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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **68 (1995)**

Heft 6

PDF erstellt am: **28.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-116752

Helv Phys Acta
Vol. 68 (1995)

0018-0238/95/060531-08$1.50+0.20/0
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Generic Simplicity of Resonances

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(26.V.1995, revised 10.IX.1995)

Abstract. We prove that a generic potential perturbation in Euclidean scattering splits the multiplicities of all resonances. Our argument can in fact be generalized to a class of non-self-adjoint Fredholm operators on an abstract Hilbert space.

1. Introduction

The purpose of this note is to show that for a generic compactly supported perturbation of the Laplacian in $\mathbb{R}^n$ the resonances are simple. The argument presented here shows in fact that for any perturbation for which the resonances are defined by complex scaling the algebraic multiplicities can be split by adding a generic compactly supported potential to the perturbation. As was pointed out to us by the referee the argument we use is more general and applies to some families of non-self-adjoint Fredholm operators of index zero – see Remark 3.2†.

Results of this type are now well-known for eigenvalues ([11, 12]) and the minor new difficulties here come from dealing with non-self adjoint operators. We were motivated by

†We would like to thank the referee for clarifying our earlier argument and for suggesting the generalized formulation of the result. The second author is also grateful to the National Science Foundation for partial support.

the fact that many statements about resonances are easy in the case of no multiplicity but become complicated in general. For instance, the correspondence between the poles of the meromorphic continuation of the resolvent and the scattering matrix for compactly supported perturbations in odd dimensions is now standard when the poles are simple (see [3]). The generic simplicity combined with the continuity of resonances in compact sets [8], shows that this correspondence persists in general. Similarly one obtains that the poles of the scattering matrix agree with multiplicities with the poles of its determinant (in the hyperbolic case where complex scaling cannot be globally used a direct argument, applicable in the Euclidean case as well, is presented in [2]).

We should stress that the above applications are probably known but we are not aware of a convenient reference. The variational formula for the resonance used here (see (3.2) below) is also probably well-known. In deriving it we were motivated by a formula of LaVita for a variation of an eigenvalue of non-self adjoint operator which we learned from [4].

We will state the result in the abstract setting introduced in [10] (see [13] for more references). Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \backslash B(0, R_0))$$

where $R_0 \geq 0$ is fixed, $B(0,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$ and the corresponding orthogonal projections are denoted by $u \mapsto u\restriction_{B(0,R_0)}$, $u \mapsto u\restriction_{\mathbb{R}^n\backslash B(0,R_0)}$. The operator

$$P : \mathcal{H} \to \mathcal{H}$$

is unbounded self-adjoint with the domain $\mathcal{D} \subset \mathcal{H}$ which satisfies

$$\mathcal{D}\restriction_{\mathbb{R}^n\backslash B(0,R_0)} \subset H^2(\mathbb{R}^n \backslash B(0, R_0)),$$

and $\{u \in H^2(\mathbb{R}^n\backslash B(0,R_0)),\ u = 0 \text{ near } B(0,R_0)\} \subset \mathcal{D}$. The crucial assumptions are

$$(Pu)\restriction_{\mathbb{R}^n\backslash B(0,R_0)} = -\Delta(u\restriction_{\mathbb{R}^n\backslash B(0,R_0)}) \text{ for all } u \in \mathcal{D} \tag{1.1}$$

$$\mathbf{1}_{B(0,R_0)}(P+i)^{-1} \text{ is compact} \tag{1.2}$$

$$\overline{P\bar{u}} = Pu. \tag{1.3}$$

The spaces $\mathcal{H}_{\text{comp}}$, $\mathcal{H}_{\text{loc}}$, $\mathcal{D}_{\text{loc}}$ are defined in the obvious way. The resolvent of P is defined as a bounded operator in the upper half-plane:

$$R(\lambda) = (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}\ ,\ \operatorname{Im}\lambda > 0\ ,\ \lambda^2 \notin \sigma_{\text{point}}(P)$$

