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Quantum Mechanics on Spaces With Finite Fundamental Group

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Abstract We consider in general terms dynamical systems with finite-dimensional, non-simply connected configuration-spaces. The fundamental group is assumed to be finite. We analyze in full detail those ambiguities in the quantization procedure that arise from the non-simply connectedness of the classical configuration space. We define the quantum theory on the universal cover but restrict the algebra of observables $\mathcal{O}$ to the commutant of the algebra generated by deck-transformations. We apply standard superselection principles and construct the corresponding sectors. We emphasize the relevance of all sectors and not just the abelian ones.

Introduction

Quantizing a system whose classical configuration space, Q, is not simply connected is ambiguous over and above other ambiguities which may already be present in the simply connected case. This paper aims to fully describe and analyze these ambiguities for the cases of finite fundamental groups without entering any discussion on problems in quantization proper. For the rest of the paper we thus assume a definite and consistent prescription for quantization on simply connected configuration spaces (or at least specific examples thereof, e.g. homogeneous spaces) to exist and focus attention to the *additional* ambiguities in the non simply-connected case. We are interested in non-abelian fundamental groups and, necessarily, their representation theory. It is to evade the unfortunate intricacies of representation theory for infinite discrete non-abelian groups that we restrict

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attention to finite groups. This at least allows a general treatment, although there are certainly many cases where specific infinite groups are of interest.

From the technical point of view the ambiguities we are interested in appear in a variety of guises, depending in particular on the quantization scheme that is employed. For example, attempting standard canonical quantization rules on $R^2 - \{0\}$ (the famous Bohm-Aharonov situation) results in unitarily inequivalent representations of the canonical commutation relations [Re]. This is possible since the point defect and its associated incompleteness prevent the representations to exponentiate to the Weyl form of the commutation relations and therefore the application of von Neumann's well known uniqueness result ([RS], theorem VIII.14). An even simpler situation that captures all the essential features involved here is given by a particle on the circle (compare remark 3.1.6;5 in [T]).

Let us go into some more details by looking at the slightly more general situation of a particle on the n-torus, T^n. We represent the torus by the cube, $K^n = \{0 \leq x_k \leq 1, k = 1,..,n\} \subset R^n$, whose opposite sides are eventually identified via translations. For the moment, however, let us work with the fundamental domain K^n. We consider the Hilbert space $L^2(K^n, d^n x)$ and in it the dense domain of absolutely continuous functions, ψ, which vanish on the boundary ∂K^n, and whose first derivatives are again in the Hilbert space. The momentum operators, $p_k = -i\frac{\partial}{\partial x_k}$, are not self-adjoint on this domain but admit self-adjoint extensions by relaxing the boundary conditions to $\psi|_{x_k=1} = \exp(i\theta_k)\psi|_{x_k=0}$, where each θ_k is some absolutely continuous but otherwise arbitrary function of the $n-1$ variables x_i, $i \neq k$. Each of the now self-adjoint operators p_k (we shall use the same symbol) exponentiates to a one-parameter unitary group: $R \ni a \rightarrow \exp(iap_k) = U_k(a)$, where $U_k(a)$ displaces ψ by an amount a in the positive x_k-direction so that values that are pushed through the boundary $x_k = 1$ reenter at $x_k = 0$ with the additional phase $\exp(-i\theta_k)$. At this point we note that our self-adjoint extensions are too general, since for non-constant θ_k the unitaries $U_k(a)$, and hence the p_k, will not mutually commute (compare section VIII.5 in [RS]). Since we want our extensions p_k to commute we restrict to constant θ_k. The inequivalent commuting extensions for the momenta are thus labelled by n angles $\theta_1, \dots, \theta_n$. If we finally identify opposite faces of K^n so as to obtain the n-torus, T^n, all the inequivalent quantizations still persist if we allow the 'functions' ψ to be sections in flat complex line-bundles-with-connection over T^n [Wo]. The fundamental group of T^n is $\mathbf{Z}^n$, and the flat line-bundles-with-connection are classified by the inequivalent one-dimensional irreducible representations thereof (see e.g chapter 5 in [Wo]). These are just labelled by the angles $\theta_1, \dots, \theta_n$ whose interpretation in the bundle picture is to fix the representation for the transition functions and also to determine the holonomies: $\exp(i\theta_k)$ is the holonomy for the loop along the x_k coordinate.

From this example it should be clear that the geometric picture underlying the possibility of inequivalent quantizations is fairly simple. It is therefore not surprising that these possibilities were first systematically studied within the path-integral formulation [LD], where different homotopy classes of paths connecting two fixed points need not carry the same weight in the path integral. (See also [Sch] for an early discussion.) Rather, they could carry relative weights given by complex numbers of unit modulus. Unitarity then implies that these weight factors must furnish some one-dimensional complex unitary representation of the fundamental group. This prescription is most conveniently formulated by employing the universal cover, $\bar{Q}$, of the configuration space Q as domain for the quantum mechanical state function [Do1-2]. At least in the case of finite coverings one may then simply work on the universal cover space. The redundancy it represents is restricted to finitely many repetitions which can easily be accounted for by appropriate normalization factors. In the case of infinite groups one may select a fundamental domain $\bar{F} \subset \bar{Q}$ for Q and chose the Hilbert space to be square integrable functions on $\bar{F}$. This is precisely what we did in the torus example above. However, in the sequel we restrict to finite coverings and here $\bar{Q}$ is more convenient to work with than $\bar{F}$. Any quantum mechanical system based on Q can be lifted to define such a system on $\bar{Q}$ so that all the operations may now be carried out on the simply connected space $\bar{Q}$. The distinguishing feature of a quantum mechanical system so obtained from a system with genuine classical configuration space $\bar{Q}$ is the absence of certain observables in the former case. For example, disjoint sets on $\bar{Q}$ which cover the same set on Q cannot give rise to different projection operators, as it would be the case if we considered a system whose configuration space were truly given by $\bar{Q}$. Hence the idea is that due to missing observables we encounter superselection rules, and that the quantization ambiguities are precisely given by the different sectors. We stress that we wish to consider *all* sectors arising in this fashion.

The plan of the paper is as follows: In section 1 we outline the underlying classical geometry thereby introducing some notation. In this setting we briefly review the known case where the fundamental group is abelian [LD]. Section 2 presents in an explicit way the geometry of the regular representation for general finite groups. In section 3 we use a finite-dimensional Hilbert space with reducible algebra of observables as a toy model to introduce some basic concepts from the theory of superselection rules in ordinary quantum mechanics. In section 4 we finally generalize the constructions mentioned in section 1 to the non-abelian case. We show how to implement the requirement of so-called abelian superselection rules which in the non abelian case is not automatic. Coherent sectors are built from sections in vector bundles for each irreducible representation of the fundamental group. Appendix A provides some explanation on how gauge theoretic concepts apply to the universal cover space and its associated vector bundles. Appendix B contains a simple quantum mechanical example with non-abelian finite fundamental group. Throughout this paper we shall not employ the summation convention for repeated indices.

1 Classical Background and Abelian Case

Let Q be a finite-dimensional manifold that serves as configuration space for some dynamical system. We denote its cotangent bundle by $T^*(Q)$. $\pi_1(Q,q)$ denotes the fundamental group of Q based at the point q. It is assumed to be finite, and hence for each q abstractly isomorphic to a finite group G. The neutral element of G will be called e. We stress that although there exist isomorphisms of $\pi_1(Q,q)$ with G for each q, there are generally no natural choices for these isomorphisms and hence no natural identifications of the fundamental groups at various points with G (see appendix A). There are, however, natural identifications of the conjugacy classes of each $\pi_1(Q,q)$ with those of G. Abelian fundamental groups may thus be identified with an abstract abelian group. In this case it makes sense to speak of *its* (meaning Q's) fundamental group, a terminology which otherwise just refers to an abstract isomorphism. The relevance of this point to our discussion should not be overlooked (compare appendix A).

Let further $\bar{Q}$ denote the universal covering manifold and $\tau : \bar{Q} \to Q$ the projection map. Points of $\bar{Q}$ are denoted by $\bar{q}, \bar{p}$, etc., where sometimes we use this notation to also indicate that $\tau(\bar{q}) = q$ etc.. $\bar{Q}$ has the structure of a G-principal bundle:

$$\begin{array}{ccc} G & \longrightarrow & \bar{Q} \\ & & \downarrow{\scriptstyle \tau} \\ & & Q \end{array} \tag{1.1}$$

where G acts on $\bar{Q}$ from the right:

$$G \times \bar{Q} \to \bar{Q}\,, \quad (g, \bar{q}) \mapsto R_g(\bar{q}) =: \bar{q}g\,, \tag{1.2}$$
$$\text{such that} \quad \tau \circ R_g = \tau \quad \forall g \in G\,. \tag{1.3}$$

Since G is discrete, τ is a local diffeomorphism and the tangent maps $\tau_{\bar{q}*} : T_{\bar{q}}(\bar{Q}) \to T_q(Q)$ are linear isomorphisms with inverse $\tau_{\bar{q}*}^{-1} : T_q(Q) \to T_{\bar{q}}(\bar{Q})$ for each $\bar{q} \in \bar{Q}$. For them (1.3) implies:

$$\left(R_{g^{-1}}\right)_{\bar{q}g\,*} \circ \tau_{\bar{q}g\,*}^{-1} = \tau_{\bar{q}*}^{-1}\,. \tag{1.4}$$

We can now lift τ to the cotangent bundles $T^*(\bar{Q})$ and $T^*(Q)$ of $\bar{Q}$ and Q (call the lift $\tilde{\tau}$) and combine it with the natural lift, $\tilde{R}_g$, of R_g into the following diagram with two commuting squares:

$$\begin{array}{ccccc} T^*(\bar{Q}) & \xrightarrow{\tilde{R}_g} & T^*(\bar{Q}) & \xrightarrow{\tilde{\tau}} & T^*(Q) \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ \bar{Q} & \xrightarrow{R_g} & \bar{Q} & \xrightarrow{\tau} & Q \end{array} \tag{1.5}$$

We denote points of the cotangent bundle by greek letters with occasionally added subscripts indicating their base point. We have

$$\tilde{R}_g(\bar{\alpha}_{\bar{q}}) := \bar{\alpha}_{\bar{q}} \circ \left(R_{g^{-1}}\right)_{\bar{q}g\,*} \quad \forall \bar{\alpha}_{\bar{q}} \in T^*_{\bar{q}}(\bar{Q})\,, \tag{1.6}$$
$$\tilde{\tau}(\bar{\alpha}_{\bar{q}}) := \bar{\alpha}_{\bar{q}} \circ \tau_{\bar{q}*}^{-1} \quad \forall \bar{\alpha}_{\bar{q}} \in T^*_{\bar{q}}(\bar{Q})\,, \tag{1.7}$$

so that, using (1.4), we get in analogy to (1.3):

$$\tilde{\tau} \circ \tilde{R}_g = \tilde{\tau}\,. \tag{1.8}$$

Let $\bar{\alpha}_{\bar{q}} \in T^*_{\bar{q}}(\bar{Q})$ and $\alpha_q \in T^*_q(Q)$, so that $\tilde{\tau}(\bar{\alpha}) = \alpha$, i.e., $\alpha_q \circ \tau_{\bar{q}*} = \bar{\alpha}_{\bar{q}}$. The canonical 1-forms on $T^*(\bar{Q})$ and $T^*(Q)$ are defined by $\bar{\sigma}_{\bar{\alpha}} := \bar{\alpha} \circ \bar{\pi}_*$ and $\sigma_\alpha := \alpha \circ \pi_*$ respectively. Then

$$\tilde{\tau}^*_{\bar{\alpha}}(\sigma_\alpha) = \sigma_\alpha \circ \tilde{\tau}_{\bar{\alpha}*} = \alpha \circ \pi_{\alpha*} \circ \tilde{\tau}_{\bar{\alpha}*} = \alpha \circ \tau_{\bar{q}*} \circ \bar{\pi}_{\bar{\alpha}*} = \bar{\sigma}_{\bar{\alpha}} \tag{1.9}$$

so that $\tilde{\tau}$ is exact-symplectic. The same holds obviously for all $\tilde{R}_g$, so that phase space functions invariant under all $\tilde{R}_g$ generate an invariant flow on $T^*(\bar{Q})$. It is easy to see that $\tilde{R}_g$-invariant ($\forall g \in G$) functions $\bar{H}$ on $T^*(\bar{Q})$ are precisely those of the form $\bar{H} = H \circ \tilde{\tau}$, where H is a function on $T^*(Q)$. Given such a function as a Hamiltonian, the dynamical descriptions using $(T^*(Q), H)$ and $(T^*(\bar{Q}), \bar{H})$ are equivalent in the following sense: pick $\alpha \in T^*(Q)$ and any $\bar{\alpha} \in T^*(\bar{Q})$ satisfying $\tilde{\tau}(\bar{\alpha}) = \alpha$. Let $\bar{\gamma}(t)$ be the uniquely determined solution curve on $T^*(\bar{Q})$ for the Hamiltonian $\bar{H}$ which satisfies $\bar{\gamma}(t=0) = \bar{\alpha}$. Then $\tilde{\tau} \circ \bar{\gamma} = \gamma$, where γ is the unique solution curve on $T^*(Q)$ for the Hamiltonian H, satisfying $\gamma(t=0) = \alpha$. In this way, the Hamiltonian description on $T^*(\bar{Q})$ using only observables of the form

$$\bar{O} = O \circ \tilde{\tau} \tag{1.10}$$

is entirely equivalent to the description on $T^*(Q)$. Note that generally the maps $\tau_{\bar{q}*}^{-1}$ allow to uniquely lift any vector field X on $T^*(Q)$ to a vector field $\bar{X}$ on $T^*(\bar{Q})$ which is invariant under the action $\tilde{R}$ of G. (The same holds, of course, for vector fields on Q and $\bar{Q}$.) Moreover, $\bar{X}$ is locally Hamiltonian if X is. The converse is not quite true, since it might happen that for some properly locally Hamiltonian X its lift, $\bar{X}$, is in fact globally Hamiltonian. It is obvious that $\bar{X}$ is complete if X is. If a (symmetry-) group S acts on $T^*(Q)$ it will generally not be true that it also acts on $T^*(\bar{Q})$. For example, let the vector field X on $T^*(Q)$ generate the circle group and suppose that its orbit loops are not contractible{1}. Then it is clear that only a cover group of the circle will act on $T^*(\bar{Q})$. Generally, there will be an action of a larger group, S_G, given by some G-extension of S{2}.

