_{l-1}(q^{-1}x)\Big)^{-1} dx, \tag{4.23}$$

and hence

$$I_{n,l} = \frac{[n]!}{[l-1]\,[l-2]\ldots[l-n]}. \tag{4.24}$$

Furthermore

$$\int_0^\infty \Big(G_{l-n}(1+q^{-n}x)\Big)^{-1} d_q x = \frac{q^n}{[l-n-1]}, \tag{4.25}$$

and finally

$$\int_0^\infty x^n \Big(G_l(x)\Big)^{-1} d_q x = \frac{[n]!\,[l-n-2]!}{[l-1]!}. \tag{4.26}$$

As a result of resolution of unity, an arbitrary vector $|\psi\rangle$ may be represented by a polynomial of degree $2j$:

$$\psi(\bar{z}) = \langle z|\psi\rangle. \tag{4.27}$$

Finally we come to the functional realization of the Hilbert space $\mathcal{F}_j$

$$\langle\psi_1|\psi_2\rangle = \int \overline{\psi_1(z)}\,\psi_2(z)\,d_q\mu(z) \tag{4.28}$$

and have the basis:

$$f_n(\bar{z}) = \langle z||n\rangle = \sqrt{\frac{[2j]!}{[n]!\,[2j-n]!}}\,\bar{z}^n. \tag{4.29}$$

It is easy to see that any set of (2j + 1) coherent states form nonorthogonal basis in $\mathcal{F}_j$.

5 Case of Quantum Qlgebra $su_q(1,1)$

In this section we consider the basic properties of the system of q-coherent states for discrete series T_k^+ (see [Bi 1989], [Ma 1989], [Ju 1991] for other details).

The basic quantities here are the operators $K_\pm$ and K_0, which act in the infinite-dimensional Hilbert space $\mathcal{H}$ with the basis

$$\{|k,\,\mu\rangle\}, \quad \mu = k,\, k+1,\, \ldots, \tag{5.1}$$

or

$$\{|n\rangle\}, \quad n = \mu - k,\, n = 0,\, 1,\, \ldots. \tag{5.2}$$

The operators $K_\pm$ and K_0 act as follows

$$K_\pm\,|k,\mu\rangle = \sqrt{[\mu \pm k]\,[\mu \mp k \pm 1]}\,|k,\,\mu \pm 1\rangle, \tag{5.3}$$

$$K_0\,|k,\mu\rangle = \mu\,|k,\mu\rangle \tag{5.4}$$

or

$$K_+|n\rangle = \sqrt{[n+1]\,[2k+n]}\,|n+1\rangle, \tag{5.5}$$

$$K_-|n\rangle = \sqrt{[n]\,[2k+n-1]}\,|n-1\rangle, \tag{5.6}$$

$$K_0\,|n\rangle = (k+n)\,|n\rangle. \tag{5.7}$$

Here $[n]$ is the Gauss symbol [Ga 1808]

$$[n] = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}, \quad [n]! = [1]\,[2]\,\ldots[n]. \tag{5.8}$$

>From (5.5) it is not difficult to obtain

$$|n\rangle = \sqrt{\frac{[2k]!}{[n]!\,[2k+n-1]!}}\,(K_+)^n\,|0\rangle. \tag{5.9}$$

>From (5.5)–(5.7) it follows that the operators $K_\pm$ and K_0 satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = [2K_0], \tag{5.10}$$

where the operator $[2K_0]$ is defined by the formula

$$[2K_0]\,|n\rangle = \lambda_n\,|n\rangle, \tag{5.11}$$

$$\lambda_n = \Big([\mu+k] + [\mu-k]\Big) = \Big([n] + [2k+n]\Big) = \begin{cases} q^{k-\mu}\,[2\mu], & \mu \geq 0; \\ -q^{k+\mu}\,[-2\mu], & \mu \leq 0. \end{cases} \tag{5.12}$$

Now following [Bi 1989] and [Ma 1989] we define the system of q-coherent states by the formula

$$||z\rangle = e_q\,(zK_+)\,|0\rangle. \tag{5.13}$$

>From (5.9) we have

$$\| z\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{[2k]!}{[n]!\,[2k+n]!}}\, z^n\, |n\rangle \tag{5.14}$$

and may calculate the norm of this state

$$\langle z\|z\rangle = F_{2k}\left(|z|^2\right) = \sum_{n=0}^{\infty} \frac{[2k+n-1]!}{[n]!\,[2k]!}\,|z|^{2n} = \left(1-|z|^2\right)^{-(2k)}. \tag{5.15}$$

