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# Representation of a Spin-1 Entity as a Joint System of Two Spin-$\frac{1}{2}$ Entities on Which we Introduce Correlations of the Second Kind

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*Abstract.* We present a model system in $\mathbb{R}^3$ for a spin-1 quantum entity. If the state of the model system corresponds to a '$|s| = 1$'-state of a spin-1 quantum entity, it behaves as a joint system of two separated spin-$\frac{1}{2}$ quantum entities. If it corresponds to a '$s = 0$'-state of a spin-1 quantum entity, it behaves as the joint system of two spin-$\frac{1}{2}$ quantum entities that are entangled by correlations of the second kind, i.e., it behaves similar as a joint system of two entangled spin-$\frac{1}{2}$ quantum entities in a singlet state.

# 1 Introduction

In [1] and [3], Aerts shows that the joint system of two separated quantum entities cannot be described as the projection lattice of the tensor product. As a consequence, alternative descriptions of joint systems should be reconsidered. In [4], Aerts presents a model system for a quantum system in a singlet state, subjected to Aspect-like measurements. In this model system, Aerts introduces the concept of *correlations of the second kind*, i.e., correlations created during the measurement process.

In section 3 of this paper, we introduce a model system in $\mathbb{R}^3$ for a spin-1 quantum entity subjected to Stern-Gerlach measurements, consisting of a joint system of two entangled[1] spin-$\frac{1}{2}$ quantum entities on which we introduce correlations of the second kind. In section 4, we study the states of this model system, and we'll find out that there are two kinds of states:

i) If the system is in a *state of the first kind*, it behaves during the measurement as the joint system of two separated spin-$\frac{1}{2}$ quantum entities.

ii) If the system is in a *state of the second kind*, it behaves during the measurement very similar as the joint system of two spin-$\frac{1}{2}$ quantum entities in a singlet state.

We consider a spin-1 entity that is prepared in a first Stern-Gerlach apparatus, and measured in a second one, and thus, the set of states that we consider is parameterized by directions in $\mathbb{R}^3$,i.e., we consider a set of coherent spin-1 states. As been shown by Aaberge in [5], the hypothesis that the rotation invariant parameters of a spin-1 system are redundant "does not have dramatic consequences for the standard applications of quantum mechanics".

# 2 Representation of a Spin-$\frac{1}{2}$ Quantum Entity

In this section, we represent the states of a spin-$\frac{1}{2}$ quantum entity on a sphere $S$.

## 2.1 The Transition Probability for Spin-$\frac{1}{2}$ Quantum Entity

The transition probability depends only on the relative position of the Stern-Gerlach apparatus in which we prepare the entity and a second Stern-Gerlach with which we measure the spin. We represent such a measurement by the Euler angles $\alpha, \beta, \gamma$, and denote it as $e_{\alpha,\beta,\gamma}$. If the initial state corresponds with a spin quantum number $s = +\frac{1}{2}$ we denote it as $p_+^0$, and if it corresponds with a spin quantum number $s = -\frac{1}{2}$ we denote it as $p_-^0$. We represent $p_+^0$ by the vector $\psi_+^0 = (1,0) \in \mathbb{C}^2$and $p_-^0$ by $\psi_-^0 = (0,1) \in \mathbb{C}^2$.The eigenstates corresponding to a measurement $e_{\alpha,\beta,\gamma}$ are the same as the ones we obtain when we rotate the initial states by an active rotation characterized by the Euler angles $\alpha, \beta, \gamma$. This active rotation is represented by a unitary operator acting on $\mathbb{C}^2$that corresponds with the following matrix (see [6] and [7]):

$$M_{\alpha,\beta,\gamma} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} cos\frac{\beta}{2} \end{pmatrix} \tag{1}$$

Thus, for the measurement $e_{\alpha,\beta,\gamma}$ we have a set of eigenstates represented by the following vectors:

$$\psi_+^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_+^0 = (e^{-i\frac{\alpha}{2}} cos\frac{\beta}{2}, e^{i\frac{\alpha}{2}} sin\frac{\beta}{2})e^{-i\frac{\gamma}{2}} \tag{2a}$$

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[1]In this paper we call entities entangled whenever they are not separated.