Let us now turn to the quantization, where the Hilbert space is built from square integrable complex functions on $\bar{Q}$. The measure $d\bar{q}$ on $\bar{Q}$ is taken as the pullback of the measure dq on Q via τ, so that, $\forall g \in G$,

$$R_g^* d\bar{q} = d\bar{q}\,. \tag{1.11}$$

In analogy to the classical case, we require: *observables must commute with the action of G on $L^2(\bar{Q}, d\bar{q})$*. For example, integral kernels of operators on $L^2(\bar{Q}, d\bar{q})$ which satisfy

$$\bar{O}(\bar{q}g, \bar{p}g) = \bar{O}(\bar{q}, \bar{p}) \quad \forall \bar{q}, \bar{p} \in \bar{Q},\ \forall g \in G \tag{1.12}$$

clearly commute with the action of G. In particular this is true for the propagator:

$$\bar{K}(\bar{q}', t'; \bar{q}, t) = \bar{K}(\bar{q}'g, t'; \bar{q}g, t)\,. \tag{1.13}$$

In [LD][D1-2] it was pointed out that the wave function on $\bar{Q}$ need not project to a well defined function on Q. Rather, one could also consider wave functions that satisfied

$$\psi^\mu(\bar{q}g) = \chi^\mu(g)\psi^\mu(\bar{q}), \tag{1.14}$$

{1} For connected $T^*(\bar{Q})$ either all or none of the orbits are contractible.
{2} G is a normal subgroup of S_G so that $S_G/G \cong S$. But if S is not a subgroup of S_G there will be no action of S on $T^*(\bar{Q})$. Since we consider only finite G, S_G will be compact if S is.

where μ labels a one-dimensional complex unitary irreducible representation of G with characters $\chi^\mu(g)$. On Q such wave functions are sections in a complex line bundle which is χ^μ-associated to the principal bundle (1.1). In general we prefer however to work instead with functions on $\bar{Q}$ satisfying (1.14), called the condition of χ^μ-equivariance (compare appendix A). We thus have the Hilbert spaces $\mathcal{H} = L^2(\bar{Q}, d\bar{q})$ and the subspaces $\mathcal{H}^\mu$ of those functions satisfying (1.14). A key point is now to establish that the observables act indeed irreducibly on each $\mathcal{H}^\mu$. This will follow from a more general result proven in chapter 4.

Let us consider the operator

$$
\begin{aligned}
T^\mu &: \mathcal{H} \to \mathcal{H}^\mu \\
(T^\mu \psi)(\bar{q}) &:= \frac{1}{n} \sum_{g \in G} \chi^\mu(g) \psi(\bar{q}g),
\end{aligned}
\tag{1.15}
$$

which is easily seen to to be self-adjoint. It satisfies

$$
T^\mu T^\nu = \delta_{\mu\nu} T^\mu \tag{1.16}
$$

due to the orthogonality of the characters. Moreover, T^μ restricts to the identity on $\mathcal{H}^\mu$. The set $\{T^\mu\}$ is thus just the collection of projection operators onto the mutually orthogonal subspaces $\{\mathcal{H}^\mu\}$ of $\mathcal{H}$. Since the propagator satisfies (1.13), we have[3]

$$
T^\mu \circ \bar{K}(t';t) = \bar{K}(t';t) \circ T^\mu = T^\mu \circ \bar{K}(t';t) \circ T^\mu =: \bar{K}^\mu(t',t)\,, \tag{1.17}
$$

where explicitly

$$
\bar{K}^\mu(\bar{q}',t';\bar{q},t) = \frac{1}{n} \sum_{g \in G} \chi^\mu(g) \bar{K}(\bar{q}'g, t'; \bar{q}, t)\,. \tag{1.18}
$$

The standard combination property for propagators, satisfied by $\bar{K}$, now implies the same for each $\bar{K}^\mu$:

$$
\int_{\bar{Q}} \bar{K}^\mu(\bar{q}',t';\bar{q}'',t'')\, \bar{K}^\mu(\bar{q}'',t'';\bar{q},t)\, d\bar{q}'' = \bar{K}^\mu(\bar{q}',t';\bar{q},t)\,. \tag{1.19}
$$

Finally, we note that due to (1.12) formulae (1.17-1.18) identically hold when $\bar{K}$ is replaced with $\bar{O}$:

$$
\begin{aligned}
T^\mu \circ \bar{O} = \bar{O} \circ T^\mu &= T^\mu \circ \bar{O} \circ T^\mu =: \bar{O}^\mu \\
\bar{O}^\mu(\bar{q}, \bar{p}) &= \frac{1}{n} \sum_{g \in G} \chi^\mu(g) \bar{O}(\bar{q}g, \bar{p})
\end{aligned}
\tag{1.20}
$$

This is essentially the framework of [LD][D1-2]. We believe, however, that starting from (1.14) (or (1.18)) is a rather ad hoc procedure and that the actual task is to construct

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[3] Composition of maps will generally be denoted by the symbol $\circ$. In the very obvious cases it will be omitted, like in (1.16).

all subspaces of $\mathcal{H}$ in which observables act irreducibly. This is not achieved by considering all $\mathcal{H}^\mu$, since generally

$$\mathcal{H} \neq \bigoplus_{\mu=1-\mathrm{dim}} \mathcal{H}^\mu \,. \tag{1.21}$$

Only for abelian groups could the equality sign hold in (1.21). In section 4 we give the generalization to non-abelian finite groups G. Similar ideas how this could be done were also formulated in a non-technical fashion in [So] and [Ba1-2]. But before attacking the actual problem, we need to present some standard facts about the regular representation of finite groups. This will be done in some detail in the next section.

2 The Geometry of the Regular Representation

Let G be a finite group of order n and unit element e. The group algebra V_G is the vector space

$$V_G := \mathrm{span}\,\{\hat{g},\, g \in G\} \tag{2.1}$$

where from now on a hat identifies an element of V_G. V_G is made into an algebra by the obvious multiplication law on the basis vectors:

$$\hat{g}\cdot\hat{h} := \widehat{g\cdot h}, \tag{2.2}$$

and linear extension. Given any two elements $\hat{v}$ and $\hat{w}$ of V_G,

$$\hat{v} = \sum_{g\in G} v(g)\hat{g}\,, \quad \hat{w} = \sum_{g\in G} w(g)\hat{g}\,, \quad v(g), w(g) \in C, \tag{2.3}$$

the components of their product are hence given by

$$(\hat{v}\cdot\hat{w})(g) = \sum_{h\in G} v(gh^{-1})w(h) = \sum_{h\in G} v(h)w(h^{-1}g). \tag{2.4}$$

The algebra V_G is called the group algebra of G and the representations of G on V_G by left or right multiplication are called the left or right regular representation respectively. Under such a regular representation V_G decomposes as (see [We] for a general discussion)

$$V_G = \bigoplus_{\mu=1}^{m} V^\mu \qquad \text{(uniquely)}, \tag{2.5a}$$

$$V^\mu = \bigoplus_{i=1}^{n_\mu} V^\mu_{i,\{L,R\}} \quad \text{(non uniquely)}, \tag{2.5b}$$

where $\mu = 1,\ldots,m$ labels all the inequivalent irreducible representations of G, and $i = 1,\ldots,n_\mu$ labels the copies of the μ-th representation. $\{L,R\}$ is understood to replace either

L or R. $V^{\mu}_{i,\{L,R\}}$ are irreducible subspaces for the left (L) and right (R) multiplications respectively. As indicated, for neither of them the decomposition of V^{μ} is unique, whereas the decomposition of V_G into the V^{μ} is unique. This will become more transparent as we proceed. It is a property of the regular representation that it contains each irreducible representation as often as its dimension, that is, $n_{\mu} = \dim V^{\mu}_i$-times (e.g. [We][Ha]). Hence

$$\dim V^{\mu} = n_{\mu}^2 \quad \text{and} \quad \sum_{\mu=1}^{m} n_{\mu} = n. \tag{2.6}$$

Performing left and right G-multiplications simultaneously, we obtain a left $G \times G$-action on V_G:

$$((g,h),\hat{v}) \mapsto \hat{g} \cdot \hat{v} \cdot \hat{h}^{-1}, \tag{2.7a}$$

which, by linear extension, yields an action of the corresponding group algebra $V_{G\times G} \cong V_G \otimes V_G$ on V_G:

$$V_{G\times G} \times V_G \ni \left(\sum_{g,h} \alpha(g,h)\, \hat{g} \otimes \hat{h}\,,\, \hat{v}\right) \overset{\rho}{\longmapsto} \sum_{g,h} \alpha(g,h)\, \hat{g} \cdot \hat{v} \cdot \hat{h}^{-1} \in V_G. \tag{2.7b}$$

The algebras of left and right multiplications are contained in $V_{G\times G}$ as subalgebras $V_G \otimes \hat{e}$ and $\hat{e} \otimes V_G$ respectively, with centralizers $V_G^c \otimes V_G$ and $V_G \otimes V_G^c$, where V_G^c denotes the centers of V_G. The images of these centralizers under ρ are isomorphic to V_G.

For what follows it will be convenient to employ a special basis of V_G which is adapted to the decomposition (2.5). We construct it by assuming we are given a complete set of unitary irreducible representation matrices $D^{\mu}_{ij}(g)$. Special choices within the unitary equivalence class can be made if required. By virtue of the orthogonality relations (e.g. [Ha]),

$$\frac{n_{\mu}}{n} \sum_{g} D^{\mu}_{ij}(g^{-1}) D^{\nu}_{kl}(g) = \delta_{\mu\nu}\delta_{il}\delta_{jk}, \tag{2.8a}$$

$$\sum_{\mu,i,j} \frac{n_{\mu}}{n} D^{\mu}_{ij}(g^{-1}) D^{\mu}_{ji}(h) = \delta_{gh}, \tag{2.8b}$$

we can use the D^{μ}_{ij} as coefficients for a new basis, $\{\hat{e}^{\mu}_{ij}\}$, of V_G, defined by

$$\hat{e}^{\mu}_{ij} := \frac{n_{\mu}}{n} \sum_{g} D^{\mu}_{ij}(g^{-1})\, \hat{g}\,, \tag{2.9a}$$

$$\text{and inversely} \quad \hat{g} = \sum_{\mu,i,j} \hat{e}^{\mu}_{ij} D^{\mu}_{ji}(g)\,. \tag{2.9b}$$

With respect to these two bases a general element $\hat{v} \in V_G$ has the expansions

$$\hat{v} = \sum_{g} v(g)\hat{g} = \sum_{\mu,i,j} v^{\mu}_{ij} \hat{e}^{\mu}_{ji} \tag{2.10a, b}$$

and from (2.9) we infer the transformation rules for the components

$$v_{ij}^{\mu} = \sum_{g} v(g) D_{ij}^{\mu}(g)\,, \tag{2.11a}$$

$$v(g) = \sum_{\mu,i,j} \frac{n_\mu}{n} v_{ij}^{\mu} D_{ji}^{\mu}(g^{-1})\,. \tag{2.11b}$$

Left and right $\hat{h}$-multiplications are now given by

$$\hat{h} \cdot \hat{e}_{ij}^{\mu} = \sum_{k} \hat{e}_{ik}^{\mu} D_{kj}^{\mu}(h), \tag{2.12a}$$

$$\hat{e}_{ij}^{\mu} \cdot \hat{h} = \sum_{k} D_{ik}^{\mu}(h) \hat{e}_{kj}^{\mu}. \tag{2.12b}$$

The rows and columns of $\hat{e}_{ij}^{\mu}$, considered as a matrix in ij, thus span left- and right-irreducible subspaces respectively, which we may take as our $V_{i,L}^{\mu}$ and $V_{i,R}^{\mu}$ in the decomposition (2.5b). For the algebra V_G this means that

$$V_{i,L}^{\mu} = \operatorname{span}\{\hat{e}_{i1}^{\mu}, \ldots, \hat{e}_{in_\mu}^{\mu}\} \quad \text{is a minimal left ideal,} \tag{2.13a}$$

$$V_{i,R}^{\mu} = \operatorname{span}\{\hat{e}_{1i}^{\mu}, \ldots, \hat{e}_{n_\mu i}^{\mu}\} \quad \text{is a minimal right ideal,} \tag{2.13b}$$

$$V^{\mu} = \bigoplus_{i}^{n_\mu} V_{i,\{L,R\}}^{\mu} \quad \text{is a minimal 2-sided ideal.} \tag{2.13c}$$

In terms of the basis $\{\hat{e}_{ij}^{\mu}\}$ the multiplication law can be easily inferred from (2.8a) and (2.9a):

$$\hat{e}_{ij}^{\mu} \cdot \hat{e}_{kl}^{\nu} = \delta_{\mu\nu} \delta_{il} \hat{e}_{kj}^{\mu}, \tag{2.14}$$

which implies that components (compare (2.10b)) just multiply like matrices:

$$(\hat{v} \cdot \hat{w})_{ij}^{\mu} = \sum_{k} v_{ik}^{\mu} w_{kj}^{\mu}. \tag{2.15}$$

Left and right multiplications by $\hat{e}_{ik}^{\mu}$ are then given by

$$\hat{e}_{ik}^{\mu} \cdot \hat{v} = \sum_{j} v_{ij}^{\mu} \hat{e}_{jk}^{\mu}, \tag{2.16a}$$

$$\hat{v} \cdot \hat{e}_{ik}^{\mu} = \sum_{j} v_{jk}^{\mu} \hat{e}_{ij}^{\mu}, \tag{2.16b}$$

which, in an obvious sense, say that left/right multiplication by $\hat{e}_{ik}^{\mu}$ results in writing the content of $V_{i,R}^{\mu}/V_{k,L}^{\mu}$ into $V_{k,R}^{\mu}/V_{i,L}^{\mu}$ and deletion of all other components.