Here $F_{2k}(x)$ is the function of degree $(-2k)$:

$$F_{2k}(x) = G_{2k}^{-1}(-x).$$

Let us give the simplest examples:

$$F_0 = 1, \quad F_1 = (1-x)^{-1}, \quad F_2 = \left(1-[2]x+x^2\right)^{-1} = \left(1-(1+q)x+x^2\right)^{-1};$$

$$F_3 = \left(1-[3]x+[3]x^2-x^3\right)^{-1} = \left((1-x)\left(1-([3]-1)x+x^2\right)\right)^{-1} \ldots \tag{5.16}$$

Here we only note that the poles of these functions are located on the circle of unit radius and that, at integer k, $x=1$ is a pole.

Let us denote the poles as $\zeta_1,\ldots,\zeta_{2k}$. Then $|\zeta_k|=1$ and $\overline{\zeta}_k$ is the pole too as ζ_k. So the normalized coherent states have the form

$$|z\rangle = \left(F_{2k}(|z|^2)\right)^{-1/2} \sum_{n=0}^{\infty} \sqrt{\frac{[2k]!}{[n]!\,[2k-n]!}}\, z^n\, |n\rangle, \tag{5.17}$$

and the scalar product of two such states is

$$\langle w|z\rangle = \frac{F_{2k}\left(\overline{w}z\right)}{\left(F_{2k}(|z|^2)\,F_{2k}(|w|^2)\right)^{1/2}}. \tag{5.18}$$

As for the standard system of coherent states for q-coherent states we also have the resolution of unity

$$\int \|z\rangle\,\langle z\|\,d_q\mu(z) = I, \tag{5.19}$$

$$d_q\mu(z) = \frac{[2k-1]}{2\pi}\left(F_{2k+2}(|z|^2)\right)^{-1} d_q(|z|^2)\,d\theta, \quad z = |z|\,e^{i\theta}. \tag{5.20}$$

To prove this, let us consider the integral

$$I_{n,l} = \int_0^1 x^n \left(F_l(x)\right)^{-1} d_qx. \tag{5.21}$$

Then after an integration by parts [Ex 1983] we have

$$I_{n,l} = \frac{q^{-n}[n]}{[l-1]} \int_0^1 x^{n-1}\left(F_{l-1}(q^{-1}x)\right)^{-1} dx \tag{5.22}$$

and hence

$$I_{n,l} = \frac{[n]!}{[l-1]\,[l-2]\,\ldots[l-n]}. \tag{5.23}$$

Furthermore

$$\int_0^1 \left(F_{l-n}(1+q^{-n}x)\right)^{-1} d_qx = \frac{q^n}{[l-n-1]} \tag{5.24}$$

and finally

$$\int_0^1 x^n \left(F_l(x)\right)^{-1} d_qx = \frac{[n]!\,[l-n-2]!}{[l-1]!}. \tag{5.25}$$

As a result of resolution of unity, an arbitrary vector $|\psi\rangle$ may be represented by a function of degree $2k$:

$$\psi(\overline{z}) = \langle z||\psi\rangle. \tag{5.26}$$

And we finally come to the functional realization of the Hilbert space $\mathcal{F}_k$:

$$\langle\psi_1|\psi_2\rangle = \int \overline{\psi_1(z)}\,\psi_2(z)\,d_q\mu(z) \tag{5.27}$$

and we have the basis

$$f_n(\bar{z}) = \langle z||n\rangle = \sqrt{\frac{[2k]!}{[n]!\,[2k-n]!}}\,\overline{z}^n. \tag{5.28}$$

So all formulae here are similar to the corresponding formulae for the case of Heisenberg-Weyl algebra. Comparing, for example the basic formulae (2.15) and (5.15) we can see that the case of Heisenberg-Weyl algebra is similar to the case of su (1,1) for $k = \frac{1}{2}$. So, for $su_q(1,1)$ algebra we have the results analogous to results of section 2.

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