and the assumptions (1.1) and (1.2) guarantee its meromorphic continuation

$$R(\lambda) : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}} \tag{1.4}$$

for $\lambda \in \mathbb{C}$ when n is odd and $\lambda \in \Lambda$, the logarithmic plane when n is even (strictly speaking [10] treats only the odd dimensional case but the proof of Theorem 1.1 there applies for n even as well). The poles of this continuation are called *resonances* or *scattering poles.* The multiplicity of a scattering pole λ_0 is defined as the rank of

$$\int_\gamma R(\lambda)\lambda d\lambda\ ,\quad \gamma : [0, 2\pi) \ni t \mapsto \lambda_0 + \varepsilon e^{it}$$

for ε sufficiently small. As we shall see this is the same as the dimension of the image of the full polar part of $R(\lambda)$ (see the definition in [7]). Let us put $\Lambda_{\theta,\psi} = \{\lambda \in \Lambda : \theta < \arg\lambda < \psi\}$.

Theorem *Let P satisfy the assumptions above and let R_1 satisfy $R_0 < R_1$. Then there exists a dense G_δ subset of $C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$, $\mathcal{U}$, such that for any $V \in \mathcal{U}$ the resonances of $P+V$ in $\Lambda_{-\pi,0}$ are simple.*

Remark 1.1. Proceeding in this generality we cannot eliminate multiplicities of either negative or embedded eigenvalues by adding a potential supported outside of the perturbation. For practically any specific perturbation that is however much easier either by the variational formula for eigenvalues or by the Fermi Golden Rule [6, 9].

2. Preliminaries

We will briefly recall the complex scaling of Section 3 of [10]. For $|\theta| < \pi$ there exists a totally real submanifold $\Gamma_\theta \subset \mathbb{C}^n$ such that $\Gamma_\theta \cap \mathbb{R}^n \subset B(0,R_1)$, $\Gamma_\theta \cap \{|z| > R_2\} = e^{i\theta}\mathbb{R}^n \cap \{|z| > R_2\}$, $R_1 \ll R_2$. Considering the Laplacian, $-\Delta = \sum_{i=1}^n D_{x_i}^2$ as a holomorphic differential operator in $\mathbb{C}^n$, $\sum D_{z_i}^2$, we obtain $-\Delta_{\Gamma_\theta} = \sum D_{z_i}^2 \restriction_{\Gamma_\theta}$ (where the restriction can be uniquely defined by, for instance, using almost analytic extensions).

The deformed space is defined as

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \backslash B(0,R_0)) \,,$$

where the measure on Γ_θ is $dz_1 \wedge \cdots \wedge dz_n \restriction_{\Gamma_\theta}$, $(z_1 \cdots z_n) \in \mathbb{C}^n$, and the deformed operator as

$$\begin{aligned} P_\theta u \restriction_{B(0,R_0)} &= P(\chi u)\restriction_{B(0,R_0)} \\ P_\theta u \restriction_{\Gamma_\theta \backslash B(0,R_0)} &= -\Delta_{\Gamma_\theta}(u \restriction_{\Gamma_\theta \backslash B(0,R_0)}) \,, \end{aligned} \tag{2.1}$$

where $\chi \in C_0^\infty(B(0,R_1))$ is equal to 1 in a neighborhood of $\overline{B(0,R_0)}$ and $u \in \mathcal{D}_\theta$ with the domain $\mathcal{D}_\theta$ defined by

$$\mathcal{D}_\theta = \{u \in \mathcal{H}_\theta : \chi u \in \mathcal{D}, (1-\chi)u \in H^2(\Gamma_\theta \backslash B(0,R_0))\} \,.$$

Section 3 of [10] gives the following

Proposition 1. *If $z \in \mathbb{C}\backslash e^{-2i\theta}\mathbb{R}$, $|\theta| < \pi$, then $(P_\theta - z) : \mathcal{D}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator of index 0.*

Hence P_θ has discrete spectrum in $\mathbb{C}\backslash e^{-2i\theta}\mathbb{R}$ and its resolvent behaves near its singularities like the resolvent of a matrix. The relation with the scattering poles (defined as poles of the meromorphic continuation of $(P-\lambda^2)^{-1}$) is given in the following proposition, quoted again from [10]:

Proposition 2. *For n even or odd, $\lambda \in \Lambda_{-\theta,0}$, $0 \le \theta < \pi$, is a scattering pole if and only if λ^2 is an eigenvalue of P_θ or $P_{-\theta}$ respectively. The multiplicity of the scattering pole λ is equal to the algebraic multiplicity of the corresponding eigenvalue λ^2.*

We recall that for n odd the poles satisfy the symmetries $\lambda \mapsto -\bar{\lambda}$ and hence $\Lambda_{-\theta,0}$, $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \varepsilon$, is sufficient for a complete study. For n even the same symmetry holds in the sense of identifying $\Lambda_{-\pi,0}$ and $\Lambda_{\pi,2\pi}$ as subset of the logarithmic plane.

If we define a bilinear form on $\mathcal{H}_\theta$

$$\mathcal{H}_\theta \ni u, v \mapsto \langle u, v\rangle_\theta \stackrel{\text{def}}{=} (u, \bar{v})_\theta , \tag{2.2}$$

where $(\bullet,\bullet)_\theta$ is the Hilbert inner product on $\mathcal{H}_\theta$, then $\langle\bullet,\bullet\rangle_\theta$ is non-degenerate and it identifies $\mathcal{H}_\theta$ antiholomorphically with $\mathcal{H}_\theta^*$. If $u, v \in \mathcal{D}_\theta$ then we have

$$\langle P_\theta u, v\rangle_\theta = \langle u, P_\theta v\rangle_\theta . \tag{2.3}$$

From the self-adjointness of P, (1.3) and the fact that the usual inner product can be used in $\mathcal{H}_{R_0}$ we only need to check this for $-\Delta_{\Gamma_\theta} = \sum_{j=1}^n D_{z_j}^2 \restriction_{\Gamma_\theta}$. That follows in turn from

$$\int_{\Gamma_\theta} D_{z_j}\tilde{u}\restriction_{\Gamma_\theta} \cdot \tilde{v}\restriction_{\Gamma_\theta} dz_1 \wedge \cdots \wedge dz_n\restriction_{\Gamma_\theta} = -\int_{\Gamma_\theta} \tilde{u}\restriction_{\Gamma_\theta} \cdot D_{z_j}\tilde{v}\restriction_{\Gamma_\theta} dz_1 \wedge \cdots \wedge dz_n\restriction_{\Gamma_\theta} , \tag{2.4}$$

where $\tilde{u}$ and $\tilde{v}$ are almost analytic extensions of u and v, $u, v \in C_0^\infty(\Gamma_\theta)$ which, as $i(D_{z_j}\tilde{u}\tilde{v} + D_{z_j}\tilde{v}\tilde{u})dz_1 \wedge \cdots \wedge dz_n = (-1)^{j+1}\partial_j(\tilde{u}\tilde{v}dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n)$, is a consequence of Stokes's theorem.

The structure of the resolvent $(P_\theta - z)^{-1}$ in a neighborhood of an eigenvalue is given by

Lemma 1. *For $z_0 \in \mathbb{C}\backslash e^{-z_i\theta}[0,+\infty]$, an eigenvalue of P_θ, one has, for some $N \in \mathbb{N}$*

$$(P_\theta - z)^{-1} = G_{\theta,z_0}(z) - \sum_{k=1}^{N} \frac{(P_\theta - z_0)^{k-1}}{(z - z_0)^k} \pi_{\theta,z_0} \tag{2.5}$$

where:

1. *$G_{\theta,z_0}(z)$ is analytic and bounded for $|z - z_0|$ sufficiently small*
2. *$\pi_{\theta,z_0} = \sum_{i,j} a_{ij}\phi_i \otimes \phi_j$ and $\phi_i \in \mathcal{D}_\theta$, $\phi_i \otimes \phi_j = \langle\bullet, \phi_i\rangle_\theta \phi_j$*
3. *$((a_{ij}))_{i,j} = (((\langle\phi_i, \phi_j\rangle_\theta)_{i,j})^{-1}$.*