Let us define $\hat{e}_{i}^{\mu} := \hat{e}_{ii}^{\mu}$ and $\hat{e}^{\mu} := \sum_i \hat{e}_{ii}^{\mu}$. It follows from (2.9$b$) that $\hat{e} = \sum_\mu \hat{e}^{\mu}$. The spaces V^{μ} form subalgebras with units $\hat{e}^{\mu}$. Left/right multiplication by $\hat{e}_{j}^{\mu}$ correspond to projection into $V^\mu_{j,R}/V^\mu_{j,L}$, as is easily seen from the following special cases of (2.14) and (2.9b):

$$\hat{e}^\mu_j \cdot \hat{e}^\nu_{ik} = \delta_{\mu\nu}\delta_{kj}\hat{e}^\nu_{ik}, \tag{2.17a}$$
$$\hat{e}^\nu_{ik} \cdot \hat{e}^\mu_j = \delta_{\mu\nu}\delta_{ij}\hat{e}^\nu_{ik}, \tag{2.17b}$$
$$\hat{e}^\mu_i \cdot \hat{e}^\nu_j = \delta_{\mu\nu}\delta_{ij}\hat{e}^\mu_i, \tag{2.18}$$
$$\sum_{\mu,i}\hat{e}^\mu_i = \hat{e}. \tag{2.19}$$

The projection into V^μ is given by right or left multiplication with $\hat{e}^\mu$. It follows that

$$A := \mathrm{span}\{\hat{e}^1_1,\ldots,\hat{e}^1_{n_1},\ldots\ldots,\hat{e}^m_1,\ldots,\hat{e}^m_{n_m}\} \tag{2.20}$$

is a maximal abelian subalgebra of V_G of dimension $\sum_{\mu=1}^m n_\mu$. Indeed, commutativity of $\hat{v} \in V_G$ with all elements of A implies that its projection into $V^\mu_{i,R}$ equals its projection into $V^\mu_{i,L}$ for all i. But the intersection $V^\mu_{i,L} \cap V^\mu_{i,R}$ is the ray spanned by $\hat{e}^\mu_i$. Thus $\hat{v}$ must be in A which shows maximality. In comparison, the centre V^c_G of V_G is also easily determined, for $\hat{v}\cdot\hat{g} = \hat{g}\cdot\hat{v}$ $\forall g \in G$ implies via (2.12) that $\sum_i v^\mu_{ji}D^\mu_{ik}(g) = \sum_i D^\mu_{ji}(g)v^\mu_{ik}$, $\forall g \in G$. Schur's Lemma then yields $v^\mu_{ij} = v^\mu\delta_{ij}$, so that

$$V^c_G = \mathrm{span}\{\hat{e}^1,\ldots,\hat{e}^m\}. \tag{2.21}$$

Note that, unless G is abelian, the centre of the group algebra contains but is not equal to the group algebra of the centre, G_c, of G. For example, for non-abelian G, $\sum_g \hat{g}$ is in V^c_G but not in the group algebra of G_c.

The projection maps $\hat{v} \mapsto \hat{e}^\mu \cdot \hat{v} = \hat{v}\cdot\hat{e}^\mu$ are homomorphisms from V_G onto the subalgebras V^μ. Left and right actions of V_G on the V^μ's thus factor through these projections. The centralizers Z^μ of V^μ are easily seen to be given by the subalgebras

$$Z^\mu = \left\{\bigoplus_{\nu\neq\mu} V^\nu\right\} \oplus \mathrm{span}\{\hat{e}^\mu\}\,. \tag{2.22}$$

Obviously we have $Z^\mu = V_G$, iff the μ-th representation is abelian. From (2.9) it follows that $\hat{g} \in Z^\mu$, iff $D^\mu_{ij}(g) = \eta(g)\delta_{ij}$. This is the case iff $g \in C^\mu := \{h \in G\,/\,D^\mu(hf) = D^\mu(fh)\ \forall f \in G\}$. C^μ is a normal subgroup of G and $D^\mu(g) = D^\mu(g_G)$ for any $g \in C^\mu$, where $g_G \subset C^\mu$ denotes the conjugacy class of g in G.

Whereas V_G decomposes unambiguously into the V^μ's for both left and right multiplication, our choice of left and right irreducible subspaces $V^\mu_{i,\{R,L\}}$ is not unique. To see what freedom there is, we prove the following

Lemma. *Let $\hat{v} \in V_G$. The following statements are equivalent. (i) $\hat{v}$ lies in a left-irreducible subspace, (ii) $\hat{v}$ lies in a right irreducible subspace, (iii) $\hat{v}$ has expansion coefficients $v^\mu_{ij} = \delta_{\mu\nu}a_ib_j$ for some complex valued n_μ-tuples $\{a_i\}$ and $\{b_i\}$.*

Proof. We shall only prove $(i) \Leftrightarrow (iii)$ since $(ii) \Leftrightarrow (iii)$ is entirely analogous. $(iii) \Rightarrow (i)$ is trivial. Conversely, assuming that $\hat{v}$ lies in a left-irreducible subspace, we know from (2.5b) that it must lie in an n_ν-dimensional subspace of some V^ν, which for the moment we call L. This explains the $\delta_{\mu\nu}$ in (iii). We set $\hat{v} = \sum_{i,j} v_{ij}\hat{e}^\nu_{ji}$. Left multiplications by $\hat{e}^\nu_{kl}$ for all $k,l \in \{1,\ldots,n_\nu\}$ produces the $n_\nu \times \mathrm{rank}\{v_{ij}\}$ linearly independent vectors $\hat{e}^\nu_{kl}\cdot\hat{v} = \sum_j v_{kj}\hat{e}^\nu_{jl}$ in L. But L is only n_ν-dimensional so that $\mathrm{rank}\{v_{ij}\} = 1 \Leftrightarrow v_{ij} = a_i b_j$ •

This shows that any other adapted basis, i.e., where each basis vector lies in an irreducible subspace, is necessarily of the form (matrix notation)

$$\hat{\eta}^\mu = M\hat{e}^\mu N^{-1}\,, \quad M,N \in GL(n_\mu, C), \tag{2.23}$$

so that the left and right actions of G are now represented equivalently to (2.12):

$$\hat{g}\cdot\hat{\eta}^\mu = \hat{\eta}^\mu(ND^\mu(g)N^{-1}), \tag{2.24a}$$
$$\hat{\eta}^\mu\cdot\hat{g} = (MD^\mu(g)M^{-1})\hat{\eta}^\mu. \tag{2.24b}$$

So far we can therefore stick to any particular choice of representation matrices in (2.9a).

If we denote by $\{e_i\}$ the standard basis in C^{n_μ}, we can employ the isomorphism $\sigma : V^\mu \to C^{n_\mu}\otimes C^{n_\mu}$, defined by

$$\sigma(\hat{e}^\mu_{ij}) := e_j \otimes e_i, \tag{2.25}$$

to identify V^μ and $C^{n_\mu}\otimes C^{n_\mu}$ for each μ. We shall occasionally use this identification without explicitly mentioning σ. As pointed out in (2.15), left and right multiplications then act only on the left and right C^{n_μ} respectively. From the previous Lemma we infer that $\hat{v}$ is an element in an irreducible subspace, iff it is a pure tensor product $a\otimes b$, $a,b \in C^{n_\mu}$ for some μ. This set of pure tensor products (also called rank=1 vectors) is not a linear space, but contains the linear spaces

$$R^\mu(a) := \mathrm{span}\{a\otimes e_1,\ldots,a\otimes e_{n_\mu}\}, \tag{2.26a}$$
$$L^\mu(a) := \mathrm{span}\{e_1\otimes a,\ldots,e_{n_\mu}\otimes a\}, \tag{2.26b}$$

which comprise all the left- and right-irreducible subspaces if a runs through all of C^{n_μ} and μ through all values of 1 to m. Two different vectors a and a' characterize the same irreducible subspace, iff $a = \alpha a'$ for some $\alpha \in C-\{0\}$. The space of left- or right-irreducible subspaces within V^μ can thus be identified with the complex projective space $CP^{n_\mu-1}$ of real dimension $2(n_\mu - 1)$.

Next we wish to introduce an inner product on V_G, denoted by $\langle\cdot|\cdot\rangle$ (antilinear in the first entry). Since right V_G-multiplications will eventually play the rôle of gauge symmetries in our application, we require it to be right invariant. This leads to the following string of equations (generally an overbar over C-valued quantities denotes complex-conjugation):

$$\langle\hat{e}^\mu_{ik}|\hat{e}^\nu_{lm}\rangle = \langle\hat{e}^\mu_{ik}\cdot g|\hat{e}^\nu_{lm}\cdot g\rangle \tag{2.27a}$$
$$= \frac{1}{n}\sum_{g\in G}\langle\hat{e}^\mu_{ik}\cdot g|\hat{e}^\nu_{lm}\cdot g\rangle \tag{2.27b}$$

$$=\frac{1}{n}\sum_{g\in G}\sum_{r,s}\bar{D}^{\mu}_{ir}(g)D^{\nu}_{ls}(g)\langle\hat{e}^{\mu}_{rk}|\hat{e}^{\nu}_{sm}\rangle \tag{2.27c}$$

$$=\delta_{\mu\nu}\delta_{il}\frac{1}{n_\mu}\sum_r\langle\hat{e}^{\mu}_{rk}|\hat{e}^{\mu}_{rm}\rangle \tag{2.27d}$$

$$=:\ \delta_{\mu\nu}\delta_{il}S^{\mu}_{km}, \tag{2.27e}$$

where we have used unitarity of the representation matrices D^μ in the second to last step for the first time. So far no choice within the equivalence class of unitary representations matrices was specified. A redefinition within the unitary equivalence class implies (matrix notation)

$$D^\mu \mapsto U^\mu D^\mu (U^\mu)^\dagger, \tag{2.28a}$$

$$\hat{e}^\mu \mapsto U^\mu \hat{e}^\mu (U^\mu)^\dagger, \tag{2.28b}$$

$$S^\mu \mapsto U^\mu S^\mu (U^\mu)^\dagger. \tag{2.28c}$$

In general we could use it to diagonalize the Hermitean matrix S^μ. We call its eigenvalues λ^μ_k, $k=1,\ldots,n_\mu$, and get from (2.27)

$$\langle\hat{e}^{\mu}_{ik}|\hat{e}^{\nu}_{lm}\rangle=\delta_{\mu\nu}\delta_{il}\delta_{km}\lambda^\mu_k\,. \tag{2.29}$$

This formula is still completely general. Choosing an inner product now corresponds to picking $\sum_{\mu=1}^m n_\mu$ coefficients λ^μ_k. For our later applications we make the particular choice:

$$\lambda^\mu_k=\lambda^\mu=\frac{n_\mu}{n^2}\,. \tag{2.30}$$

Independence of the lower index is in fact a necessary and sufficient condition to make the right-invariant inner product also left-invariant. It also means that we actually did not restrict our choice of unitary representation matrices at all, so that all redefinitions (2.28) are still at our disposal. Proportionality of λ^μ to n_μ implies that $\hat{g}$ and $\hat{h}$ are orthogonal for for $g\neq h$. Indeed, using (2.8b) and (2.9b), we obtain

$$\langle\hat{g}|\hat{h}\rangle=\sum_{\mu,j}D^\mu_{jj}(hg^{-1})\lambda^\mu==\frac{1}{n}\delta_{gh}\,. \tag{2.31}$$

A linear operator on V_G is said to be right-invariant if its matrix elements satisfy the analogous condition to (2.27a). If O is such an operator, we have in analogy to (2.27)

$$O^{\mu,\nu}_{ik,lm}:=\langle\hat{e}^{\mu}_{ik}|O|\hat{e}^{\nu}_{lm}\rangle=\delta_{\mu\nu}\delta_{il}O^\mu_{km}, \tag{2.32a}$$

$$O^\mu_{km}:=\frac{1}{n_\mu}\sum_r\langle\hat{e}^{\mu}_{rk}|O|\hat{e}^{\mu}_{rm}\rangle. \tag{2.32b}$$

On the other hand, using the completeness relation (where we now employ Dirac's notation of |bra⟩ and ⟨ket| vectors)

$$1=\sum_{\mu,i,k}|\hat{e}^\mu_{ik}\rangle\frac{1}{\lambda^\mu}\langle\hat{e}^\mu_{ik}|, \tag{2.33}$$

we can write

$$O = \mathbf{1}O\mathbf{1} = \sum_{\mu,i,k,m} |\hat{e}^{\mu}_{ik}\rangle \frac{1}{\lambda^{\mu}} O^{\mu}_{km} \frac{1}{\lambda^{\mu}} \langle \hat{e}^{\mu}_{im}|, \tag{2.34}$$

so that O's action on $\hat{v}$ can be reformulated, using (2.10b) and (2.29), as a left multiplication

$$O\hat{v} = \sum_{\mu,i,k,m} \frac{1}{\lambda^{\mu}} O^{\mu}_{km} v^{\mu}_{mi} \hat{e}^{\mu}_{ik} \tag{2.35a}$$

$$= \left(\sum_{\mu,i,k} \frac{1}{\lambda^{\mu}} O^{\mu}_{ik} \hat{e}^{\mu}_{ki} \right) \cdot \hat{v} =: \hat{o} \cdot \hat{v}\,. \tag{2.35b}$$

Clearly, O is Hermitean, iff $O^{\mu}_{ik} = \bar{O}^{\mu}_{ki}$. (2.35) says that any right-invariant Hermitean operator is given by left multiplication with an element $\hat{o} \in V_G$ whose coefficients with respect to the bases $\{\hat{e}^{\mu}_{ij}\}$ and $\{\hat{g}\}$ satisfy respectively

$$o^{\mu}_{ij} = \bar{o}^{\mu}_{ji} \Leftrightarrow o(g) = \bar{o}(g^{-1})\,. \tag{2.36}$$

Since the algebra V_G acts as operators on its underlying vector space, these last relations have intrinsic meaning on V_G once an inner product is introduced. In fact, any inner product on V_G defines a $*$-operation $V_G \to V_G$, which is antilinear and satisfies $* \circ * = 1$, through, say, left multiplication:

$$\langle \hat{v} | \hat{o} \cdot \hat{w} \rangle =: \langle \hat{o}^* \cdot \hat{v} | \hat{w} \rangle\,. \tag{2.37}$$

Alternatively, we could have defined the $*$-operation via right multiplication which in the general case would have led to a different $*$-map. However, if the inner product is right- and left-invariant, the two definitions for the $*$-operations agree. In this case it follows immediately from (2.37) that

$$\Big\{ \sum_g o(g)\hat{g} \Big\}^* = \sum_g \bar{o}(g^{-1})\hat{g} \quad \text{and} \quad \Big\{ \sum_{\mu,i,j} o^{\mu}_{ij} \hat{e}^{\mu}_{ji} \Big\}^* = \sum_{\mu,i,j} \bar{o}^{\mu}_{ji} \hat{e}^{\mu}_{ji}\,. \tag{2.38}$$

In particular, $\hat{g}^* = \hat{g}^{-1}$ and $\hat{e}^{\mu\,*}_{ij} = \hat{e}^{\mu}_{ji}$, which, by (2.13), implies that $\{V^{\mu}\}^* = V^{\mu}$ and $\{V^{\mu}_{i,R}\}^* = V^{\mu}_{i,L}$.

