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The SLLN for the Free-Energy of a Class of Neural Networks

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Abstract. We first show the self-averaging property in the sense of almost sure convergence for the free energy of the spin glass model and of the Hopfield model with an infinite number of patterns. Then we prove the strong law of large number(SLLN) of the free energy in the Hopfield type model with finite number of patterns. Here the Hopfield type model implies that the interaction among neurons is higher order, the patterns embedded in the neural network are assumed to be independent random variables rather than only taking value $+1$ and -1 and i.i.d. The model with weighted patterns is certainly included in. The SLLN of the free energy in the Little model is proved. The convergence rate for above two cases is also estimated.

1 Introduction

We deal with the problem of convergence of the free energy f_N, associated to a large class of important models of disordered systems, to its mean value (Ef_N) with respect of the probability distribution of the disorder when the number N of components of the system goes to infinity. We consider models of systems with a random interaction described by an assigned probability distribution. We call self-averaging the above convergence. This property is important because the free energy contains the main information about the system and we want to know whether its main features depend on the particular choice of the disorder or not. Also we look for what kind of convergence one has i.e. if $f_N \to Ef_N$ in probability or almost everywhere with respect to the disorder. The disordered systems we are concerned with are the neural networks systems(Hopfield model [2] [7]) and the spin glass model[13]. The neural neworks models are interesting because they exhibit the property of associative memory which is fundamental for the development of artificial intelligence. The spin glass model is connected to the neural networks since they share many properties with each other.

The self-averaging property for the free energy of the Hopfield model with a finite number of patterns has been proved in [3] and there the convergence was shown to be almost everywhere. Also the large deviation has been established for this model[9] [4]. In [17] the self-averaging property in probability is shown for the Hopfield model with an infinite number of patterns p such that $p/N \to \alpha$ as $N \to \infty$, α a constant. This property was used in [16] for obtaining further important results. Pastur and Shcherbina [15] proved the self-averaging property also in the mean square sense for the free energy of the spin glass model. We improve the results in [17] and [15] showing the self-averaging property in the almost sure sense for the free energy of the spin glass model and the Hopfield model with $p/N \to \alpha$, as $N \to \infty$, which is the content of section 2 and section 3.

In section 4 and section 5, we consider the property of the free energy in the *Hopfield type model*, i.e. the patterns embedded in the model is only assumed to be independent and the interaction among neurons is higher order ([1] [8] [10] [14]). Based upon the similar idea as above, under the restriction of the finiteness of the expectation of the fourth order of the imbedded patterns, we proved the SLLN, or self-average in the a.s. sense, for the free energy of the model. It is well known that the equilibrium of the Hopfield network is achieved in terms of asynchronous dynamics[2][11][12]. If we realize the retrieving dynamics in terms of the synchronous dynamics, which is called Little network, the free energy of it is different from that of the Hopfield type network. In the present paper, we also showed the SLLN, or self-average property in the a.s. sense, for the free energy of the Little network. The convergence rate is estimated in above two cases.

2 Models and Results for Hopfield and Spin Glass Model

2.1 Hopfield Model

For a given nonnegative number α, suppose that $\xi_i^\mu, i = 1, \cdots, N, \mu = 1, \cdots, p$ with

$$\lim_{N\to\infty} \frac{p}{N} = \alpha \tag{2.1}$$

are i.i.d. random variables taking values in $\{-1, 1\}$. The connection strength T_{ij} is obtained by the Hebb learning law [2] [6] [7], i.e.

$$T_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu. \tag{2.2}$$

The Hamiltonian for the Hopfield model is defined by

$$H_N^h(\xi, S) = -\frac{1}{2}\sum_{i\neq j} T_{ij} S_i S_j = -\frac{1}{2N}\sum_{i\neq j}\sum_{\mu} \xi_i^\mu \xi_j^\mu S_i S_j \tag{2.3}$$

where $S = (S_1, \cdots, S_N)$, S_i is the neural activity of the i-th neuron, $S_i = \pm 1, i = 1, \cdots, N$. The partition function and free energy are given by

$$Z_N^h(\xi) = \sum_{S\in\{-1,1\}^N} \exp(-\beta H_N^h(\xi, S)) \tag{2.4}$$

and

$$f_N^h(\xi) = -\frac{1}{\beta N}\log Z_N^h(\xi) \tag{2.5}$$

respectively. The purpose of the present paper is to prove the self-averaging property of the free function $f_N^h(\xi)$.