$$\psi_-^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_-^0 = (-e^{-i\frac{\alpha}{2}} sin\frac{\beta}{2}, e^{i\frac{\alpha}{2}} cos\frac{\beta}{2})e^{i\frac{\gamma}{2}} \tag{2b}$$

The vectors in eq(2a) and eq(2b) that correspond to different values of $\gamma$ (for fixed $\alpha$ and $\beta$) represent the same states. As a consequence, we omit the superscript $\gamma$ in the notations for the vectors and the measurements. We represent the states corresponding to the vectors in eq(2a) and eq(2b) respectively by $p_+^{\alpha,\beta}$ and $p_-^{\alpha,\beta}$. Thus we have $p_+^{0,0} = p_+^0$ and $p_-^{0,0} = p_-^0$. We also have:

$$p_+^{\alpha,\beta} = p_-^{\alpha+\pi,\pi-\beta} \tag{3}$$

We denote the probability to obtain a state $p_+^{\alpha,\beta}$ in a measurement $e_{\alpha,\beta}$ on an entity in a state $p_+^0$ as $P_{+,+}^{\alpha,\beta}$, and the probability to obtain $p_-^{\alpha,\beta}$ in a measurement $e_{\alpha,\beta}$ on an entity in a state $p_+^0$ as $P_{+,-}^{\alpha,\beta}$. Analogously we define $P_{-,+}^{\alpha,\beta}$ and $P_{-,-}^{\alpha,\beta}$. We have:

$$P_{+,+}^{\alpha,\beta} = | < \psi_+^0 | \psi_+^{\alpha,\beta} > |^2 = cos^2\frac{\beta}{2} = \frac{1 + cos\beta}{2} \tag{4a}$$

$$P_{-,-}^{\alpha,\beta} = \frac{1 + cos\beta}{2} \tag{4b}$$

$$P_{+,-}^{\alpha,\beta} = P_{-,+}^{\alpha,\beta} = sin^2\frac{\beta}{2} = \frac{1 - cos\beta}{2} \tag{4c}$$

### 2.2 Representation of the Spin-$\frac{1}{2}$ States in $S$

The set of states of a spin-$\frac{1}{2}$ entity is given by (see eq.(3)):

$$\Sigma_{\frac{1}{2}} = \{p_+^{\alpha,\beta} | \alpha \in [0, 2\pi], \beta \in [0, \pi]\} \tag{5}$$

Let $S$ be a unit sphere in $\mathbb{R}^3$ with its center in the origin. We represent every state $p_+^{\alpha,\beta} \in \Sigma_{\frac{1}{2}}$ by the point in $S$ with coordinates $(cos\alpha sin\beta, sin\alpha sin\beta, cos\beta)$. It is clear (as a consequence of the definition of the Euler angles), that the representation of $\Sigma_{\frac{1}{2}}$ in $S$ is one to one and onto.

## 3 A Model System for a Spin-1 Quantum Entity.

In this section we'll introduce the model system for a spin-1 quantum entity. First we represent the states of a spin-1 quantum entity in $S \times S$, i.e., we represent every spin state as two points on a sphere[2].

[2]The representation introduced in this paper is not the same as the Majorana representation for all spin-1 states, presented in [8]. One can easily verify (by applying the construction for the Majorana representation presented in [9]) that our representation could not be extended in a continuous way to a one to one representation of all spin-1 states.

## 3.1 The Transition Probability for Coherent Spin-1 States

We proceed along the same lines as in section 2.1. We denote a measurement characterized by the Euler angles $\alpha, \beta, \gamma$ as $e_{\alpha,\beta,\gamma}$. If the initial state of the entity corresponds with a spin quantum number $s = +1$, we denote it as $p_+^0$. If $s = 0$, we denote it as $p_0^0$ and if $s = -1$ as $p_-^0$. We represent $p_+^0$ by the vector $\psi_+^0 = (1,0,0) \in \mathbb{C}^3$,$p_0^0$ by $\psi_0^0 = (0,1,0) \in \mathbb{C}^3$and $p_-^0$ by $\psi_-^0 = (0,0,1) \in \mathbb{C}^3$.The eigenstates corresponding to a measurement $e_{\alpha,\beta,\gamma}$ are the same as the ones we obtain when we rotate the initial states by an active rotation characterized by the Euler angles $\alpha, \beta, \gamma$. This active rotation is represented by a unitary operator acting on $\mathbb{C}^3$following matrix (see [6] and [7]):