Algebras with such a $*$-operation are called H^* algebras and elements invariant under $*$ are called self-adjoint, or Hermitean. The elements $\hat{e}^{\mu}_{i}$ introduced earlier correspond to mutually orthogonal Hermitean idempotents, as do the elements $\hat{e}^{\mu}$. The latter ones are however decomposable into the former, which are themselves indecomposable (i.e. so-called primitive idempotents). The subalgebras V^{μ} are mutually orthogonal H^* subalgebras. In particular, the V^{μ}'s are also minimal 2-sided H^* ideals. (The split (2.5a) is thus still valid in the sense of H^* algebras.) In contrast, since the subspaces $V^{\mu}_{i,\{L,R\}}$ are not invariant under $*$, they do not form any H^* ideals.

This basically concludes our presentation of the group algebra. In the fourth section we shall discuss the decomposition of the quantum mechanical state space according to an inherited V_G-action. Not surprisingly, it will be very similar to the decomposition of V_G under the regular representation. In fact, we can immediately build a finite-dimensional toy model with all the essential features. This we will do first in order to introduce some general concepts and notations in a simple context. Then we turn to the general quantum mechanical case.

3 General Concepts and a Toy Model

Consider the n-dimensional Hilbert space $\mathcal{H} = V_G$ with inner product $\langle\cdot|\cdot\rangle$ and the right regular representation of V_G on $\mathcal{H}$. As we have seen, it is useful with respect to V_G's action to represent $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_{\mu=1}^{m} \mathcal{H}^{\mu} = \bigoplus_{\mu=1}^{m} C^{n_\mu} \otimes C^{n_\mu}. \tag{3.1}$$

We call $B(\mathcal{H})$ the algebra of bounded (a redundant adjective in finite dimensions) linear operators, which here is isomorphic to the matrix algebra $M(n, C)$. We wish to regard G as a gauge group with gauge algebra V_G, that is, we require observables to commute with the action of the group G (such transformations are called supersymmetries in [JM]). The algebra of observables, $\mathcal{O}$, is thus defined as the *commutant* of (the right-) V_G *in* $B(\mathcal{H})$, denoted by V_G'. Quite generally, given any set $S \subset B(\mathcal{H})$, the set of operators commuting elementwise with S forms an algebra, called the commutant, S', of S. The double commutant, S'', is easily seen to be just the algebra generated by S. It is stated in (2.35*b*) that $\mathcal{O}$ is isomorphic to the algebra of left V_G multiplications, which, as e.g. expressed by (2.15), one may identify with a direct sum of matrix algebras:

$$\mathcal{O} = \bigoplus_{\mu=1}^{m} M(n_\mu, C)\,, \tag{3.2}$$

where each matrix algebra $M(n_\mu, C)$ acts on the left C^{n_μ}-factor in (3.1). The representation of $\mathcal{O}$ in $\mathcal{H}$ is thus highly reducible. Whenever the algebra of observables is represented in a reducible fashion, the pair $(\mathcal{H}, \mathcal{O})$ is said to contain *superselection rules.* In what follows, we shall investigate more into the structure of these rules. More precisely, we are interested in the geometric structure of those subsets of $\mathcal{H}$ that represent pure states, where this has always to be understood *relative to* $\mathcal{O}$. As a word of principle, and as indicated by the word 'relative' , we do not wish to regard states as being attributed with any more status over and above that which suffices to answer all the questions contained in $\mathcal{O}$.

The centre of $\mathcal{O}$ is the m-dimensional algebra generated by the projection operators, $T^\mu : \mathcal{H} \to \mathcal{H}^\mu$, given by left or right multiplications with $\hat{e}^\mu$. Obviously, vectors representing pure states must always lie in some $\mathcal{H}^\mu$, for, given the sum of two nonzero vectors $v \in \mathcal{H}^\mu$ and $w \in \mathcal{H}^\nu$, where $\mu \neq \nu$, the density matrix for the pure state $|v\rangle + |w\rangle$, considered as a positive linear functional (the expectation value) on $\mathcal{O}$, is identical to the mixed state $|v\rangle\langle v| + |w\rangle\langle w|$. However, the converse is not true unless μ labels an abelian representation. Let us therefore focus on a higher-dimensional $\mathcal{H}^\mu$. It may be considered as the composite state space of two systems, called left and right, with individual state spaces C^{n_μ}, and where we have no observables for the right system. To say that a vector in $\mathcal{H}^\mu$ represent a pure state now means the following: express it as a density matrix, form the reduced density matrix for the left system by tracing out the right system, then this reduced density matrix is pure. We know from elementary quantum mechanics that this is the case iff the original vector in $\mathcal{H}^\mu$ was a pure tensor product (i.e. of rank one). Taken together with the lemma above, we arrive at the following statement: a vector in $\mathcal{H}$ represents a pure state, iff it lies in a left invariant subspace $L^\mu(b)$. We can represent it by a matrix with components $a_i b_j$. Observables act on the left index, gauge transformations on the right. $\mathcal{O}$ acts irreducibly on $L^\mu(b)$ in which any two rays can be separated by $\mathcal{O}$. However, for each such ray there is a unique ray in each $L^\mu(b')$, $b' \neq b$, which gives the *same* state for $\mathcal{O}$. We have thus seen that, with respect to $\mathcal{O}$, the different left invariant subspaces are indistinguishable so that a pure state is represented by a ray in each left invariant subspace. This is equivalent to saying that a pure state corresponds uniquely to a whole right invariant subspace $R^\mu(a)$. That higher than one-dimensional subspaces should represent quantum mechanical states has already been discussed in the mid 60's in the context of parastatistics [MG], where these subspaces were called *generalized rays*. There is nothing inconsistent with this kind of higher-dimensional redundancy. For example, the superposition principle takes the following form: three states (generalized rays) $R^\mu(a)$, $R^\mu(a')$, and $R^\mu(a'')$ are said to be linearly dependent, iff a lies in the plane determined by a' and a''. Alternatively, given three rays in each left invariant subspace, then the three states they define are said to be linearly dependent, iff in each left invariant subspace the rays lie in a plane. It is clear that this is either simultaneously true in all or none of the subspaces. This definition coincides with the more abstract prescription given in [Ho].

Although there is nothing wrong with generalized rays, they do seem to carry unnecessary redundancy as far as the representation of $\mathcal{O}$ is concerned{4}. This can be expressed in rational terms in a variety of ways. For example, in ordinary quantum mechanics, one often hears *Dirac's requirement: There exists a complete set of commuting observables* [Di]. Let us call them $\{A_i\}$. Here, by definition, completeness means that a set of simultaneous eigenvalues determine a ray uniquely. This statement works for finite-dimensional Hilbert spaces but has to be replaced in infinite dimensions, where, because of continuous spectra, the proper notion of eigenvectors does not exist. But this can be cured by a slight reformulation [J]: Let $\mathcal{A} = \{A_i\}'' \subset \mathcal{O}$ be the abelian algebra generated by the set $\{A_i\}$. The set is said to be complete, iff $\mathcal{A}$ is a maximal abelian subalgebra{5} of $B(\mathcal{H})$, that is, iff $\mathcal{A}' = \mathcal{A}$. See [J][JM] for more details and [Wi2] for a recent review. The generally valid replacement for Dirac's formulation is *Jauch's requirement: $\mathcal{O}$ contains a maximal abelian subalgebra of $B(\mathcal{H})$*{6}. It is clear that in our case the failure to meet these requirements has to do with the existence of different rays that cannot be separated by $\mathcal{O}$, or equivalently, that $\mathcal{O}$ does not contain all the projectors onto rays representing

{4} However, note that $\mathcal{H}$ and $\mathcal{O}$ were not independently given: $\mathcal{O}$ was defined as the commutant of V_G *in* $B(\mathcal{H})$.

{5} The commutant $\{A_i\}'$ is always a von Neumann algebra, that is, equal to its double commutant.

{6} Standard formulations in the literature usually do not make this explicit reference to $B(\mathcal{H})$. We put it to emphasize the dependence of this statement on $\mathcal{H}$

pure states. That this is entirely due to the non-commutativity of the gauge group G is made manifest by an equivalent formulation of Jauch's requirement, due to Wightman [Wi1]. It is also known as the requirement (or hypothesis) of *commutative (or abelian) superselection rules.* We call it *Wightman's requirement: The commutant $\mathcal{O}'$ of $\mathcal{O}$ in $B(\mathcal{H})$ is abelian.* We emphasize that $\mathcal{O}$ was assumed to be a von Neumann algebra{7}. See e.g. [GMN] for a simple proof of the equivalence. It tells us that we cannot keep a non-commutative gauge group if we want to get rid of generalized rays.

Although generalized rays do no harm, they are also not necessary for the formulation of a quantum mechanical state space incorporating all the pure states for $\mathcal{O}$. We demonstrate this "elimination of the generalized ray" [HT] in our model, which highlights in an elementary fashion the last remark of the previous paragraph. The method is simple: we truncate $\mathcal{H}$ by selecting an $a \in C^{n_\mu}$, say $a = e_1$, and keep only $L^\mu(a) =: \mathcal{H}^\mu_{\mathrm{tr}}$ for each μ. Within this space we would then have the standard bijection between pure states and rays representing them. This amounts to truncating the Hilbert space representing states for $\mathcal{O}$ to

$$\mathcal{H}_{\mathrm{tr}} = \bigoplus_{\mu=1}^{m} \mathcal{H}^\mu_{\mathrm{tr}}, \tag{3.3}$$

where of course $\mathcal{H}^\mu_{\mathrm{tr}} = \mathcal{H}^\mu$, iff μ is abelian. Note that no pure state has been lost. Only redundancies have been eliminated. Pure states are in bijective correspondence with rays in the subset

$$\bigcup_{\mu=1}^{m} \mathcal{H}^\mu_{\mathrm{tr}} \subset \mathcal{H}. \tag{3.4}$$

In fact, the space of rays in this subset is just the disjoint union of the spaces of rays in each $\mathcal{H}^\mu_{\mathrm{tr}}$. The representations of $\mathcal{O}$ on $\mathcal{H}$ and $\mathcal{H}_{\mathrm{tr}}$ differ only by trivial multiplicities. In both cases $\mathcal{O}$ is isomorphic to

$$\mathcal{O} = \bigoplus_{\mu=1}^{m} M(n_\mu, C) = \bigoplus_{\mu=1}^{m} B(\mathcal{H}^\mu_{\mathrm{tr}})\,. \tag{3.5}$$

But in the first case each $M(n_\mu, C)$ appears with multiplicity n_μ. Representations related in this fashion are therefore called *phenomenologically equivalent* [BLOT]. The price for this elimination is that the symmetry group does not act on $\mathcal{H}_{\mathrm{tr}}$ anymore. What remains from the gauge algebra V_G is a residual action of its centre V_G^c which is now generated by the projections $T^\mu : \mathcal{H}_{\mathrm{tr}} \to \mathcal{H}^\mu_{\mathrm{tr}}$. Clearly the commutant $\mathcal{O}'$ of $\mathcal{O}$ in $B(\mathcal{H}_{\mathrm{tr}})$ just satisfies Wightman's requirement. Equivalently, Jauch's requirement is satisfied, since projectors onto rays are now all in $\mathcal{O}$ and any abelian subalgebra generated by a complete set of orthogonal projectors is maximal in $B(\mathcal{H}_{\mathrm{tr}})$. In a sense, $\mathcal{H}$ was too big for $\mathcal{O}$ and $\mathcal{H}_{\mathrm{tr}}$ is the most economical way to represent the pure states of $\mathcal{O}$. As we have seen, the projectors onto different $L^\mu(a)$ were not in $\mathcal{O}$, only the sum of projectors onto the mutually orthogonal $L^\mu(e_i)$ was.

{7} The von Neumann property of $\mathcal{O}$ is not necessary to prove the implication Wightman $\Rightarrow$ Jauch, but for the converse, therefore showing that without the von Neumann property Wightman's requirement is logically weaker.

Finally we note that there is a way to satisfy the Jauch-Wightman requirement *and* have the full gauge group G being reduced by the state space, and that is to just truncate the sum in (3.3) to include only abelian representations. This in fact is an often adopted point of view since it conforms with two seemingly obvious requirements. It has e.g. been used to "prove" the impossibility of parastatistics in a quantum mechanical framework [GMN]. In this work we reject this rather ad hoc procedure on the grounds that it unnecessarily discards the potentially interesting non-abelian sectors (i.e. those for which μ labels a non-abelian representation). For example, non-abelian sectors are in fact used in the theory of deformed nuclei. This is explained in appendix B. Generally speaking, it is a perfectly legitimate procedure to use the gauge group to find all the sectors and then, in order to conform with the Jauch-Wightman requirement, sacrifice its action up to an abelian residue. Whoever wants to have the gauge group still acting might work with generalized rays. This viewpoint is also expressed in [MG] and [HT].

Note that whereas it is true that only the centre of the gauge algebra acts on $\mathcal{H}_{\rm tr}$ a larger part of it does act on a specific $\mathcal{H}^{\mu}_{\rm tr}$ considered in isolation. Precisely that subalgebra of V_G acts on $\mathcal{H}^{\mu}_{\rm tr}$ which commutes with V_G under the μ-th representation. In the previous section this subalgebra has been called Z^{μ} (compare (2.22)). As discussed there, the corresponding part of the gauge group that still acts on $\mathcal{H}^{\mu}_{\rm tr}$ is given by C^{μ}. The way it acts is obvious, since commutativity allows us to write it as left-multiplication.