In [17], it is verified that

$$E(Ef_N^h(\xi) - f_N^h(\xi))^2 \to 0 \tag{2.6}$$

as N goes to infinity, here we prove that

Theorem 1 $\forall \alpha \geq 0$, *we have*

$$Ef_N^h(\xi) - f_N^h(\xi) \to 0 \quad a.s. \tag{2.7}$$

Because of the boundness of the free energy, we see that the conclusion in Theorem 1 implies the Theorem 1 in [17], i.e. conclusion (2.6). Combining Theorem 1 and Theorem 2 in [17], we obtain immediately that

Corollary 1 *If $p/N \to 0$ as N goes to infinity, then*

$$\lim_{N\to\infty} f_N^h(\xi) = \min_C[-\frac{1}{\beta}\log 2\cosh\beta C + \frac{C^2}{2}], \quad a.s.$$

2.2 Spin Glass Model

The Hamiltonian for the spin glass model is

$$H_N^s(J,S) = -\frac{1}{2}\sum_{i\neq j} J_{ij}S_iS_j \tag{2.8}$$

where the $J's$ are independent Gaussian random variables with mean zero and variance $1/N$

$$J_{ij} = J_{ji},\ \ i,j = 1,\cdots,N. \tag{2.9}$$

Let $Z_N^s(J)$ be the partition function defined by

$$Z_N^s(J) = \sum_S \exp(-\beta H_N^s(J,S)) \tag{2.10}$$

and $f_N^s(J)$ be the free energy, i.e.

$$\begin{aligned} f_N^s(J) &= -\frac{1}{\beta N}\log\sum_S \exp(-\beta H_N^s(J,S)) \\ &= -\frac{1}{\beta N}\log Z_N^s(J). \end{aligned} \tag{2.11}$$

We have the following theorem similar to Theorem 1(see [15], Theorem 2).

Theorem 2 *For the free energy of the spin glass model, we have*

$$Ef_N^s(\xi) - f_N^s(\xi) \to 0 \quad a.s.$$

The conclusion in the Theorem 2 implies the Theorem 2 in [15] because of the boundness of the free energy of the spin glass model.

3 Proofs of Theorem 1 and Theorem 2

In order to prove the Theorem 1, we first rewrite $Ef_N^h(\xi) - f_N^h(\xi)$ in terms of a sum of a sequence of martingale difference.

$$\begin{aligned} Ef_N^h(\xi) - f_N^h(\xi) &= Ef_N^h(\xi) - E(f_N^h(\xi)|\mathcal{F}_1(N)) + E(f_N^h(\xi)|\mathcal{F}_1(N)) \\ &\quad + \cdots + E(f_N^h(\xi)|\mathcal{F}_{N-1}(N)) - f_N^h(\xi) \\ &= \sum_{k=1}^{N}[E(f_N^h(\xi)|\mathcal{F}_{k-1}(N)) - E(f_N^h(\xi)|\mathcal{F}_k(N))] \end{aligned} \tag{3.1}$$

where the sigma-algebras $\{\mathcal{F}_k(N), k = 1,\cdots,N\}$ are defined in the following way:

$$\begin{aligned} \mathcal{F}_0(N) &= \{\Phi,\Omega\}, \\ \mathcal{F}_k(N) &= \{\xi_j^\mu,\ j\leq k,\ \ \mu = 1,\cdots,p\},\ \ k = 1,\cdots,N. \end{aligned} \tag{3.2}$$

Note that the sigma algebra $\mathcal{F}_k(N)$ dependes on N since $p \sim [\alpha N]$.