$$M_{\alpha,\beta,\gamma} = \begin{pmatrix} \frac{1+cos\beta}{2}e^{-i\frac{\alpha+\gamma}{2}} & -\frac{sin\beta}{\sqrt{2}}e^{-i\frac{\alpha}{2}} & \frac{1-cos\beta}{2}e^{-i\frac{\alpha-\gamma}{2}} \\ \frac{sin\beta}{\sqrt{2}}e^{-i\frac{\gamma}{2}} & cos\beta & -\frac{sin\beta}{\sqrt{2}}e^{i\frac{\gamma}{2}} \\ \frac{1-cos\beta}{2}e^{i\frac{\alpha-\gamma}{2}} & \frac{sin\beta}{\sqrt{2}}e^{i\frac{\alpha}{2}} & \frac{1+cos\beta}{2}e^{i\frac{\alpha+\gamma}{2}} \end{pmatrix} \tag{6}$$

Thus, for the measurement $e_{\alpha,\beta,\gamma}$ we have a set of eigenstates represented by the following vectors:

$$\psi_+^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_+^0 = \Big(\frac{1+cos\beta}{2}e^{-i\frac{\alpha}{2}}, \frac{sin\beta}{\sqrt{2}}, \frac{1-cos\beta}{2}e^{i\frac{\alpha}{2}}\Big)e^{-i\frac{\gamma}{2}} \tag{7a}$$

$$\psi_0^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_0^0 = \Big(-\frac{sin\beta}{\sqrt{2}}e^{-i\frac{\alpha}{2}}, cos\beta, \frac{sin\beta}{\sqrt{2}}e^{i\frac{\alpha}{2}}\Big) \tag{7b}$$

$$\psi_-^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_-^0 = \Big(\frac{1-cos\beta}{2}e^{-i\frac{\alpha}{2}}, -\frac{sin\beta}{\sqrt{2}}, \frac{1+cos\beta}{2}e^{i\frac{\alpha}{2}}\Big)e^{i\frac{\gamma}{2}} \tag{7c}$$

As been motivated in section 2.1, we represent the states corresponding to the vectors in eq(7a), eq(7b) and eq(7c), respectively by $p_+^{\alpha,\beta}$, $p_0^{\alpha,\beta}$ and $p_-^{\alpha,\beta}$, and we omit the subscript $\gamma$ in the notation of the measurement. We have:

$$p_+^{\alpha,\beta} = p_-^{\alpha,\pi+\beta} \tag{8}$$

$$p_0^{\alpha,\beta} = p_0^{\alpha,\pi+\beta} \tag{9}$$

Using similar notations as introduced in section 2.1, we find the transition probabilities through square of the Hilbert in-product on $\mathbb{C}^3$:

$$P_{0,0}^{\alpha,\beta} = |<\psi_0^0|\psi_0^{\alpha,\beta}>|^2 = cos^2\beta \tag{10a}$$

$$P_{+,+}^{\alpha,\beta} = P_{-,-}^{\alpha,\beta} = \Big(\frac{1+cos\beta}{2}\Big)^2 \tag{10b}$$

$$P_{+,-}^{\alpha,\beta} = P_{-,+}^{\alpha,\beta} = \Big(\frac{1-cos\beta}{2}\Big)^2 \tag{10c}$$

$$P_{+,0}^{\alpha,\beta} = P_{0,+}^{\alpha,\beta} = P_{-,0}^{\alpha,\beta} = P_{0,-}^{\alpha,\beta} = \frac{sin^2\beta}{2} \tag{10d}$$

Again the transition probabilities depend only on the angle $\beta$.