4 The Non-Abelian Case

As in section 1, we denote by $\mathcal{H}$ the Hilbert space $L^2(\bar{Q}, d\bar{q})$ with right invariant measure $d\bar{q}$. The right action of G on $\bar{Q}$ induces a right action of G on $\mathcal{H}$, defined by

$$(g, \psi) \rightarrow T_g\psi := \psi \circ R_{g^{-1}}. \tag{4.1}$$

It is an isometry due to the right-invariance of the measure. Linear extension yields a right V_G-action on $\mathcal{H}$:

$$(\hat{v}, \psi) \rightarrow T_{\hat{v}}\psi := \sum_{g\in G} v(g)\, \psi \circ R_{g^{-1}}. \tag{4.2}$$

We also introduce a second Hilbert space, $\hat{\mathcal{H}}$, as completion of V_G-valued, equivariant functions on $\bar{Q}$ which are square integrable. The point of doing this is that this Hilbert space is unitarily isomorphic to $\mathcal{H}$ (see (4.9) below) but displays the representation properties under the action of V_G in a more direct way. Equivariance means

$$\hat{\psi} \circ R_g = \hat{g}^{-1} \cdot \hat{\psi}. \tag{4.3}$$

The inner product on $\hat{\mathcal{H}}$, denoted by $(\cdot|\cdot)$, is given by

$$(\hat{\psi}|\hat{\phi}) := \int_{\bar{Q}} \langle \hat{\psi}(\bar{q})|\hat{\phi}(\bar{q})\rangle \, d\bar{q}, \tag{4.4}$$

where $\langle\cdot|\cdot\rangle$ is defined by (2.31). Expanding $\hat{\psi} \in \hat{\mathcal{H}}$ in components,

$$\hat{\psi} = \sum_{h \in G} \hat{h}\,\psi_h, \tag{4.5}$$

then (4.3) implies for the component functions

$$\psi_h \circ R_g = \psi_{gh}\,. \tag{4.6}$$

We now define the linear maps

$$\mathcal{F} : \mathcal{H} \to \hat{\mathcal{H}}, \quad \psi \mapsto \mathcal{F}(\psi) := \sum_{g \in G} \hat{g}\,\psi \circ R_g, \tag{4.7a}$$

$$\mathcal{E} : \hat{\mathcal{H}} \to \mathcal{H}, \quad \hat{\psi} \mapsto \mathcal{E}(\hat{\psi}) := \psi_e, \tag{4.7b}$$

where ψ_e is the component of $\hat{e}$ in the expansion (4.5). It is easy to check that $\mathcal{F}(\psi)$ is indeed equivariant. We have

$$\mathcal{E} \circ \mathcal{F} = \mathrm{Id}\Big|_{\mathcal{H}}, \quad \mathcal{F} \circ \mathcal{E} = \mathrm{Id}\Big|_{\hat{\mathcal{H}}}. \tag{4.8}$$

The first equation is obvious, the second follows from (4.6). Hence $\mathcal{E} = \mathcal{F}^{-1}$. Moreover, we have (an overbar over ψ denotes complex conjugation)

$$\int_{\bar{Q}} \langle \mathcal{F}(\psi)(\bar{q}) | \mathcal{F}(\phi)(\bar{q})\rangle \, d\bar{q} = \sum_{g,h} \int_{\bar{Q}} \langle \hat{g} | \hat{h}\rangle \, \bar{\psi}(\bar{q}g)\phi(\bar{q}h)\, d\bar{q} = \int_{\bar{Q}} \bar{\psi}(\bar{q})\phi(\bar{q})\, d\bar{q}, \tag{4.9}$$

where we used (2.31) in the last step. Hence $\mathcal{F}$ establishes an unitary isomorphism between $\mathcal{H}$ and $\hat{\mathcal{H}}$. The action T of V_G on $\mathcal{H}$ can now be transferred to an action $\hat{T}$ of V_G on $\hat{\mathcal{H}}$ via

$$(\hat{v}, \hat{\psi}) \to \hat{T}_{\hat{v}}(\hat{\psi})\,, \quad \hat{T}_{\hat{v}} := \mathcal{F} \circ T_{\hat{v}} \circ \mathcal{E} \tag{4.10}$$

which yields, using (4.7) and (4.6),

$$\begin{aligned}
\hat{T}_{\hat{v}}(\hat{\psi}) &= \mathcal{F} \circ T_{\hat{v}}(\psi_e) = \mathcal{F}\left(\sum_h v(h)\psi_e \circ R_{h^{-1}}\right) \\
&= \sum_{g,h} \hat{g} v(h)\psi_e \circ R_{h^{-1}} \circ R_g = \sum_{g,h} \hat{g} v(h)\psi_e \circ R_{gh^{-1}} \\
&= \sum_{f,h} \hat{f} \cdot \hat{h}\, v(h)\psi_f = \hat{\psi} \cdot \hat{v}\,.
\end{aligned} \tag{4.11}$$

Hence V_G's action on $\hat{\mathcal{H}}$ just corresponds to pointwise right multiplication. Note that a pointwise left multiplication is not defined within $\hat{\mathcal{H}}$ since the resulting function would generally not be equivariant. But there is such an action of left multiplications if one restricts to the centre V_G^c.

Linear operators $\bar{O}$ on $\mathcal{H}$ whose integral kernels satisfy (1.12) ((1.13) for propagators) define linear Operators $\hat{O}$ on $\hat{\mathcal{H}}$ via $\hat{O} := \mathcal{F} \circ \bar{O} \circ \mathcal{E}$ (for propagators: $\hat{K}(t';t) := \mathcal{F} \circ \bar{K}(t';,t) \circ \mathcal{E}$). As in (4.11), we can easily derive the following explicit expressions

$$(\hat{O}\hat{\psi})(\bar{q}') = \int_{\bar{Q}} \bar{O}(\bar{q}';\bar{q})\hat{\psi}(\bar{q})\, d\bar{q}; \tag{4.12a}$$

$$(\hat{K}(t',t)\hat{\psi})(\bar{q}') = \int_{\bar{Q}} \bar{K}(\bar{q}',t';\bar{q},t)\hat{\psi}(\bar{q})\, d\bar{q}\,, \tag{4.12b}$$

which show that these operators just act componentwise on the functions $\hat{\psi}$, thus displaying manifestly the commutativity with the right V_G-action:

$$\hat{O} \circ \hat{T}_{\hat{v}} = \hat{T}_{\hat{v}} \circ \hat{O}\,, \quad \hat{K}(t',t) \circ \hat{T}_{\hat{v}} = \hat{T}_{\hat{v}} \circ \hat{K}(t',t). \tag{4.13}$$

Since the algebra V_G now acts on the infinite-dimensional space $\hat{\mathcal{H}}$ (or $\mathcal{H}$), we slightly adapt the basic notations from the previous section. $B(\hat{\mathcal{H}})$ is the C^*-algebra of all bounded linear operators on $\hat{\mathcal{H}}$ (similarly with $\mathcal{H}$). Through the implementation (4.11), V_G is mapped linearly and anti-homomorphically (because of the right-multiplication) onto a subalgebra of $B(\hat{\mathcal{H}})$, which we call $\mathcal{V}_G$. It is not difficult to show that $\mathcal{V}_G$ is in fact a von Neumann algebra. A proof may be found in [GMN]{8}. The actions of $\hat{g}$ or $\hat{e}^{\mu}_{ij}$ on $\hat{\mathcal{H}}$ according to (4.11) are denoted by the linear operators $\hat{T}_g$ or $\hat{T}^{\mu}_{ij}$ respectively. Accordingly, the linear operators corresponding to right $\hat{e}^{\mu}_{i}$- and $\hat{e}^{\mu}$-multiplications are projection operators which we call $\hat{T}^{\mu}_{i}$ and $\hat{T}^{\mu}$. They satisfy

$$\hat{T}^{\mu}_{ij}\hat{T}^{\nu}_{kl} = \delta_{\mu\nu}\delta_{jk}\hat{T}^{\mu}_{il}, \tag{4.14}$$

which follows directly from (4.11) and (2.14).

All the H^*-structural properties of V_G are inherited by $\mathcal{V}_G$, which makes it at the same time an H^* and a von Neumann algebra. From the definition of the scalar product (4.4) it is obvious that the two $*$-involutions so defined coincide. In particular, $\hat{T}^{\mu}_{i}$ and $\hat{T}^{\mu}$ are self-adjoint idempotents, i.e., projection operators. The image of the subalgebras V^{μ}, V^{μ}_{c}, Z^{μ} and A will be called $\mathcal{V}^{\mu}$, $\mathcal{V}^{\mu}_{c}$, $\mathcal{Z}^{\mu}$ and $\mathcal{A}$ respectively. For any subset $S \subset B(\hat{\mathcal{H}})$, S' is the commutant which is in fact a von Neumann algebra. S'' is called the von Neumann algebra generated by S, which is equal to S in case S is already a von Neumann algebra.

Let us now look at the Hilbert space $\hat{\mathcal{H}}$. We define the *algebra of observables*, $\mathcal{O}$, by $\mathcal{O} := (\mathcal{V}_G)'$. Its commutant then satisfies $\mathcal{O}' = \mathcal{V}_G$. Further, the projection operators $\hat{T}^{\mu}$ and $\hat{T}^{\mu}_{i}$ define a split analogous to (2.5)

$$\hat{\mathcal{H}} = \bigoplus_{\mu=1}^{m} \hat{\mathcal{H}}^{\mu}, \tag{4.15a}$$

$$\hat{\mathcal{H}}^{\mu} = \bigoplus_{i=1}^{n_\mu} \hat{\mathcal{H}}^{\mu}_{i}, \tag{4.15b}$$

{8} Although this reference is primarily concerned with the symmetric group, the proof given there works literally for any finite group.

where $\hat{\mathcal{H}}^\mu = \hat{T}^\mu \hat{\mathcal{H}}$ and $\hat{\mathcal{H}}_i^\mu = \hat{T}_i^\mu \hat{\mathcal{H}}$. The functions in these Hilbert spaces are just given by the V^μ- and $L^\mu(e_i)$-valued functions in $\hat{\mathcal{H}}$ respectively.

The second split of course inherits the non-uniqueness from (2.5*b*). Under a redefinition (2.28) we just have to analogously conjugate the matrix $\hat{T}^\mu$ by U^μ. For example, given a normalized $a = \sum_i a_i e_i \in C^{n_\mu}$, we can choose it as the first basis vector of a new basis $e'_i = \sum_j U^\mu_{ij} e_j$ with $a_i = U^\mu_{1i}$. The projection operator onto $L^\mu(a)$-valued functions is then given by $\hat{T}^\mu(a) := \sum_{i,j} a_i \bar{a}_j \hat{T}^\mu_{ij}$.

The operators and propagators in (4.12) now project into each subspace:

$$\hat{O}^\mu := \hat{T}^\mu \circ \hat{O} \circ \hat{T}^\mu, \tag{4.16a}$$

$$\hat{O}^\mu_i := \hat{T}^\mu_i \circ \hat{O} \circ \hat{T}^\mu_i, \tag{4.16b}$$

where, since $O \in \mathcal{O}$, the left projection operators are not really necessary. The analogous formulae hold for the propagator. It is then obvious that the projected propagators in $\hat{\mathcal{H}}^\mu_i$ satisfy the standard combination rule:

$$\int_{\bar{Q}} \hat{K}^\mu_i(\bar{q}', t'; \bar{q}'', t'')\, \hat{K}^\mu_i(\bar{q}'', t''; \bar{q}, t)\, d\bar{q}'' = \hat{K}^\mu_i(\bar{q}', t'; \bar{q}, t), \tag{4.17}$$

and the analogous relations for $\hat{K}^\mu$ by summing over i. The latter ones are then exactly the non-abelian versions of (1.19), only expressed in terms of $\hat{\mathcal{H}}$ rather than $\mathcal{H}$. Here, in the non-abelian case, we have a finer splitting due to the n_μ-fold multiplicity (labeled by the index i) of the μ-th representation.

Clearly, everything said for $\hat{\mathcal{H}}$ can be easily translated to $\mathcal{H}$ using the unitary equivalence (4.7). For example, the projection maps T^μ_i, $T^\mu(a)$ and the projected integral kernels of propagators and operators take the form

$$T^\mu_i \psi = \frac{n_\mu}{n} \sum_{g \in G} D^\mu_{ii}(g)\, \psi \circ R_g\,, \tag{4.18a}$$

$$\bar{K}^\mu_i(\bar{q}', t'; \bar{q}, t) = \frac{n_\mu}{n} \sum_{g \in G} D^\mu_{ii}(g)\, \bar{K}(\bar{q}' g, t'; \bar{q}, t)\,, \tag{4.19}$$

$$\bar{O}^\mu_i(\bar{q}', \bar{q}) = \frac{n_\mu}{n} \sum_{g \in G} D^\mu_{ii}(g)\, \bar{O}(\bar{q}' g, \bar{q})\,, \tag{4.20}$$

and equivalently (by summing these expressions over i) for T^μ and O^μ. As explained above, the most general expression for a projector is given for some normalized $a \in C^{n_\mu}$ by

$$T^\mu(a)\psi = \frac{n_\mu}{n} \sum_{g,i,j} a_i \bar{a}_j D^\mu_{ij}(g)\, \psi \circ R_g\,. \tag{4.18b}$$

In the same way (4.19) and (4.20) can be written in terms of A. All these expressions form the non-abelian generalization of (1.15), (1.18) and (1.20). An application of (4.18) appears in appendix B. As already mentioned, (4.17) hold literally for $\bar{K}$ instead of $\hat{K}$.

In the present setting this is obvious from construction, though it can of course also be verified explicitly from (4.19) and (2.8a). For many of the general aspects we consider here it is however more convenient to work with $\hat{\mathcal{H}}$ rather than $\mathcal{H}$.

Coming back to the definition of observables on $\hat{\mathcal{H}}$, they do not only include those of the form (4.12a), but also right multiplications with elements in the centre $\mathcal{V}_G^c$ of $\mathcal{V}_G$, that is, the algebra generated by $\{\hat{T}^1, \ldots, \hat{T}^m\}$. We now state the main structural properties of the pair $(\hat{\mathcal{H}}, \mathcal{O})$ in the following

Theorem. *(i) $\mathcal{O}$ is completely reducible. The subspaces $\hat{\mathcal{H}}_i^\mu$ are minimal invariant relative to $\mathcal{O}$. (ii) A subspace $\hat{\mathcal{H}}' \subset \hat{\mathcal{H}}$ reduces $\mathcal{O}$ and $\mathcal{V}_G$, iff $\hat{\mathcal{H}}' = \bigoplus_{\mu\in J} \hat{\mathcal{H}}^\mu$, where J is a subset of $\{1, \ldots, m\}$. (iii) A minimal invariant subspace $\hat{\mathcal{H}}_i^\mu$ reduces $\mathcal{Z}^\mu$. It reduces $\mathcal{V}_G$, iff the μ-th representation is abelian.*

Proof. (i) Suppose $\hat{\mathcal{H}}_i^\mu = \hat{T}_i^\mu \hat{\mathcal{H}}$ were reducible under $\mathcal{O}$. Then there existed two orthogonal self-adjoint idempotents $\hat{S}_i^\mu$ and $\hat{P}_i^\mu$ with $\hat{T}_i^\mu = \hat{S}_i^\mu + \hat{P}_i^\mu$ and $\hat{S}_i^\mu, \hat{P}_i^\mu \in \mathcal{O}' = \mathcal{V}_G$ (since $\mathcal{V}_G$ is von Neumann). But this cannot be, since from the structural properties of V_G we know that the T_i^μ are already minimal idempotents. (ii) From (i) we have $\hat{\mathcal{H}}_i^\mu = \hat{T}_i^\mu \hat{\mathcal{H}} \subset \hat{\mathcal{H}}'$ for some pair μ, i. By hypothesis $\hat{T}_{ki}^\mu \hat{\mathcal{H}}_i^\mu \subset \hat{\mathcal{H}}'$ for any k, and these subspaces are clearly non null. Using (4.14), the left side can be rewritten as $\hat{T}_{ki}^\mu \hat{T}_i^\mu \hat{\mathcal{H}} = \hat{T}_k^\mu \hat{T}_{ki}^\mu \hat{\mathcal{H}} \subset \hat{\mathcal{H}}_k^\mu$. Hence there is a non trivial intersection $\hat{\mathcal{H}}' \cap \hat{\mathcal{H}}_k^\mu \; \forall k$, which by (i) implies $\bigoplus_k \hat{\mathcal{H}}_k^\mu = \hat{\mathcal{H}}^\mu \subset \hat{\mathcal{H}}'$. (iii) It reduces $\mathcal{Z}^\mu$ since it commutes with $\mathcal{V}_G$. To reduce $\mathcal{V}_G$ it is clear from (i) and (ii) that μ must be such that the range of i is only 1, i.e., $n_\mu = 1$. But this is the case iff the μ-th representation is abelian •