Furthermore, let $\psi_k(N) = 1/\beta E(\log Z_N^h(\xi)|\mathcal{F}_k(N)) - 1/\beta E(\log Z_N^h(\xi)|\mathcal{F}_{k-1}(N))$, we have

$$Ef_N(\xi) - f_N(\xi) = \frac{1}{N}\sum_{k=1}^{N} \psi_k(N). \tag{3.3}$$

Hence for fixing N and α, $\{\mathcal{F}_k(N)\}$ is a family of increasing sigma algebra with respect to k and $\{\psi_k(N), \mathcal{F}_k(N), k = 1, \cdots, N\}$ is a sequence martingale difference.

Lemma 1 *$\forall q \geq 0$, an integer, $E\psi_k(N)^{2q} \leq C, k = 1, \cdots, N$ where C is a constant independent of N.*

Proof As in [17], define the Hamiltonians

$$H_k(\xi, S) = -\frac{1}{2N}\sum_{\mu}\sum_{\substack{i,j;i\neq j\\ i,j\neq k}} \xi_i^\mu \xi_j^\mu S_i S_j \tag{3.4}$$

obtained by dropping the terms with S_k in $H_N^h(\xi, S)$, and

$$\tilde{H}_k(\xi, S; t) = H_k(\xi, S) + tR_k(\xi, S) \tag{3.5}$$

where $t \in R^1$,

$$R_k(\xi, S) = -\frac{1}{N}\sum_{\mu}\sum_{j\neq k} \xi_k^\mu S_k \xi_j^\mu S_j. \tag{3.6}$$

Hence $\tilde{H}_k(\xi, S; 1) = H_N^h(\xi, S), \tilde{H}_k(\xi, S; 0) = H_k(\xi, S)$. Let

$$\tilde{f}_k(\xi, t) = -\frac{1}{\beta}\log\frac{Z_N(\tilde{H}_k(\xi, S; t))}{Z_N(\tilde{H}_k(\xi, S; 0))} = -\frac{1}{\beta}\log\frac{Z_N(\tilde{H}_k(\xi, S; t))}{Z_N(H_k(\xi, S))} \tag{3.7}$$

for $Z_N(H)$ representing the partition function with the Hamiltonian H defined similar to that in section 2.

¿From the independence of $\tilde{H}_k(\xi, S; 0)$ and $\xi_k^\mu, \mu = 1, \cdots, p$, we see that

$$\frac{1}{\beta}E(\log Z_N(\tilde{H}_k(\xi, S; 0))|\mathcal{F}_k(N)) = \frac{1}{\beta}E(\log Z_N(\tilde{H}_k(\xi, S; 0))|\mathcal{F}_{k-1}(N)). \tag{3.8}$$

Hence we can rewrite $\psi_k(N)$ as follows

$$\begin{aligned}
\psi_k(N) &= \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_k(N)) - \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_{k-1}(N)) \\
&= \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_k(N)) - \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_{k-1}(N)) \\
&\quad -\frac{1}{\beta}E(\log Z_N(\tilde{H}_k(\xi, S; 0))|\mathcal{F}_k(N)) + \frac{1}{\beta}E(\log Z_N(\tilde{H}_k(\xi, S; 0))|\mathcal{F}_{k-1}(N)) \\
&= -E(\tilde{f}_k(\xi, 1)|\mathcal{F}_k(N)) + E(\tilde{f}_k(\xi, 1)|\mathcal{F}_{k-1}(N))
\end{aligned} \tag{3.9}$$

which, togehter with the basic inequality $(a+b)^{2q} \le 2^{2q-1}(a^{2q}+b^{2q})$ for $a \ge 0, b \ge 0$ and the Jensen inequality $E((\tilde{f}_k(\xi,1))^{2q}|\mathcal{F}_k(N)) \ge [E(|\tilde{f}_k(\xi,1)||\mathcal{F}_k(N))]^{2q}$, imply that

$$\begin{aligned} E\psi_k^{2q}(N) &\le 2^{2q-1}E([E(|\tilde{f}_k(\xi,1)||\mathcal{F}_k(N))]^{2q} + [E(|\tilde{f}_k(\xi,1)||\mathcal{F}_{k-1}(N))]^{2q}) \\ &\le 2^{2q-1}E(E(\tilde{f}_k(\xi,1)^{2q}|\mathcal{F}_k(N)) + E(\tilde{f}_k(\xi,1)^{2q}|\mathcal{F}_{k-1}(N))) \\ &\le 2^{2q}E(\tilde{f}_k(\xi,1))^{2q}. \end{aligned} \tag{3.10}$$