### 3.2 Representation of the Coherent Spin-1 States in $S \times S$

We define the following two subsets of the set of coherent spin states:

$$\Sigma_+ = \{p_+^{\alpha,\beta} | \alpha \in [0, 2\pi], \beta \in [0, \pi]\} \tag{11a}$$

$$\Sigma_0 = \{p_0^{\alpha,\beta} | \alpha \in [0, 2\pi], \beta \in [0, \pi]\} \tag{11b}$$

We call the states contained in $\Sigma_+$ states of the first kind, the states contained in $\Sigma_0$ states of the second kind. The set of all coherent spin-1 states is (see eq(8), eq(9), eq(11a) and eq(11b)):

$$\Sigma_1 = \Sigma_+ \cup \Sigma_0 \tag{12}$$

If $p_+^{\alpha,\beta} \in \Sigma_+$ we represent this state by two identical points in $S$ (one point of $S \times S$) with coordinates $(cos\alpha sin\beta, sin\alpha sin\beta, cos\beta)$. If $p_0^{\alpha,\beta} \in \Sigma_0$ we represent this state by two points in $S$ with respective coordinates $(cos\alpha sin\beta,\ sin\alpha sin\beta,\ cos\beta)$ and $(-cos\alpha sin\beta,\ -sin\alpha sin\beta,\ -cos\beta)$. It is clear that this representation of $\Sigma_1$ in $S \times S$ is one to one (see eq.(9)).

### 3.3 The Model System

In this subsection we introduce a model system. Then we prove that this model system is a representation for a spin-1 quantum entity.

Suppose that the states of model system can be represented by the same subset of $S \times S$, as we used in the previous section for the representation of the coherent states of a spin-1 quantum entity. A state represented as two identical points on the sphere with coordinates $v$ will be denoted as $p_{v,v}$, and a state represented as two diametrically opposite points with coordinates $v$ and $-v$ as $p_{v,-v}$. We choose a set of coordinates such $v = (0, 0, 1)$. We define a measurement $e_u$ on the system in a state $p_{v,v}$ in the following way:

i) We consider the system as a joint system of two separated spin-$\frac{1}{2}$ quantum entities in a state $p_v$.

ii) On both entities perform a measurement[3] with eigenstates $p_u$ and $p_{-u}$.

We define the measurement $e_u$ on the system in a state $p_{-v,v}$ in the following way:

i) We consider the system as a joint system of two spin-$\frac{1}{2}$ quantum entities that are entangled, one of them in a state $p_v$, the other in a state $p_{-v}$.

ii) On one of these two entities, that we denote as $\mathcal{S}_1$, we perform a measurement with eigenstates $p_u$ and $p_{-u}$. Let $u = (u_1, u_2, u_3)$. If, as a consequence of this measurement, we obtain a state $p_u$ for $\mathcal{S}_1$, then the state of the other entity (denoted as $\mathcal{S}_2$) changes to $p_{v'}$, where $v' = (u_1, u_2, -u_3)$. If we obtain a state $p_{-u}$ for $\mathcal{S}_1$, then the state of $\mathcal{S}_2$ changes to $p_{v'}$.

[3]We consider measurements on spin-$\frac{1}{2}$ quantum entities as presented in section 2.

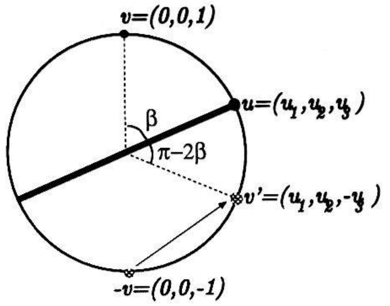


Figure 1: Illustration of the $e_u$ measurements on the model system in a state $p_{v,-v}$. In this illustration, the black spots represent position of $\mathcal{S}_1$.

iii) We perform a measurement with eigenstates $p_u$ and $p_{-u}$ on $\mathcal{S}_2$.