To conform with the Jauch-Wightman requirement, we proceed exactly as in the previous section. For each μ we truncate the Hilbert space so as to contain only one summand in (4.15b), say $\hat{\mathcal{H}}_1^\mu =: \hat{\mathcal{H}}_{\rm tr}^\mu$, and obtain

$$\hat{\mathcal{H}}_{\rm tr} = \bigoplus_{\mu=1}^{m} \hat{\mathcal{H}}_{\rm tr}^\mu. \tag{4.21}$$

Accordingly, the algebra of observables can now be written as

$$\mathcal{O} = \bigoplus_{\mu=1}^{m} B(\hat{\mathcal{H}}_{\rm tr}^\mu) \tag{4.22}$$

which is the general form of the algebra of observables in any theory with standard{9} superselection rules [BLOT]. Its representation on $\hat{\mathcal{H}}_{\rm tr}$ is phenomenologically equivalent to

{9} Superselection rules are said to be standard, if they are commutative, and in addition the linear span of the pure states lies dense in the Hilbert space. The latter condition is known as the condition of *discrete* superselection rules, since it ensures the decomposability into a discrete direct sum (rather than a direct integral) of irreducible representations (possibly with multiplicities) of the algebra of observables. In short: commutative + discrete = standard. For the nomenclature, see e.g. [BLOT]. However, the condition of discreteness is often violated even in standard quantum mechanics. For example, the mass superselection rule in Galilean invariant quantum mechanics is continuous, since each mass value defines a separate sector (compare [Gi]).

its representation on $\hat{\mathcal{H}}$, but pure states are now in bijective correspondence with rays in the set

$$\bigcup_{\mu=1}^{m} \hat{\mathcal{H}}^{\mu}_{\mathrm{tr}}. \tag{4.23}$$

In each sector $\hat{\mathcal{H}}^{\mu}_{\mathrm{tr}}$ the group C^{μ} is still acting. All these features are just like in the finite-dimensional model.

It is important to note that the definition $\mathcal{O} = \mathcal{V}'_G$ yields a richer set of observables than those coming from quantizing functions on the non-redundant classical phase space $T^*(Q)$. This is obvious from (4.12a), since the operators do not act on the "internal" vector space. But since $\mathcal{O}$ acts irreducibly in the sectors $\hat{\mathcal{H}}^{\mu}_{\mathrm{tr}}$, as asserted by the theorem above, there must be additional observables for the non-abelian sectors [So][Ba2]. For example, for non-abelian sectors, any localization on the true configuration space Q still does not specify in any way the direction of the "internal" vector. In order to fix it, additional observables must be employed. These observables cannot simply be given by pointwise left V_G-multiplication, for, as we have seen above, only elements of Z^{μ} act on $\hat{\mathcal{H}}^{\mu}_{\mathrm{tr}}$, where they are necessarily proportional to the identity operator. However, if we first apply some localization to the system in configuration space, we can indeed define observables acting on the "internal" space. Let us explain this in more detail.

Let $U \subset Q$ be a closed connected[10] subset and $\bar{U} \subset \bar{Q}$ a connected covering set. We call U admissible if $\bar{U} \cap \bar{U}g = \emptyset \; \forall g \neq e$. Here, $\bar{U}g$ is the right translation of $\bar{U}$ by g. We call $\hat{\psi}$ U-localized, iff its support is contained in the interior of $\bigcup_g \bar{U}g$. This defines a linear subspace $\hat{\mathcal{H}}_U$ of U-localized states. Note that the variety of admissible subsets U is very big. In particular they contain all contractible subsets of Q. Also, the set may be chosen such as to leave a complement with arbitrarily small volume. However, physically it might be more relevant to think of the admissible sets as being rather small portions of Q on which realistic "filters" project. Any localized state is completely determined by its restriction to $\bar{U}$. Let $\chi_{\bar{U}}$ be the characteristic function of $\bar{U}$, and $\chi_{\bar{U}g} = \chi_{\bar{U}} \circ R_{g^{-1}}$ those of the translated sets. We set $\hat{\psi}_{\bar{U}g} = \chi_{\bar{U}g}\hat{\psi}$. Equivariance (4.3) implies that $\hat{\psi}_{\bar{U}g} = \hat{g}^{-1} \cdot \hat{\psi}_{\bar{U}} \circ R_{g^{-1}}$. A projection operator $P_U : \hat{\mathcal{H}} \to \hat{\mathcal{H}}_U$ is then given by

$$P_U(\hat{\psi}) := \sum_{g \in G} \hat{\psi}_{\bar{U}g} = \sum_{g \in G} \hat{g} \cdot \hat{\psi}_{\bar{U}} \circ R_g \,, \tag{4.24}$$

and functions $\hat{\psi} \in \hat{\mathcal{H}}_U$ are determined by their restriction $\hat{\psi}_{\bar{U}}$. Since $\tau : \bar{U} \to U$ is a diffeomorphism, we can also use the pullback $\hat{\psi}_U := \hat{\psi}_{\bar{U}} \circ \tau^{-1}$ on Q. Now, on $\hat{\mathcal{H}}_U$ we can define a left V_G-action as follows: for $\hat{v} = \sum_h v(h)\hat{h}$ we set

$$(\hat{v}, \hat{\psi}) \mapsto \gamma_{\hat{v}}(\hat{\psi}) := \sum_{h \in G} v(h) \sum_{g \in G} \hat{g} \cdot \hat{h} \cdot \hat{\psi}_{\bar{U}} \circ R_g \,. \tag{4.25}$$

It is easily seen that this is indeed a map from $\hat{\mathcal{H}}_U$ to $\hat{\mathcal{H}}_U$, in particular, $\gamma_{\hat{v}}(\hat{\psi})$ is equivariant. Moreover, this action commutes with $\mathcal{V}_G$ since it clearly commutes with right V_G

[10] Connectedness is not a relevant requirement and may without gain or loss just as well be dropped. It does simplify the argument however.

multiplications. It therefore also defines an action on U-localized states in $\hat{\mathcal{H}}_{\rm tr}$ and each sector $\hat{\mathcal{H}}^{\mu}_{\rm tr}$ separately. For general (i.e. non localized) states, observables may be defined by first projecting with P_U on any admissible U and then applying $\gamma_{\hat{v}}$:

$$O_{\hat{v}} := \gamma_{\hat{v}} \circ P_U. \tag{4.26}$$

One easily verifies that this is a self-adjoint operator iff $\hat{v} = \hat{v}^*$. On the local representative $\hat{\psi}_U$ on $U \subset Q$ this just corresponds to left $\hat{v}$-multiplication. This construction seems to implement some ideas presented in [So][Ba2]. It would be interesting to explicitly construct and interpret these observables in simple models.

Everything we have said could be rephrased in terms of the possibly more familiar language of vector bundles over Q. Sections of this bundle could be represented by locally defined functions like $\hat{\psi}_U$. This is explained in detail in the following appendix A. We have deliberately avoided this language in order to always deal with globally defined functions (on $\bar{Q}$). In particular, the left G-action defined on localized states through (4.25) should not be confused with gauge transformations. We refer to appendix A for more details.

Finally we make a few comments on the implementation of symmetries. The issue is whether we can always assume the symmetries to respect the sector structur, that is, whether symmetries that initially act on $\mathcal{H}$ are reduced by the subspaces $\mathcal{H}^{\mu}$ and $\mathcal{H}^{\mu}(a) = T^{\mu}(a)\mathcal{H}$. If the unitary symmetry operators commute with $\mathcal{V}_G$, i.e., are elements in $\mathcal{O}$, all subspaces that reduce $\mathcal{O}$ also reduce the symmetry group and there is no problem with its implementation in the sectors. This is the case for continuous groups whose generators should correspond to physical quantities and therefore commute with $\mathcal{V}_G$ (in the sense of section VIII.5 in [RS]). But there are discrete symmetries which do not commute with $\mathcal{V}_G$, like time-reversal. In fact, if the complex conjugate representation, $\bar{D}^{\mu}$, of D^{μ} is not equivalent to D^{μ}, i.e., $\bar{D}^{\mu} = D^{\lambda}$, $\lambda \neq \mu$, complex conjugation will connect two different sectors. The operation of time-reversal is therefore not implementable in these sectors. They are said to 'break' time-reversal invariance. For abelian sectors this is the case iff the representation is not real [Sch]. Conversely, if we have $\bar{D}^{\mu} = U^{\dagger} D^{\mu} U$, then (4.18$b$) shows $\overline{T^{\mu}(a)\psi} = T^{\mu}(\overline{Aa})\bar{\psi}$. Since the truncated Hilbert space $\mathcal{H}^{\mu}_{\rm tr}$ can be identified with any of the $\mathcal{H}^{\mu}(a)$, which are mutually isomorphic in a natural way, we can use this isomorphism to map back $\mathcal{H}^{\mu}(\overline{Aa})$ to $\mathcal{H}^{\mu}(a)$ and thus define the operator of time-reversal on $\mathcal{H}^{\mu}_{\rm tr}$. We avoid to write down the details at this point which immediately follow from our general discussion in section 3. We conclude that the μ-th sector breakes time-reversal invariance, iff the representation D^{μ} is inequivalent to its complex conjugate. (For a general criterion see chapter 5-5 in [Ha].)

Appendix A

In this appendix we recall some basic features of principal bundles and their associated vector bundles as applied to the universal covering space. As already stated in section 1, the universal covering space $\bar{Q}$ is the total space of a principal fibre bundle with structure group $G \cong \pi_1(\bar{Q}, \bar{q})$, base Q and projection $\tau : \bar{Q} \to Q$. G acts on $\bar{Q}$ via right multiplications:

$R_g(\bar{q}) = \bar{q}g$, so that $\tau(\bar{q}g) = \tau(\bar{q})$ for all $g \in G$. The action is transitive on each fibre $\tau^{-1}(q)$. Discreteness of the fibres implies that $\tau_* : T_{\bar{q}}(Q) \to T_q(Q)$ and $R_{g*} : T_{\bar{q}}(\bar{Q}) \to T_{\bar{q}g}(\bar{Q})$ are both isomorphisms. We can thus trivially regard $T_{\bar{q}}(\bar{Q})$ as its own horizontal subspace. This defines a naturally given connection as follows: given a loop, $\gamma : [0,1] \to Q$, based at $\gamma(0) = \gamma(1) = q$, we have for each $\bar{q} \in \tau^{-1}(q)$ a unique (horizontal) lift, $\bar{\gamma} : [0,1] \to \bar{Q}$, such that $\bar{\gamma}(0) = \bar{q}$ and $\bar{\gamma}(1) = \bar{q}g$ for some uniquely determined $g \in G$. Since G is discrete, g depends only on the homotopy class $[\gamma] \in \pi_1(Q,q)$. This defines the family of maps

$$I_q : \pi_1(Q,q) \to G, \quad [\gamma] \mapsto g =: I_q([\gamma]). \tag{A.1}$$

Choosing a different point, $\bar{q}' = \bar{q}h \in \tau^{-1}(q)$, the lift of γ starting at $\bar{q}'$ is now given by $\bar{\gamma}' = R_h \circ \bar{\gamma}$, which ends at $\bar{\gamma}'(1) = \bar{\gamma}(1)h = \bar{q}gh = \bar{q}'h^{-1}gh$, so that

$$I_{\bar{q}h} = \mathrm{Ad}(h^{-1}) \circ I_{\bar{q}}. \tag{A.2}$$

Moreover, $I_{\bar{q}}([\gamma_1][\gamma_2])$ is defined by lifting $[\gamma_1\gamma_2]$[11]: lifting γ_1 takes one from $\bar{q}$ to $\bar{q}' = \bar{q}I_{\bar{q}}([\gamma_1])$, and the lift of γ_2 then from $\bar{q}'$ to $\bar{q}'I_{\bar{q}'}([\gamma_2])$, which, using $(A.2)$, is equal to $\bar{q}I_{\bar{q}}([\gamma_2])I_{\bar{q}}([\gamma_1])$. Hence each $I_{\bar{q}}$ defines an anti-isomorphism:

$$I_{\bar{q}}([\gamma_1][\gamma_2]) = I_{\bar{q}}([\gamma_2])I_{\bar{q}}([\gamma_1]). \tag{A.3}$$

As already mentioned in section 1, there is generally no natural isomorphism between the fundamental groups at different points and G. For example, looping the basepoint along γ results in a conjugation with $[\gamma]$ (see e.g. [St], paragraph 16). Identifications with an abstract group G are therefore only defined up to inner automorphisms. This at least provides a natural identification of conjugacy classes of all $\pi_1(Q,q)$ with those of G. Unless one refers to a basepoint, it generally does not make sense to talk about *the* fundamental group, or a specific element thereof. But it does make sense to speak of *a particular* conjugacy class. For example, if $g \in G_c$ (the centre), it makes sense to call it a *particular* element of *the* fundamental group. If restricted to the centre, the maps $I_{\bar{q}h}$ are independent of h, as $(A.2)$ shows. Right multiplication by the central element g might therefore be interpreted as "parallelly transporting each element of $\bar{Q}$ along the loop g". For elements not in the centre this notion is not defined.