On the other hand, we note that

$$\frac{d^2}{dt^2}\tilde{f}_k(\xi,t) \le 0, \tag{3.11}$$

and $\tilde{f}_k(\xi,0) = 0$. One thus obtain that

$$\tilde{f}_k(\xi,1)' \le \tilde{f}_k(\xi,1) \le \tilde{f}_k(\xi,0)'. \tag{3.12}$$

We can use the following formula

$$\begin{aligned} \tilde{f}_k(\xi,1)' &= \frac{\sum_S R_k(\xi,S)\exp(-\beta\tilde{H}_k(\xi,S;1))}{Z_N(\tilde{H}_k(\xi,S;1))} \\ &= -\frac{1}{N}\frac{\sum_S\sum_\mu\sum_{j\ne k}\xi_k^\mu S_k\xi_j^\mu S_j\exp(-\beta\tilde{H}_k(\xi,S;1))}{Z_N(\tilde{H}_k(\xi,S;1))} \\ &= -\frac{1}{N}E_1(\sum_\mu\sum_{j\ne k}\xi_k^\mu S_k\xi_j^\mu S_j) \end{aligned} \tag{3.13}$$

where we use E_1 to represent the expectation with respect to the random probability measure

$$P_\xi(S) = \frac{\exp(-\beta\tilde{H}_k(\xi,S;1))}{Z_N(\tilde{H}_k(\xi,S;1))} \tag{3.14}$$

on $\{-1,1\}^N$ and

$$\begin{aligned} \tilde{f}_k(\xi,0)' &= \frac{\sum_S R_k(\xi,S)\exp(-\beta\tilde{H}_k(\xi,S;0))}{Z_N(\tilde{H}_k(\xi,S;0))} \\ &= -\frac{1}{N}\frac{\sum_S\sum_\mu\sum_{j\ne k}\xi_k^\mu S_k\xi_j^\mu S_j\exp(-\beta\tilde{H}_k(\xi,S;0))}{Z_N(\tilde{H}_k(\xi,S;0))} \\ &= -\frac{1}{N}E_0(\sum_\mu\sum_{j\ne k}\xi_k^\mu S_k\xi_j^\mu S_j) \end{aligned} \tag{3.15}$$

where E_0 represents the expectation with respect to the random probability measure

$$\tilde{P}_\xi(S) = \frac{\exp(-\beta\tilde{H}_k(\xi,S;0))}{Z_N(\tilde{H}_k(\xi,S;0))}. \tag{3.16}$$

Next we are going to estimate two terms $\tilde{f}_k(\xi,1)'$ and $\tilde{f}_k(\xi,0)'$ above. Because of the symmetry of the H and of its E-expectation with respect to the indexes, we see that

$$\begin{aligned}
E(f_k(\xi,1)')^{2q} &= E((E_1\sum_{l\neq k} T_{kl}S_kS_l)^{2q}) \\
&\leq E(E_1(\sum_{l_1,\cdots,l_{2q}\neq k} T_{kl_1}S_{l_1}\cdots T_{kl_{2q}}S_{l_{2q}})) \\
&= \frac{1}{N}E(E_1(\sum_{l_1,\cdots,l_{2q-2}\neq k}\sum_{l_{2q-1},l_{2q}} T_{kl_1}\cdots T_{kl_{2q}}S_{l_1}\cdots S_{l_{2q}})) \qquad (3.17) \\
&\leq E(E_1(\sum_{l_1,\cdots,l_{2q-2}\neq k} ||T||^2 T_{kl_1}\cdots T_{kl_{2q-2}}S_{l_1}\cdots S_{l_{2q-2}})) \\
&\leq \cdots \leq E||T||^{2q}
\end{aligned}$$