There are three possible outcome states for $e_u$: $p_{u,u}$, $p_{-u,-u}$ or $p_{-u,u}$. Denote by $\beta$ the angle between the vectors $u$ and $v$. We can calculate the transition probabilities. Denote the probability to become a state $p_{u,u'}$ in the measurement $e_u$ on the system in a state $p_{v,v'}$ as $P_{v,v',u,u'}$. We have: $P_{v,v,u,u} = (\frac{1+cos\beta}{2})^2$, $P_{v,v,-u,-u} = (\frac{1-cos\beta}{2})^2$ and $P_{v,v,u,-u} = \frac{1+cos\beta}{2} \cdot \frac{1-cos\beta}{2} + \frac{1-cos\beta}{2} \cdot \frac{1+cos\beta}{2} = \frac{sin^2\beta}{2}$, Following section 3.2, we can identify the state $p_{v,v}$ with the state $p_+^0 \in \Sigma_1$ of a spin-1 quantum entity, and the state $p_{u,u}$ with the state $p_+^{\alpha,\beta} \in \Sigma_1$ and the state $p_{u,-u}$ with the state $p_o^{\alpha,\beta} \in \Sigma_1$. According to eq.(10b), eq.(10c) and eq.(10d), we find the same transition probabilities for the model system as for a spin-1 quantum entity, for an initial state contained in $\Sigma_+$.

For an initial state $p_{v,-v}$ we have to take the two possibilities into account: $\mathcal{S}_1$ is in a state $p_v$ or $\mathcal{S}_2$ is in a state $p_v$. For example: the probability to become a state $p_{u,u}$ in the measurement $e_u$ on the entity in a state $p_{v,-v}$, if $\mathcal{S}_1$ is in a state $p_v$, is equal to the the probability to obtain the state $p_u$ in the measurement on $\mathcal{S}_1$, multiplied by the probability to obtain a state $p_u$ in the measurement on $\mathcal{S}_2$. The probability to obtain $p_u$ in this second measurement is $\frac{1}{2}(1 + cos[\pi - 2\beta])$. We have:

$$P_{v,-v,u,u} = \frac{1}{2} \cdot \frac{1+cos\beta}{2} \cdot \frac{1+cos[\pi-2\beta]}{2} + \frac{1}{2} \cdot \frac{1-cos\beta}{2} \cdot \frac{1+cos[\pi-2\beta]}{2} = \frac{1+cos[\pi-2\beta]}{4} = \frac{sin^2\beta}{2}$$

$$\begin{aligned} P_{v,-v,u,-u} &= \frac{1}{2} \cdot \frac{1+cos\beta}{2} \cdot \frac{1-cos[\pi-2\beta]}{2} + \frac{1}{2} \cdot \frac{1-cos\beta}{2} \cdot \frac{1-cos[\pi-2\beta]}{2} \\ &+ \frac{1}{2} \cdot \frac{1-cos\beta}{2} \cdot \frac{1-cos[\pi-2\beta]}{2} + \frac{1}{2} \cdot \frac{1+cos\beta}{2} \cdot \frac{1-cos[\pi-2\beta]}{2} = \frac{1-cos[\pi-2\beta]}{2} = cos^2\beta \end{aligned}$$

and

$$P_{v,-v,-u,-u} = \frac{1}{2} \cdot \frac{1-cos\beta}{2} \cdot \frac{1+cos[\pi-2\beta]}{2} + \frac{1}{2} \cdot \frac{1+cos\beta}{2} \cdot \frac{1+cos[\pi-2\beta]}{2} = \frac{1+cos[\pi-2\beta]}{4} = \frac{sin^2\beta}{2}$$

Thus, if we identify $p_{v,-v}$ with the state $p_0^0 \in \Sigma_1$ of a spin-1 quantum entity, we can compare eq.(10a) and eq.(10d) with $P_{v,-v,u,u}$, $P_{v,-v,u,-u}$ and $P_{v,-v,-u,-u}$. Again we find the same transition probabilities for the model system as for a spin-1 quantum entity. Thus, this model system is a representation for a spin-1 quantum entity with $\Sigma_1$ as set of states.