Proof. As P_θ is Fredholm, z_0 is an isolated eigenvalue and for $\varepsilon_0 > 0$ small enough, we define:

$$\pi_{\theta,z_0} = \frac{1}{2\pi i}\int_\gamma (z - P_\theta)^{-1} dz \ \text{ with } \ \gamma : [0, 2\pi) \ni t \mapsto \gamma(t) = z_0 + \varepsilon_0 e^{it} . \tag{2.6}$$

The Fredholm property guarantees that the range Ran π_{θ,z_0} is finite dimensional and an argument based on Cauchy's Theorem and on the resolvent identity gives $\pi_{\theta,z_0}^2 = \pi_{\theta,z_0}$ and $[\pi_{\theta,z_0}, P_\theta] = 0$. Hence for $z \notin \sigma(P_\theta)$ $(P_\theta - z)^{-1} = (P_\theta - z)^{-1}\pi_{\theta,z_0} + G_{\theta,z_0}(z)$ where

$$\begin{aligned} G_{\theta,z_0}(z) &= (P_\theta - z)^{-1} - \frac{1}{2\pi i}\int_\gamma (P_\theta - z)^{-1}(z' - P_\theta)^{-1} dz' \\ &= (P_\theta - z)^{-1} + \frac{1}{2\pi i}\int_\gamma \left(-\frac{1}{z' - z}(P_\theta - z)^{-1} + \frac{1}{z' - z}(P_\theta - z')^{-1}\right) dz' \end{aligned}$$

So if we choose z inside the disk of boundary γ, we get $G_{\theta,z_0}(z) = 1/(2\pi i)\int_\gamma (z' - z)^{-1}(P_\theta - z')^{-1}dz'$ which gives (1) and (2.5).

Since P_θ is symmetric with respect to $\langle\bullet,\bullet\rangle_\theta$, (2.6) shows that so is π_{θ,z_0}, that is $\langle\pi_{\theta,z_0}u, v\rangle_\theta = \langle u, \pi_{\theta,z_0}v\rangle_\theta$. If $\phi_1, \cdots, \phi_M$ form a basis of Ran π_{θ,z_0}, we can write π_{θ,z_0} as $\sum_{i=1}^M \phi_i \otimes \tilde{\phi}_i$, in the sense of (2), where $\tilde{\phi}_i \in \mathcal{H}_\theta$. The symmetry of the bilinear form shows that $\tilde{\phi}_i = \sum_{j=1}^M a_{ij}\phi_j$ where $(a_{ij})_{i,j}$ is a symmetric matrix of full rank. Checking the projection property of π_{θ,z_0} immediately gives $((a_{ij}))_{i,j} = (((\langle\phi_i, \phi_j\rangle_\theta)_{i,j})^{-1}$ which is (3). □

Remark 2.1. There exist some natural choices for the basis $\{\phi_j\}_{1\leq j\leq M}$. For instance, we can take a basis in which the matrix of $P_\theta\pi_{\theta,z_0}$ takes the Jordan normal form (see, for instance [6], Sect.I.5.4). In that case the matrix $(a_{ij})_{1\leq i,j\leq M}$ will in general be rather far from the identity matrix. On the other hand the non-degeneracy and the symmetry of the form $\langle\bullet,\bullet\rangle_\theta$ allow a basis for which $a_{ij}=\delta_{ij}$. Then the resulting matrix for $P_\theta\pi_{\theta,z_0}$ is not in the Jordan normal form unless $N=1$.

3. Proof of Theorem

Let $W\in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ and set $\widetilde{P}_\theta=P_\theta+W$ which is well defined as $\Gamma_\theta\cap\mathbb{R}^n\supset B(0,R_1)$. For the same reason,

$$\widetilde{P}_\theta=(P+W)_\theta\ ,$$

and it is clear that $\widetilde{P}_\theta$ is Fredholm with index 0 and symmetric with respect to $\langle\bullet,\bullet\rangle_\theta$.