Since $\bar{Q}$ is a principal bundle, we can also apply the concept of gauge transformations. These are given by diffeomorphisms $F : \bar{Q} \to \bar{Q}$ such that $F \circ R_g = R_g \circ F$ (bundle automorphisms), and $\tau \circ F = \tau$ (projecting to the identity on Q). It is easy to see that any such function F can be written in the form $F(\bar{q}) = \bar{q}f(\bar{q})$, with a uniquely determined smooth function $f : \bar{Q} \to G$ satisfying $f \circ R_g = \mathrm{Ad}(g^{-1}) \circ f$. In that sense gauge transformations uniquely correspond to Ad-equivariant, G-valued functions on $\bar{Q}$. The composition $F = F_1 \circ F_2$ corresponds to the function $f = f_1 f_2$, where juxtaposition on the right hand side means pointwise multiplication in G. However, since in our case G is discrete, the G-valued function f must be constant. Ad-equivariance then implies that it must assume values in the centre G_c of G. The group of gauge transformations is therefore

[11] We adopt the standard convention that products of paths are read from the left, that is, $\gamma_1\gamma_2$ is γ_1 followed by γ_2. If we read it from right to left, like maps, the $I_{\bar{q}}$ would be isomorphisms in $(A.3)$

given by right G_c-multiplications. *In particular, the group of gauge transformations does not contain the gauge group if G is non-abelian.*

Vector Bundles

Let V be a (complex) vector space and D^μ an (irreducible) representation of G on V. We can associate to the principal bundle $\bar{Q}$ a vector bundle, $E(Q, V, G, \mu)$, with base Q, fibre V, structure group G, and total space E:

$$\begin{aligned} &E = \{\bar{Q} \times V\}/\sim, \\ \text{where} \quad &(\bar{q}, v) \sim (\bar{p}, w) \Leftrightarrow \exists g \in G \,/\, \bar{p} = \bar{q}g, \; w = D^\mu(g^{-1})v. \end{aligned} \tag{A.4}$$

We denote the equivalence class of $(\bar{q}, v)$ by $[\bar{q}, v]$, and have the inherited projection map $\tau_E : E \to Q$, $\tau_E([\bar{q}, v]) := \tau(\bar{q}) = q$. Parallel transportation of $[\bar{q}, v] \in \tau_E^{-1}(q)$ along a curve γ in Q from $\gamma(0) = q$ to $\gamma(1) = p$ is defined as follows: Take the horizontal lift $\bar{\gamma}$ of γ on $\bar{Q}$, such that $\bar{\gamma}(0) = \bar{q}$. This defines a curve $\tilde{\gamma}$ in E via $\tilde{\gamma} := [\bar{\gamma}, v]$. Its end point, $\tilde{\gamma}(1) = [\bar{\gamma}(1), v] \in \tau_E^{-1}(p)$, then defines the parallel transport of $[\bar{q}, v]$. In particular, if γ is a loop at $q \in Q$, we have, using the notation above, $\tilde{\gamma}(1) = [\bar{\gamma}(1), v] = [\bar{q}g, v] = [\bar{q}, D^\mu(g)v] = [\bar{q}, D^\mu(I_{\bar{q}}([\gamma]))v]$. This defines a family of holonomy maps H_q:

$$\begin{aligned} &H_{\bar{q}} : \; \pi_1(Q, q) \to \mathrm{End}(\tau_E^{-1}(q)), \\ &H_{\bar{q}}([\gamma])([\bar{q}, v]) := [\bar{q}, D^\mu(I_{\bar{q}}([\gamma]))v], \end{aligned} \tag{A.5}$$

which is an anti-homomorphism, due to $(A.3)$. Note that the right action R_g on $\bar{Q}$ does generally not define an action – hypothetically denoted by γ_g – on E, since in this case $\gamma_g([\bar{q}, v]) := [\bar{q}g, v] = [\bar{q}, D^\mu(g)(v)]$ should equal $\gamma_g([\bar{q}h, D^\mu(h^{-1})v] = [\bar{q}hg, D^\mu(h^{-1})v] = [\bar{q}, D^\mu(hgh^{-1})v]$ for all h. This is the case iff $g \in C^\mu$, where $C^\mu = \{h \in G \,/\, D^\mu(hg) = D^\mu(gh) \, \forall g \in G\}$; in words, C^μ is the largest subgroup of G which under D^μ maps into the centre of $D^\mu(G)$. One also easily verifies that $D^\mu(g) = D^\mu(g_G) \; \forall g \in C^\mu$, where g_G is the conjugacy class of g. Thus, although there is generally no action of G on E, there is such an action of C^μ:

$$\gamma_g : \; [\bar{q}, v] \to [\bar{q}g, v] = [\bar{q}, D^\mu(g)v] \; \forall g \in C^\mu . \tag{A.6}$$

Allowing some abuse of language, we may say that this corresponds to a parallel transportation along a loop at q representing $g \in C^\mu$. As explained above we should actually refer to the whole class g_G, but the ambiguity in assigning a particular member of g_G to each $\pi_1(Q, q)$ is projected out due to D^μ being constant on g_G.

Finally, given a cross section σ in E, we can define an action of C^μ on σ. To see this explicitly, recall that for each section σ there is a unique D^μ-equivariant function $\bar{\sigma}$ on $\bar{Q}$:

$$\bar{\sigma} : \bar{Q} \to V, \quad \bar{\sigma} \circ R_g = D^\mu(g^{-1})\,\bar{\sigma}, \tag{A.7}$$

defined implicitly by

$$\sigma(q) = [\bar{q}, \bar{\sigma}(\bar{q})]. \tag{A.8}$$

Alternatively, sections in E can be described locally on Q. Given a local section $\lambda : U \to \bar{Q}$ on an open subset $U \subset Q$, we have the locally defined V-valued function on U:

$$\sigma_\lambda : U \to V\,, \quad \sigma_\lambda := \bar{\sigma} \circ \lambda\,. \tag{A.9}$$

On U it satisfies $[\lambda(q), \sigma_\lambda(q)] = \sigma(q)$. Any other local section, $\lambda' : U \to \bar{Q}$, is necessarily of the form $\lambda' = R_h \circ \lambda$ for some $h \in G$. We then have, using $(A.9)$,

$$\sigma_{\lambda'} = D^\mu(h^{-1})\sigma_\lambda. \tag{A.10}$$

Now, an action (also denoted by γ_g) of C^μ on the section σ is just given by the obvious choice

$$(\gamma_g\sigma)(q) := [\bar{q}g, \bar{\sigma}(\bar{q})] = [\bar{q}, D^\mu(g)\bar{\sigma}(\bar{q})]. \tag{A.11}$$

Equivalently, expressed in terms of $\bar{\sigma}$ or the local representative σ_λ, we have

$$(\gamma_g\bar{\sigma})(\bar{q}) = \bar{\sigma}(\bar{q}g^{-1}) = D^\mu(g)\bar{\sigma}(\bar{q}), \tag{A.12}$$

$$(\gamma_g\sigma_\lambda)(q) = D^\mu(g)\sigma_\lambda(q). \tag{A.13}$$

As above, we could – again with some abuses of language – say that $\gamma_g\sigma$ is the result of "parallelly transporting the section σ along a loop representing (the class of) g in the fundamental group".

Quite generally, in gauge theory one cannot use the local formula $(A.13)$ as definition of an action of the gauge group. *The gauge group simply does not act on the space of sections in the general case.* However, in special circumstances meaning can be given to a definition in the form $(A.13)$ in the following way: Let $U \subset Q$ and λ as before and Γ_U the linear space of sections $\sigma : Q \to E$ whose support is contained in U. We now use the distinguished section λ to *define* a G-action on Γ_U via $(A.13)$. With respect to a different section $\lambda' = R_h \circ \lambda$ the so defined action reads

$$\gamma_g\sigma_{\lambda'} = D^\mu(h^{-1}gh)\sigma_{\lambda'}\,. \tag{A.14}$$

The best way to see that this defines indeed a G-action on Γ_U is to express it in terms of the globally defined (on $\bar{Q}$) equivariant functions $\bar{\sigma}$ and check that the result is again equivariant. To do this, let $\lambda(U) = \bar{U} \subset \bar{Q}$ and recall that the restriction $\bar{\sigma}\big|_{\bar{U}}$ determines all other restrictions $\bar{\sigma}\big|_{\bar{U}g}$ by equivariance. From $(A.7)$ one has $\bar{\sigma}\big|_{\bar{U}g} = D^\mu(g^{-1})\bar{\sigma}\big|_{\bar{U}} \circ R_{g^{-1}}$. That σ is in Γ_U means here that $\bar{\sigma}$ has support in $\bigcup_{g\in G}\bar{U}g \subset \bar{Q}$. We define the function $\bar{\sigma}_{\bar{U}}$ on $\bar{Q}$ to equal the restriction $\bar{\sigma}\big|_{\bar{U}}$ within $\bar{U}$ and be identically zero otherwise. We can then express $\bar{\sigma}$ as a sum of terms with disjoint support:

$$\bar{\sigma} = \sum_{h\in G} D^\mu(h)\bar{\sigma}_{\bar{U}} \circ R_h\,. \tag{A.15}$$

Since $\bar{\sigma}_{\bar{U}}$ is essentially σ_λ, the action defined by $(A.13)$ now reads

$$\gamma_g\bar{\sigma} = \sum_{h\in G} D^\mu(hg)\bar{\sigma}_{\bar{U}} \circ R_h, \tag{A.16}$$

which is again equivariant. What happened here is that in the support component $\bar{U}h$ the function $\bar{\sigma}$ is multiplied with $D^{\mu}(h^{-1}gh)$, as required by $(A.14)$. Here the additional conjugation is necessary for the result to be equivariant. This definition would be contradictory if the support were not inside the disjoint regions $\bar{U}g$. This is the reason why we had to restrict to Γ_U

There is a certain danger to misunderstand this construction in the following way: the restriction to Γ_U effectively truncates the principal bundle $\bar{Q}$ to $\tau^{-1}(U)$ which is itself a trivial bundle. Given a distinguished section λ in this truncated bundle there is a induced trivialization $\tau^{-1}(U) \to U \times G$ given by{12} $\lambda(q)h \mapsto (q,h)$. Then there is a left action γ_g of G defined by $\gamma_g(q,h) = (q,gh)$ or $\gamma(\lambda(q)h) = \lambda(q)gh$. This clearly defines a gauge transformation F of $\tau^{-1}(U)$ which is easily seen to induce the action γ_g on sections. This suggests the incorrect conclusion that our action γ_g is really nothing but a gauge transformation. The point is that the map F will not extend from $\tau^{-1}(U)$ to $\bar{Q}$, so that we are *not* dealing with a gauge transformation on $\bar{Q}$ or E.

Appendix B

A simple mechanical system with finite non-abelian fundamental group is the non-symmetric rotor. It serves, for example, as a dynamical model for the collective rotational degrees of freedom of deformed nuclei [BM]. In this appendix we explicitly construct the sectors by applying formula (4.18) to the standard basis functions. This leads precisely to the known symmetry classification of collective rotational modes of nuclei but interprets it in the present formalism. In particular, the only sector for odd-A nuclei corresponds to non-abelian representations of the fundamental group. This relevant sector would have been lost if one restricted to abelian representations. This example therefore serves to illustrate our discussion at the end of section 3.

The different configurations for the non-symmetric rotor are easily visualized as the different orientations of a solid ellipsoid with pairwise different major axes. Its symmetries are generated by π-rotations about any two of the three major axes and form the group $Z_2 \times Z_2$. The configuration space is thus given by $SO(3)/Z_2 \times Z_2$, but it is more conveniently represented by $SU(2)/D_8^*$, where D_8^* is the preimage of $Z_2 \times Z_2$ under the 2-1 projection $SU(2) \to SO(3)$. D_8^* is called the binary dihedral group of order eight and is conveniently defined using unit quaternions: $D_8^* = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$, where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = \mathrm{k}$ and cyclic. The configuration space is thus defined by $Q := SU(2)/D_8^*$. Since $SU(2) \cong S^3$ is simply connected, we have $\bar{Q} = S^3$ and $\pi_1(Q) \cong D_8^*$.

We consider the Hilbert space $\mathcal{H} = L^2(S^3, d\bar{q})$ where $d\bar{q}$ is the measure induced by the kinetic energy metric of the rotor. Such a metric is invariant under left $SU(2)$ and right D_8^* multiplications{13}, and so is the measure $d\bar{q}$. Let $\{R^{\Lambda}_{MN}\}$, $2\Lambda = 0, 1, 2..$, denote the

{12} Any element in $\tau^{-1}(U)$ can be uniquely written as $\lambda(q)h$.

{13} We adopt the standard convention that left multiplications correspond to rotations in the space-fixed and right multiplications to rotations in the body-fixed frame. The identifications under D_8^* are therefore done using the right multiplications.

representation matrices for $SU(2)$. We use the standard convention to label the $2\Lambda+1$ values for the indices $M, N, ..$ by $\{-\Lambda, -\Lambda+1, \ldots, \Lambda\}$. We can now expand each $\psi \in \mathcal{H}$ in the form[14]:

$$\psi = \sum_{M,N,\Lambda} C^{\Lambda}_{MN} R^{\Lambda}_{MN} . \tag{B.1}$$

D^*_8 has four one-dimensional irreducible representations, D^0, D^1, D^2, D^3, and one two-dimensional one, D^4. The one-dimensional representations are labelled by three $\{-1,1\}$-valued numbers, (r_1, r_2, r_3), where $D^\mu(\pm\mathrm{i}) = r_1$, $D^\mu(\pm\mathrm{j}) = r_2$, and $D^\mu(\pm\mathrm{k}) = r_3$, so that:

$$(r_1, r_2, r_3) = \begin{cases} (1,1,1) & \text{for } \mu = 0, \\ (1,-1,-1) & \text{for } \mu = 1, \\ (-1,1,-1) & \text{for } \mu = 2, \\ (-1,-1,1) & \text{for } \mu = 3. \end{cases} \tag{B.2}$$

One sees that it is in fact sufficient to uniquely characterize a one-dimensional representation by two of the three r_i's. We shall take r_2 and r_3. The two-dimensional representation, D^4, can be defined using the standard Pauli-matrices $\{\tau_1, \tau_2, \tau_3\}$:

$$\begin{aligned} D^4(\pm 1) &= \pm i\mathbf{1}, \\ D^4(\pm\mathrm{i}) &= \mp i\tau_1, \\ D^4(\pm\mathrm{j}) &= \mp i\tau_2, \\ D^4(\pm\mathrm{k}) &= \mp i\tau_3. \end{aligned} \tag{B.3}$$

Using standard results from finite group theory[15] one easily finds that for even Λ, D^0 occurs $(\frac{\Lambda}{2}+1)$-times and $D^{1,2,3}$ each $\frac{\Lambda}{2}$-times, whereas for odd Λ D^0 occurs $\frac{\Lambda-1}{2}$-times and $D^{1,2,3}$ each $\frac{\Lambda+1}{2}$-times. D^4 is of course not contained in representations with integer Λ. Conversely, for $\Lambda = \frac{\text{odd}}{2}$ only the two-dimensional representation D^4 occurs, namely $(\Lambda + \frac{1}{2})$-times. All representations are equivalent to their complex conjugates. This is trivial for the one-dimensional ones, which are real, and for D^4 we have $\bar{D}^4 = U^\dagger D^4 U$ with $U = i\tau_2$.