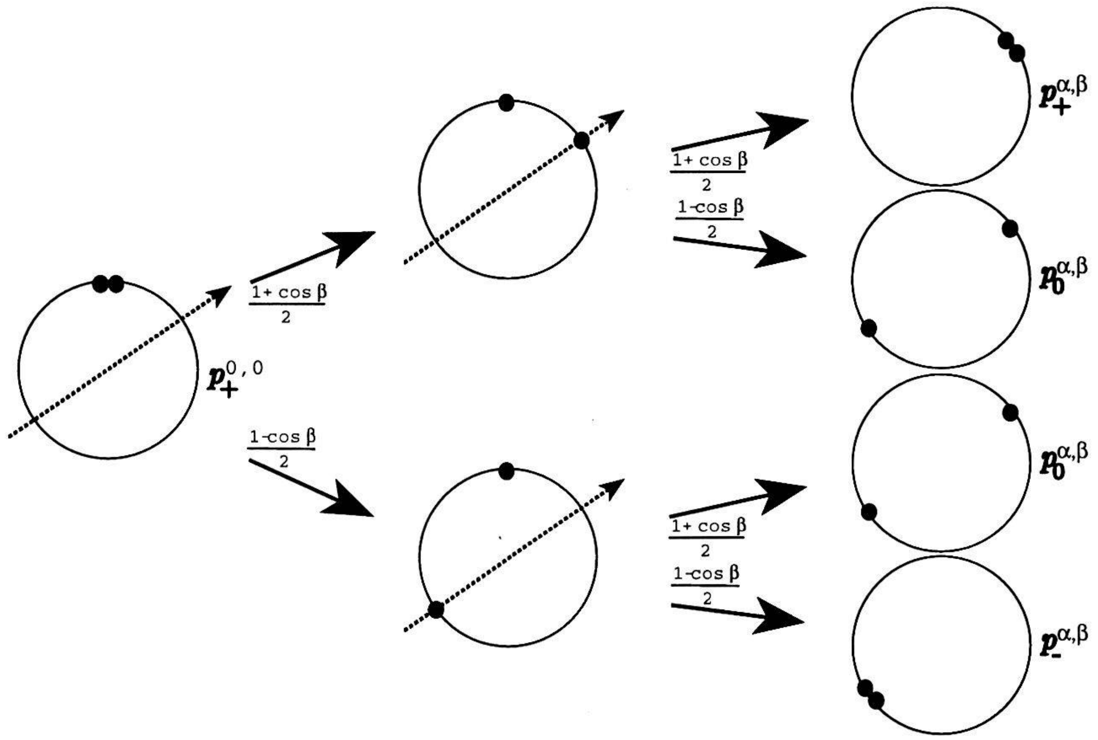


Figure 2: Illustration going with the calculation of the transition probabilities of the model system for spin-1 for an initial state in $\Sigma_1$.