We define a family of open sets (the eigenvalues of P_θ are discrete by the Fredholm property), E_θ^r, which consist of all potentials $W\in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ for which all the eigenvalues of $\widetilde{P}_\theta=P_\theta+W$ in $\Lambda_\theta\cap\{|z|\leq r\}$ are simple. Then we have a G_δ set

$$E_\theta=\bigcap_{n\in\mathbb{N}}E_\theta^n.$$

The theorem in Sect.1 will follow directly from:

Proposition 3. *If $F_\theta=C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})\backslash E_\theta$ then the interior of F_θ is empty.*

By the Baire category theorem and the discreteness of the spectrum we only need to prove that if $W\in F_\theta$ and z_0 an eigenvalue of $\widetilde{P}_\theta$ then for any $\varepsilon>0$ there exists $V\in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ with $\|V\|_\infty\leq\varepsilon$ such that $\widetilde{P}_\theta+V$ has only simple eigenvalues in a neighbourhood of z_0.

Thus we take $W\in F_\theta$ and let z_0 be a multiple eigenvalue of $\widetilde{P}_\theta=P_\theta+W$. For $V\in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)})$ satisfying $\|V\|_\infty\leq\varepsilon$ we define $\widetilde{P}_\theta^V=\widetilde{P}_\theta+V$. If $D(z_0,\delta)$ is the disk of center z_0 and radius δ (δ chosen small enough so that $\sigma(\widetilde{P}_\theta)\cap D(z_0,\delta)=\{z_0\}$) and with γ its boundary, then for $z\in\gamma$,

$$z-\widetilde{P}_\theta^V=(z-\widetilde{P}_\theta)\left(\mathbf{1}+\widetilde{G}_{\theta,z_0}(z)V-\sum_{k=1}^N\frac{(\widetilde{P}_\theta-z_0)^{k-1}}{(z-z_0)^k}\pi_{\theta,z_0}V\right).$$

Consequently, for $\|V\|_\infty$ small enough, $(z-\widetilde{P}_\theta^V)$, $z\in\gamma$, is invertible with a bounded inverse. We define

$$\widetilde{\pi}_\theta^V=\frac{1}{2\pi i}\int_\gamma(z-\widetilde{P}_\theta^V)^{-1}dz$$

which is a finite rank projection, and, moreover, it is analytic in V for $\|V\|_\infty<\varepsilon$. The analyticity is meant in the following sense: $\widetilde{\pi}_\theta^{V_1+zV_2}$ is analytic in τ for $\tau\in\mathbb{C}$, V_1, $V_2\in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ such that $\|V_1+\tau V_2\|_\infty<\varepsilon$. We also note that for $\|V\|_\infty<\varepsilon,\|V'\|_\infty<\varepsilon$ (with ε small enough),

$$\|\widetilde{\pi}_\theta^V-\widetilde{\pi}_\theta^{V'}\|\leq C\|V-V'\|_\infty$$

Hence the rank of $\tilde{\pi}_\theta^V$ is constant and equal to M, say, and $\tilde{\pi}_\theta^V$ is the projector on the generalized eigenspace associated to the eigenvalues of $\tilde{P}_\theta^V$ contained in $D(z_0, \delta)$.

To prove the proposition, we will proceed by induction. Either of the following two cases occurs

(1) $\forall\, \varepsilon\ \exists\, V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$, $\|V\|_\infty < \varepsilon$ such that $\tilde{P}_\theta^V$ has at least two distinct eigenvalues in $D(z_0,\delta)$

or

(2) $\exists\, \varepsilon\ \forall\, V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$, $\|V\|_\infty < \varepsilon$, there exists a unique eigenvalue $z(V)$ for $\tilde{P}_\theta^V$ in $D(z_0,\delta)$, that is $\exists\, 1 \le k \le M$ and $z(V)$ such that