We are interested in the projection operators $T^\mu(a)$, written down in (4.18*b*). We first deal with the abelian cases $\mu = 0,1,2,3$. Here $R^\Lambda(\mathrm{jk}) = R^\Lambda(\mathrm{kj})$. It is convenient to introduce the four projector matrices:

$$\begin{aligned} P^\Lambda_{r_2}(\mathrm{j}) &= \tfrac{1}{2}\left[\mathbf{1} + r_2 R^\Lambda(\mathrm{j})\right], \\ P^\Lambda_{r_3}(\mathrm{k}) &= \tfrac{1}{2}\left[\mathbf{1} + r_3 R^\Lambda(\mathrm{k})\right]. \end{aligned} \tag{B.4}$$

[14] In order to properly normalize the basis functions we would have to multiply each R^Λ_{MN} with a factor proportional to $\sqrt{I_1 I_2 I_3 (2\Lambda+1)/16\pi^2}$. The moments of inertia, I_i, appear because they need to be cancelled from the measure derived from the kinetic energy metric.

[15] Here we just use the formula $n^\Lambda_\mu = \frac{1}{8}\sum_{g\in D^*_8} \bar{\chi}^\mu(g)\chi^\Lambda(g)$ for the number of times D^μ is contained in R^Λ. χ^μ and $\chi^\Lambda(g) = \sin((\Lambda+\frac{1}{2})\alpha)/\sin(\frac{\alpha}{2})$ are the characters of $D^\mu(g)$ and $R^\Lambda(g)$ respectively, and α is the rotation angle of g.

We easily find

$$T^{\mu}R^{\Lambda}=R^{\Lambda}P^{\Lambda}_{r_2}(\mathrm{j})P^{\Lambda}_{r_3}(\mathrm{k})\,, \tag{B.5}$$

where the right hand side is understood as multiplication of the matrix-valued function R^{Λ} from the right with the projector matrices. Using Wigner's formula for R^{Λ}_{MN} (see e.g. [Wi], chapter XV, formula (27)) one has $R^{\Lambda}_{MN}(\mathrm{j})=(-1)^{\Lambda+N}\delta_{-M,N}$ and $R^{\Lambda}_{MN}(\mathrm{k})=(-1)^{N}\delta_{MN}$. The projections of the basis functions can now be written in the final form

$$(T^{\mu}R^{\Lambda})_{MN}=\tfrac{1}{2}(1+r_3(-1)^N)\left(R^{\Lambda}_{MN}+r_2(-1)^{\Lambda+N}R^{\Lambda}_{M,-N}\right) \tag{B.6}$$

$$=\left(R^{\Lambda}_{MN}+r_2(-1)^{\Lambda+N}R^{\Lambda}_{M,-N}\right)\begin{cases} N\geq 0 \text{ and even} & \text{for } r_3=1,\\ N\geq 1 \text{ and odd} & \text{for } r_3=-1.\end{cases} \tag{B.7}$$

These are precisely the bases used in [BM] to describe collective rotational modes of even-A nuclei. (Compare formula (4-276) in [BM].) From Wigners formula one has $\bar{R}^{\Lambda}_{MN}=(-1)^{M-N}R^{\Lambda}_{-M,-N}$, which for the basis functions $B^{\Lambda}_{MN}:=R^{\Lambda}_{MN}+r_2(-1)^{\Lambda+N}R^{\Lambda}_{M,-N}$ implies $\bar{B}^{\Lambda}_{MN}=r_2(-1)^{\Lambda+M}B^{\Lambda}_{-M,N}$. This defines the operation of time-reversal – given by complex conjugation – within each sector.

We now turn to the two-dimensional representation D^4. Here we only have to consider $\Lambda=\frac{\text{odd}}{2}$. Again we straightforwardly use formula (4.18b) with some normalized $a\in C^2$ Applied to the functions R^{Λ}_{MN}, one obtains

$$T^4(a)R^{\Lambda}=R^{\Lambda}P^{\Lambda}(a)\,, \tag{B.8}$$

where the right side is again understood as matrix multiplication with

$$\begin{aligned}P^{\Lambda}(a)=\tfrac{1}{2}\Big[&|a_1|^2\left(\mathbf{1}-iR^{\Lambda}(\mathrm{k})\right)+|a_2|^2\left(\mathbf{1}+iR^{\Lambda}(\mathrm{k})\right)\\ &+a_1\bar{a}_2\left(-iR^{\Lambda}(\mathrm{i})-R^{\Lambda}(\mathrm{j})\right)+\bar{a}_1a_2\left(-iR^{\Lambda}(\mathrm{i})+R^{\Lambda}(\mathrm{j})\right)\Big]\,.\end{aligned} \tag{B.9}$$

It is not difficult to check explicitly that this is indeed a projection operator. Using Wigner's formula{16} for $R^{\Lambda}(\mathrm{j})$, $R^{\Lambda}(\mathrm{k})$ and the relation $R^{\Lambda}(\mathrm{i})=R^{\Lambda}(\mathrm{j})R^{\Lambda}(\mathrm{k})$ we find

$$R^{\Lambda}_{MN}(\mathrm{i})=\exp(i\pi N)(-1)^{\Lambda+N}\,\delta_{M,-N}=i(-1)^{\Lambda+\frac{1}{2}}\,\delta_{M,-N}\,, \tag{B.10}$$

$$R^{\Lambda}_{MN}(\mathrm{j})=(-1)^{\Lambda+N}\,\delta_{M,-N}\,, \tag{B.11}$$

$$R^{\Lambda}_{MN}(\mathrm{k})=\exp(i\pi N)\,\delta_{MN}=(-i)(-1)^{N+\frac{1}{2}}\,\delta_{MN}\,, \tag{B.12}$$

where the first expressions on the right hand sides are valid for all Λ and the second expressions specialize to $2\Lambda=$ odd (and hence $2N,2M=$ odd). Using them we obtain

$$\tfrac{1}{2}\Big[\mathbf{1}\mp iR^{\Lambda}(\mathrm{k})\Big]_{MN}=\tfrac{1}{2}\left(1\mp(-1)^{N+\frac{1}{2}}\right)\,\delta_{MN}\,, \tag{B.13}$$

$$\tfrac{1}{2}\Big[-iR^{\Lambda}(\mathrm{i})\mp R^{\Lambda}(\mathrm{j})\Big]_{MN}=\tfrac{1}{2}(-1)^{\Lambda+\frac{1}{2}}\left(1\mp(-1)^{N-\frac{1}{2}}\right)\,\delta_{M,-N}\,, \tag{B.14}$$

{16} We use Wigner's convention which agrees with $R^{\frac{1}{2}}(\mathrm{i},\mathrm{j},\mathrm{k})=-i\tau_{1,2,3}$. It differs from other conventions in use by a factor $(-1)^{M-N}$, adopted e.g. in [Ha], formula (9-76).

which inserted into $(B.9)$ gives for $(B.8)$:

$$T^4(a)R^\Lambda_{MN} = \begin{cases} \bar{a}_1\left(a_1 R^\Lambda_{MN} + (-1)^{\Lambda+\frac{1}{2}} a_2 R^\Lambda_{M,-N}\right), & \text{for } 2N = 1 \bmod 4 \\ \bar{a}_2\left(a_2 R^\Lambda_{MN} + (-1)^{\Lambda+\frac{1}{2}} a_1 R^\Lambda_{M,-N}\right), & \text{for } 2N = -1 \bmod 4\,. \end{cases} \tag{B.15a, b}$$

If for fixed Λ we let M, N run through all $(2\Lambda+1)^2$ values, the right hand side of $(B.15)$ contains $\frac{1}{2}(2\Lambda+1)^2$ linearly independent functions. For $a_2 = 0$ they are $\{R^\Lambda_{MN},\ 2N = 1 \bmod 4\}$ and $\{R^\Lambda_{MN},\ 2N = -1 \bmod 4\}$ for $a_1 = 0$. If $a_1 a_2 \neq 0$ the set of functions in $(B.15a)$ and $(B.15b)$ are the same up to an overall factor. The general representation of the truncated Hilbert space, $\mathcal{H}^{\mu=4}_{\text{tr}}$, is therefore given by

$$\begin{aligned} \mathcal{H}^{\mu=4}_{\text{tr}} = \text{span}\Big\{ & a_1 R^\Lambda_{MN} + (-1)^{\Lambda+N} a_2 R^\Lambda_{M,-N}\,, \\ & 2\Lambda = \text{odd},\ -\Lambda \leq M, N \leq \Lambda,\ 2N = 1 \bmod 4\Big\}\,, \end{aligned} \tag{B.16}$$

where we could set $(-1)^{\Lambda+\frac{1}{2}} = (-1)^{\Lambda+N}$ due to $N = \frac{1}{2} \bmod 2$. But this is precisely the basis used in [BM] to describe collective rotational modes for odd-A nuclei. (Compare formula (4-293) in [BM].)

Finally, if we set $B^\Lambda_{MN}(a) := a_1 R^\Lambda_{MN} + (-1)^{\Lambda+N} a_2 R^\Lambda_{M,-N}$, we have $\bar{B}^\Lambda_{MN}(a) = -(-)^{\Lambda+M} B^\Lambda_{-M,N}(i\tau_2\bar{a})$. Since the canonical isomorphism $I : \mathcal{H}^4_{\text{tr}}(a) \to \mathcal{H}^4_{\text{tr}}(a')$ is just given by $I(B^\Lambda_{MN}(a)) := B^\Lambda_{MN}(a')$, the operation of time reversal, $\mathcal{T}$, can be defined by $\mathcal{T}(B^\Lambda_{MN}) := (-1)^{\Lambda+M} B^\Lambda_{-M,N}$ (plus antilinear extension) on the set of basis functions B^Λ_{MN} for fixed a.

This concludes the presentation of a relatively simple example for the usage of non-abelian sectors within familiar quantum mechanics.

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References

[Ba1] Balachandran, A.P. (1989). Classical Topology and Quantum Phases. In: Geometrical and Algebraic Aspects of Nonlinear Field Theory, S. de Filippo, M. Marinaro, G. Marmo and G. Vilasi (Editors). Elsevier Science Publishers B.V., North Holland.

[Ba2] Balachandran, A.P., Marmo, G., Skagerstam, B.S., Stern, A. (1991). Classical Topology and Quantum States. World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong.

[BLOT] Bogolubov, N.N., Logunov, A.A., Oksak, A.I., Todorov, I.T. (1990). General Principles of Quantum Field Theory. Kluwer Academic Publishers, Dordrecht-Boston-London.

[BM] Bohr, A., Mottelson, B. (1975). Nuclear Strukture, Vol.II: Nuclear Deformations. W.A. Benjamin, INC., Reading Massachusetts.

[Di] Dirac, P.A.M. (1982). The Principles of Quantum Mechanics. Fourth edition, Clarendon Press, Oxford.

[Do1] Dowker, S. (1972). Quantum Mechanics and Field Theory on Multiply Connected and Homogeneous spaces. *J. Phys.* A, **5**, 936-943.

[Do2] Dowker, S. (1979). Selected Topics in Topology and Quantum Field Theory. Lectures delivered at Center for Relativity, Austin, January-May 1979.

[Gi] Giulini, D. (1995). On Galilei Invariance in Quantum Mechanics and the Bargmann Superselection Rule. University of Freiburg, Preprint Thep 95/15, and quant-ph 9508002.

[GMN] Galindo, A., Morales, A., Nuñez-Lagos, R. (1962). Superselection Principle and Pure States of n-Identical Particles. *Jour. Math. Phys.*, **3**, 324-328.

[Ha] Hamermesh, M. (1964). Group Theory and its Application to Physical Problems. Addison-Wesley Publ. Comp., Inc. Reading Massachusetts. Second (corrected) printing.

[HT] Hartle, J.B., Taylor, J.R. (1969). Quantum Mechanics and Paraparticles. *Phys. Rev.*, **178**, 2043-2051.

[Ho] Horuzhy, H. (1976). Superposition Principle in Algebraic Quantum Theory. *Theor. Math. Phys.*, **23**, 413-421.

[Ja] Jauch, J.M. (1960). Systems of Observables in Quantum Mechanics. *Helv. Phys. Acta* **33**, 711-726.

[JM] Jauch, J.M., Misra, B. (1961). Supersymmetries and Essential Observables. *Helv. Phys. Acta* **34**, 699-709.

[LD] Laidlaw, M., DeWitt, C. (1971). Feynman Functional Integrals for Systems of Indistinguishable Particles. it Phys. Rev. D, **3**, 1375-1378.

[MG] Messiah, A.M., Greenberg, O.W. (1964). Symmetrization Postulate and its Experimental Foundations. *Phys. Rev.*, **136B**, 248-267.

[Re] Reeh, H. (1988). A Remark Concerning Canonical Commutation Relations. *Jour. Math. Phys.*, **29**, 1535-1536.

[RS] Reed, M., Simon, B. (1972). Methods of Modern Mathematical Physics. Vol. I: Fuctional Analysis. Academic Press, New York - San Francisco - London.

[Sch] Schrödinger, E. (1938). Die Mehrdeutigkeit der Wellenfunktion. Annalen der Physik (Leipzig) **32**, 49-55.

[So] Sorkin, R. (1989). Classical Topology and Quantum Phases. In: Geometrical and Algebraic Aspects of Nonlinear Field Theory, S. de Filippo, M. Marinaro, G. Marmo and G. Vilasi (Editors). Elsevier Science Publishers B.V., North Holland.

[St] Steenrod, N. (1974). The Topology of Fibre Bundles; ninth printing. Princeton University Press, Princeton, New Jersey.

[T] Thirring, W. (1981). A Course in Mathematical Physics, Vol. 3: Quantum Mechanics of Atoms and Molecules. Springer-Verlag, New York, Wien.

[We] Weyl, H. (1981). Gruppentheorie und Quantenmechanik. Wissenschaftliche Buchgesellschaft, Darmstadt

[Wi] Wigner, E. (1931). Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren. Friedr. Vieweg & Sohn Akt.-Ges., Braunschweig.

[Wi1] Wightman, A.S. (1959). Relativistic Invariance and Quantum Mechanics (Notes by A. Barut). *Nuovo Cimento, Suppl.*, **14**, 81-94.

[Wi2] Wightman, A.S. (1995). Superselection Rules; Old and New. *Nouvo Cimento*, **110 B**, 751-769.

[Wo] Woodhouse, N. (1980). Geometric Quantization. Claredon Press, Oxford.