where $||T||$ is the largest eigenvalue of the mtrix T. ¿From Lemma 4.2[17], we have that

$$E||T||^{2q} \leq C_1 \qquad (3.18)$$

where C_1 is a constant independent of N. Furthermore, we have a similar estimate for $\tilde{f}_k(\xi,0)'$

$$\begin{aligned}
E(\tilde{f}_k(\xi,0)')^{2q} &\leq \frac{1}{N^{2q}}E(E_0(\sum_{l_1,\cdots,l_{2q}\neq k}\sum_{\mu_1,\cdots,\mu_{2q}} \xi_k^{\mu_1}\xi_{l_1}^{\mu_1}S_{l_1}\cdots\xi_k^{\mu_{2q}}\xi_{l_{2q}}^{\mu_{2q}}S_{2q})) \\
&\leq \frac{1}{N^{2q}}\sum_{\mu_1,\cdots,\mu_{2q}} 1 + \frac{1}{N^{2q}}E(E_0(\sum_{l_1,\cdots,l_{2q}}\sum_{\mu_1,\cdots,\mu_q} S_{l_1}\xi_{l_1}^{\mu_1}S_{l_2}\xi_{l_2}^{\mu_1} \qquad (3.19) \\
&\quad \cdots S_{l_{2q-1}}\xi_{l_{2q-1}}^{\mu_q}S_{l_{2q}}\xi_{l_{2q}}^{\mu_q})) \\
&\leq C_3\alpha^{2q} + C_4E||T||^q \\
&\leq C_2
\end{aligned}$$

where C_2, C_3 and C_4 are all constants independent of N. So we have finally proven the Lemma.

Now we are able to prove Theorem 1.

Proof of Theorem 1 To prove the theorem, according the definition of almost surely convergence and equation (3.3), we only need to consider that

$$\begin{aligned}
P(\sup_{k_1\leq N}\frac{1}{N}|\sum_{k=1}^{N}\psi_k(N)| \geq \epsilon) &\leq \sum_{k_1\leq N} P(\frac{1}{N}|\sum_{k=1}^{N}\psi_k(N)| \geq \epsilon) \qquad (3.20) \\
&\leq \sum_{k_1\leq N}\frac{E(\sum_{k=1}^{N}\psi_k(N))^4}{\epsilon^4N^4}.
\end{aligned}$$

¿From the Burkholder inequality[5], we know that

$$\begin{aligned}
E(\sum_{k=1}^{N}\psi_k(N))^4 &\leq b_2E(\sum_{k=1}^{N}\psi_k(N)^2)^2 \\
&\leq b_2E(\sum_{k=1}^{N}\psi_k(N)^4)(\sum_{k=1}^{N}1) \qquad (3.21) \\
&\leq b_2(\sum_{k=1}^{N}E\psi_k(N)^4)N
\end{aligned}$$

where $b_2 = 18(4)^{3/2}(3^{1/2})$ is a constant independent of N. From Lemma 1 and the inequality above, we yield that

$$\begin{aligned} P(\sup_{k_1\leq N}\tfrac{1}{N}|\textstyle\sum_{k=1}^N \psi_k(N)| \geq \epsilon) &\leq \textstyle\sum_{k_1\leq N} \dfrac{E(\sum_{k=1}^N \psi_k(N))^4}{\epsilon^4 N^4} \\ &\leq b_2 \textstyle\sum_{k_1\leq N} \dfrac{C_1+C_2}{\epsilon^4 N^2} \\ &\longrightarrow 0 \end{aligned} \tag{3.22}$$

as k_1 goes to infinity.

Proof of Theorem 2 We omit the proof of it here since it is similar to that of Theorem 1.