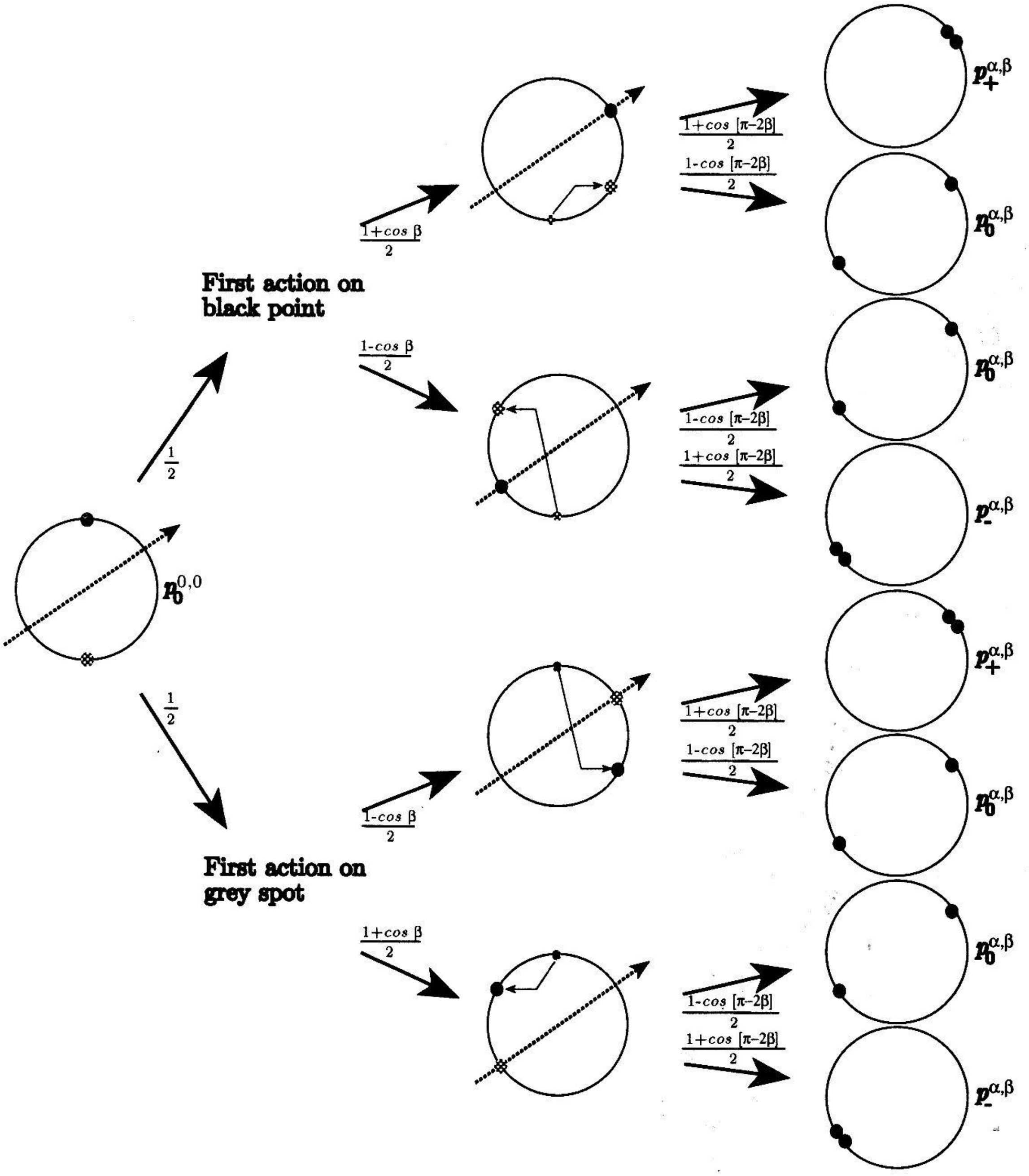


Figure 3: Illustration going with the calculation of the transition probabilities of the model system for spin-1, for an initial state in $\Sigma_0$.