$$(\tilde{P}_\theta^V - z(V))^k \tilde{\pi}_\theta^V = 0. \tag{3.1}$$

If the first case occurs, we choose a V which splits the eigenvalues and set $\tilde{P}_\theta^1 = \tilde{P}_\theta^V$. We then perturb $\tilde{P}_\theta^1$ by some potential V_1: $\tilde{P}_\theta^{1,V} = \tilde{P}_\theta^1 + V_1$, where $\|V_1\|_\infty \le \varepsilon/2$ and $V_1 \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$. For each of the distinct eigenvalues of $\tilde{P}_\theta^1$, we apply the same procedure as for z_0 choosing $\|V_1\|_\infty$ small enough so that distinct eigenvalues remain separated. Applying the same argument inductively, after at most M such steps we get either only simple eigenvalues or we encounter case (2) to which we now turn. We will show that it cannot in fact occur.

Lemma 2. *Assume that case (2) above holds. Then, for ε small enough, $z(V)$ is analytic in V (for $\|V\|_\infty < \varepsilon$) in the sense that, for V_1, $V_2 \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ and τ such that $\|V_1 + \tau V_2\|_\infty < \varepsilon$*

$$z(V_1 + \tau V_2) \text{ is analytic in } \tau.$$

Proof. From (3.1) we deduce that $z(V)$ is the unique eigenvalue of of $\tilde{P}_\theta^V \tilde{\pi}_\theta^V$. Recalling that the rank of $\tilde{\pi}_\theta^V$, M, is constant for V small, we obtain that $z(V) = \operatorname{tr} \tilde{P}_\theta^V \tilde{\pi}_\theta^V / M$. Hence the analyticity of $z(V)$ follows from that of $\tilde{P}_\theta^V \tilde{\pi}_\theta^V$. □

For $V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$, $\|V\|_\infty < \varepsilon$, and assuming that (2) holds, we define

$$k(V) = \inf\{k \,:\, (\tilde{P}_\theta^V - z(V))^k \tilde{\pi}_\theta^V = 0\},$$

so that $k(V)$ satisfies

$$(\tilde{P}_\theta^V - z(V))^{k(V)} \tilde{\pi}_\theta^V = 0 \text{ and } (\tilde{P}_\theta^V - z(V))^{k(V)-1} \tilde{\pi}_\theta^V \neq 0\ .$$

The function $k(V)$ is lower semi-continuous and we have the obvious bounds $1 \le k(V) \le M$.

Let $k_\varepsilon = \sup\{k(V) \,:\, V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R}),\ \|V\|_\infty < \varepsilon\}$. Then for any $\varepsilon > 0$ small enough there exists V_ε such that $V_\varepsilon \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$, $\|V_\varepsilon\|_\infty < \varepsilon$ and $k(V_\varepsilon) = k_\varepsilon$. Hence there exists $\eta_\varepsilon > 0$ such that for all $V' \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ with $\|V' - V_\varepsilon\|_\infty < \eta_\varepsilon$, $k(V') = k_\varepsilon$. Consequently, we may now assume that $k(V)$ is constant, that is the maximal size of the Jordan blocks of $\tilde{P}_\theta^V \tilde{\pi}_\theta^V$ is constant equal to k, say. We first treat the easier case where the geometric multiplicity is equal to the algebraic one (i.e $k = 1$):

Lemma 3. *Let us assume that for some $\varepsilon_0 > 0$ and for all*

$$V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R}), \quad \|V\|_\infty < \varepsilon_0,$$
$$(\widetilde{P}_\theta^V - z(V))\widetilde{\pi}_\theta^V = 0.$$

Let $\{\phi_i\}_{1\le i\le M}$ be a sequence of eigenvectors of $\widetilde{P}_\theta$ and let $z(\varepsilon) = z(\varepsilon V)$ for some fixed $V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$. Then

$$\langle V\phi_i, \phi_j\rangle_\theta = \dot{z}(0)\langle \phi_i, \phi_j\rangle_\theta \,. \tag{3.2}$$

Proof. Abusing notation slightly let $\widetilde{P}_\theta^\varepsilon = \widetilde{P}_\theta^{\varepsilon V}$ for some fixed $V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$ such that $\|V\|_\infty < 1$. We put $\phi_j^\varepsilon = \widetilde{\pi}_\theta^\varepsilon\phi_j$ so that $(\widetilde{P}_\theta^\varepsilon - z(\varepsilon))\phi_j^\varepsilon = 0$. We now differentiate this identity and this yields

$$(V - \dot{z}(\varepsilon))\phi_j^\varepsilon + (\widetilde{P}_\theta^\varepsilon - z(\varepsilon))\left(\widetilde{\pi}_\theta^\varepsilon\phi_j^\varepsilon\right)^{\cdot} = 0\,.$$

Pairing with ϕ_i^ε under the bilinear form $\langle\bullet,\bullet\rangle_\theta$ gives

$$\langle V\phi_j^\varepsilon, \phi_i^\varepsilon\rangle_\theta - \dot{z}(\varepsilon)\langle\phi_j^\varepsilon, \phi_i^\varepsilon\rangle_\theta = -\langle\left(\phi_j^\varepsilon\right)^{\cdot}, (\widetilde{P}_\theta^\varepsilon - z(\varepsilon))\phi_i^\varepsilon\rangle_\theta = 0\,.$$

We now set $\varepsilon = 0$ and obtain the 'variational formula' (3.2) □

An analogue of this standard argument works also when the algebraic multiplicity exceeds the geometric one:

Lemma 4. *Let us assume that for some $\varepsilon_0 > 0$, $k \in \mathbb{N}$, $k > 1$ and for all*

$$V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R}), \quad \|V\|_\infty < \varepsilon_0,$$
$$(\widetilde{P}_\theta^V - z(V))^k\widetilde{\pi}_\theta^V = 0, \quad (\widetilde{P}_\theta^V - z(V))^{k-1}\widetilde{\pi}_\theta^V \neq 0\,.$$

Let $\psi \neq 0$ be an eigenfunction of the form $\psi = (\widetilde{P}_\theta - z(0))^{k-1}h$. Then for all V above

$$\langle V\psi, \psi\rangle_\theta = 0\,. \tag{3.3}$$

Proof. Using the same notation as in the proof of Lemma 3 we put $\psi^\epsilon = (\widetilde{P}_\theta^\epsilon - z(\epsilon))^{k-1}\widetilde{\pi}_\theta^\epsilon h$ which is now an eigenfunction of $\widetilde{P}_\theta^\epsilon$ depending analytically on ϵ. As in Lemma 3 the differentiation of the eigenequation gives (3.2) with $\phi_i = \phi_j = \psi$. But now we also have

$$\langle\psi,\psi\rangle_\theta = \langle(\widetilde{P}_\theta - z(0))^{k-1}h, \psi\rangle_\theta = \langle(\widetilde{P}_\theta - z(0))^{k-2}h, (\widetilde{P}_\theta - z(0))\psi\rangle_\theta = 0\,,$$

and (3.3) follows. □

Lemmas 3 and 4 exclude case (2): if $k = 1$ then (3.2) for all $V \in C_0^\infty(B(0,R_1)\backslash\overline{B(0,R_0)};\mathbb{R})$, with $\|V\|_\infty < 1$ would imply that $\phi_i\restriction_{B(0,R_1)\backslash\overline{B(0,R_0)}} = 0$ for some $1 \le i \le M$ since we can take $\langle\phi_i, \phi_j\rangle = \delta_{ij}$ (see Remark 2.1) . Since $(-\Delta_{\Gamma_\theta} + W - z_0)\phi_i\restriction_{B(0,R_1)\backslash\overline{B(0,R_0)}} = 0$, unique continuation for second order elliptic operators (see for instance [5], Sect.17.2) implies that $\phi_i\restriction_{\Gamma_\theta\backslash\overline{B(0,R_0)}} = 0$ and thus $\phi_i \in L^2(\Gamma_\theta)$ for all θ. Hence z_0 is an eigenvalue of P, and $z_0 \neq \lambda^2$ for any $\lambda \in \Lambda_{0,\pi}$. The same contradiction is clearly obtained from (3.3) in the case when $k > 1$.