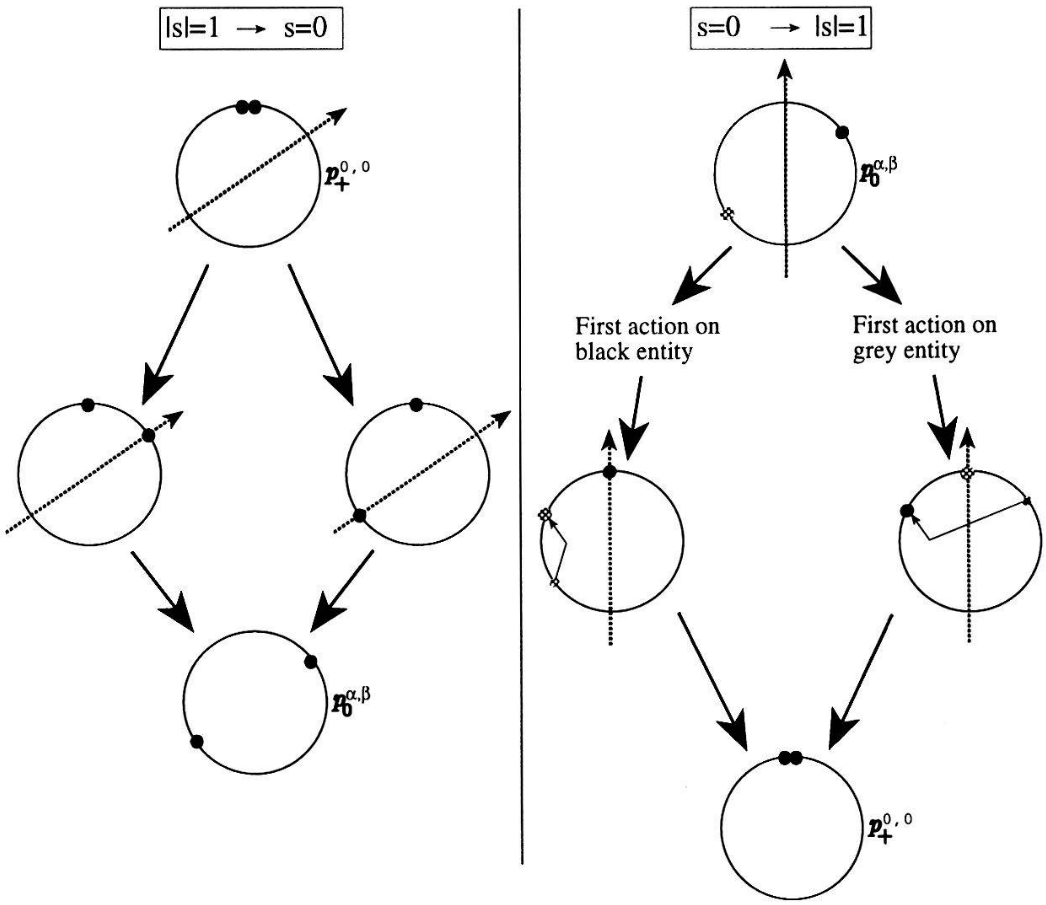


Figure 4: Two state transitions corresponding with two completely different mechanisms, but with the same probability of appearence. The symmetry of the transition probabilty does not necessarly imply a process that is reversibel from a geometrical point of view.

# 4 The States of the Model System

As we saw in section 3.3, $\Sigma_+$ corresponds to the states of a joint system of two separarted spin-$\frac{1}{2}$ quantum entities, and thus, *a spin-1 quantum entity in a state in $\Sigma_+$ behaves as two separated spin-$\frac{1}{2}$ quantum entities.*

In our model system, the correlations are created during the measurement, and thus, *they are of the second kind.* In [4], Aerts shows that due to the presence of correlations of the second kind, the singlet state is a new state, not contained in $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)(\mathcal{H}(X)$ are the rays of $X)$, but in $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.Thus, it are these kinds of correlations that are responsible for the 'quantum nature' of the singlet state. All this indicates a different approach towards the description of joint systems: *if a joint system consists of two separated entities, the states are represented by the cartesian product of the state spaces of these entities; if due to the interaction between the measurement apparatus and the joint system, correlations of the second kind are created, new states occure.* The states in our model system, are also new states: we know that they can be representated in $\mathcal{H}(\mathbb{C}^3)$,a subspace of $\mathcal{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$,and one can verify that, although we have represented the initial states of the model system in $S \times S$, these states do not correspond to the subset $\mathcal{H}(\mathbb{C}^2) \times \mathcal{P}(\mathbb{C}^2)$of $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

The symmetry of the transition probability of the presented model system seems to be 'accidental', in the sense that the two transitions $p_+^{0,0} \longrightarrow p_0^{\alpha,\beta}$ (or $p_+^{\alpha,\beta} \longrightarrow p_0^{0,0}$, or $p_-^{0,0} \longrightarrow p_0^{\alpha,\beta}$) and $p_0^{\alpha,\beta} \longrightarrow p_+^{0,0}$ (or $p_0^{0,0} \longrightarrow p_+^{\alpha,\beta}$, or $p_0^{\alpha,\beta} \longrightarrow p_-^{0,0}$) correspond to processes with a completely different geometry. The geometry of the transition $p_+^{0,0} \longrightarrow p_0^{\alpha,\beta}$ corresponds to the one of a measurement on a joint system of two separated spin-$\frac{1}{2}$ quantum entities. The transition $p_0^{\alpha,\beta} \longrightarrow p_+^{0,0}$ corresponds to a measurement on a joint system of two entangled entities.

# 5 Conclusion

In this paper, we showed that a spin-1 quantum entity can be interpreted as a joint system of two entangled spin-$\frac{1}{2}$ quantum entities. Moreover, by introducing correlations of the second kind on this joint system of two spin-$\frac{1}{2}$ quantum entities, we find states in $\mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ that are not contained in $\mathcal{H}(\mathbb{C}^2) \times \mathcal{H}(\mathbb{C}^2)$. Thus, this model system indicates a new way of looking at a spin-1 quantum entity, and motivates a new approach towards the description of joint systems.

# 6 Acknowledgements

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