Remark 3.1. The formula (3.2) is the obvious analogue of the standard variational formula for eigenvalues of a self-adjoint operator and that analogy is particularly valid when the resonances are simple and some differentiability is allowed (here it was guaranteed by Lemma 2 under the degeneracy hypothesis). But as the bilinear form is not positive definite there is no control on the speed of motion of the resonance. The standard example is $\Delta + \epsilon V$, $V \in \mathbb{C}_0^\infty(\mathbb{R}^n)$ where as $\epsilon \to 0$ the resonances escape to (imaginary) infinity in 'finite time'. Even more striking examples come from considering a pair of δ functions potential on $\mathbb{R}$ (see [1]) where the same phenomenon occurs with the non-zero limiting potential[†].

Remark 3.2. The argument above works in much greater generality, in particular without the assumption (1.3) – however, one does not have then the exact analogues of the statements of the self-adjoint case with the inner product replaced by the indefinite form (see (3.2)). We could also consider a family of (unbounded) operators, $H(V)$, on a Hilbert space $\mathcal{H}$, depending smoothly on a parameter V in a Banach space $\mathcal{B}$. We would then assume that for z in an open set $\Omega \subset \mathbb{C}$, $H(V) - z$ is a Fredholm operator with index zero. The simplest abstract condition replacing the unique continuation argument above is

$$\forall\, W \in \mathcal{B}\ (dH_V(W)u, v)_{\mathcal{H}} = 0,\ H(V)u = zu,\ H(V)^*v = \bar{z}v,\ z \in \Omega \implies u = v = 0.$$

We can now use the same argument as in Lemmas 2 and 3 but with the Hilbert space inner product and with pairing with the eigenfunctions of the adjoint of $H(V)$ (which were equal to $\bar{\phi}_j$ and $\bar{\psi}$ above so that we could use $\langle \bullet, \bullet \rangle_\theta$ and drop the complex conjugate). Hence, generically, in the sense of a G_δ dense subset of V's in $\mathcal{B}$ the spectrum of $H(V)$ in Ω is simple.

It would be somewhat cumbersome to devise an optimal abstract setting and the one described in this remark does not completely apply to Theorem in Sect.1 (see Remark 1.1 and for instance the case of scattering on finite volume surfaces with hyperbolic ends – see Sect.1 of [10] or [2]).

References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics.* Springer Texts and Monographs in Physics, Springer Verlag, 1988

[2] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces.* to appear.

[3] P. Lax and R. Phillips, *Scattering Theory.* Academic Press 1969.

[4] J. Helton and J. Ralston, *The first variation of the scattering matrix.* J. Diff. Eq. **21**(1976), 378-394.

[5] L. Hörmander, *The Analysis of Linear Partial Differential Operators.* vol.3, Springer Verlag, 1985.

[6] T. Kato, *Perturbation Theory for Linear Operators.* Springer Verlag, 1966.

[7] R.B. Melrose, *Lectures on Geometric Scattering Theory.* Cambridge University Press, 1995.

[8] S.V. Petras, *On the continuous dependence of the poles of the scattering matrix on the coefficients of an elliptic operator.* Proc. Steklov Inst. Math. **159** (1983), 135-139.

[9] M. Reed and B. Simon, *Methods of Mathematical Physics.* vol.4, Academic Press, 1978.

[10] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles.* J. of Amer. Math. Soc. **4**(4)(1991), 729-769.

[11] K. Uhlenbeck, *Generic properties of eigenfunctions.* Amer. J. Math. **98**(1976), 1059-1078.

[12] S. Zelditch, *On the generic spectrum of a riemannian cover.* Ann. Inst. Fourier **40**(1990), 407-442.

[13] M. Zworski, *Counting scattering poles.* in *Spectral and Scattering Theory.* M. Ikawa, ed. Marcel Dekker, 1994.

[†]This was kindly pointed out to us by R. Froese.