4 Models and Results for the Hopfield type model and the Little model

Suppose that $\{\xi_i^\mu, \mu = 1,\cdots,p, i = 1,\cdots,N\}$ are random variables and $\{\xi_i^\mu, \mu = 1,\cdots,p\}$ are independent for different i, where p is fixed and finite. In the terminology of neural netwrok $\{\xi_i^\mu, i = 1,\cdots,N\}$ is called the μ-th pattern, embedded in the neural network in terms of the Hebb law as in section 2, i.e.

$$T_{i_1\cdots i_r} = \frac{1}{N^{r-1}}\sum_{\mu=1}^p \xi_{i_1}^\mu \xi_{i_2}^\mu \cdots \xi_{i_r}^\mu \tag{4.1}$$

and to be retrieved, where $i_1,\cdots,i_r = 1,\cdots,N$, r is a fixed number. For each $S \in \{-1,1\}^N$, we associate a Hamiltonian

$$\begin{aligned} H_N(\xi,S) &= -\sum_{i_1\neq i_2\neq\cdots\neq i_r} T_{i_1\cdots i_r} S_{i_1} S_{i_2}\cdots S_{i_r} \\ &= -\sum_{i_1\neq i_2\neq\cdots\neq i_r} \frac{1}{N^{r-1}}\sum_{\mu=1}^p \xi_{i_1}^\mu \xi_{i_2}^\mu \cdots \xi_{i_r}^\mu S_{i_1} S_{i_2}\cdots S_{i_r} \end{aligned} \tag{4.2}$$

to it. So $T_{i_1\cdots i_r}, i_1,\cdots,i_r = 1,\cdots,N$ are r-order interaction among neurons. Let $Z_N(\xi)$ be the partition function defined by

$$Z_N(\xi) = \sum_{S\in\{-1,1\}^N} \exp[-\beta H_N(\xi,S)] \tag{4.3}$$

and $f_N(\xi)$ be the free energy, i.e.

$$\begin{aligned} f_N(\xi) &= -\frac{1}{\beta N}\log Z_N(\xi) \\ &= -\frac{1}{\beta N}\log \sum_{S\in\{-1,1\}^N} \exp[-\beta H_N(\xi,S)]. \end{aligned} \tag{4.4}$$

Now we state one of our main results of the present section

Theorem 3 *i)If $E|\xi_i^\mu|^4 \le M < \infty, \forall i, \mu$, we have*

$$\lim_{N\to\infty} |f_N(\xi) - Ef_N(\xi)| = 0, \qquad a.e.$$

ii)If $E|\xi_i^\mu|^{2q} \le M < \infty,\ i = 1,\cdots,N,\ \mu = 1,\cdots,p$ and q an integer, we have

$$E|f_N(\xi) - Ef_N(\xi)|^{2q} \le \frac{b_q 2^{2q} p^{2q} M^r}{N^q}.$$

where $b_q = 18(2q)^{3/2}(2q-1)^{1/2}$.

As a simple Corollary, we obtain the self-average property for the Hopfield model with higher order weighted patterns.

Corollary 2 *If $\xi_i^\mu = (a_\mu)^{\frac{1}{r}}\eta_i^\mu, \eta_i^\mu = +1$ or, -1 $i = 1,\cdots,N, \mu = 1,\cdots,p$ and i.i.d.,where $a_\mu, \mu = 1,\cdots,p$ are positive constants, we have*

$$\lim_{N\to\infty} |f_N(\xi) - Ef_N(\xi)| = 0, \qquad a.e.$$

and

$$E|f_N(\xi) - Ef_N(\xi)|^{2q} \le \frac{b_q 2^{2q} p^{2q} M^r}{N^q}$$

where $b_q = 18(2q)^{3/2}(2q-1)^{1/2}$.

Next, we consider the SLLN for the Little Network, in this case the Hamiltonian is[2]

$$\overline{H}_N(\xi, S) = -\frac{1}{\beta}\sum_{i=1}^N \log[\cosh\beta\sum_{j=1}^N (T_{ij}S_j)] \tag{4.5}$$

for $S = (S_1,\cdots,S_N)$ and T_{ij} defined as before, i.e.

$$T_{ij} = \begin{cases} \frac{1}{N}\sum_\mu \xi_i^\mu \xi_j^\mu, & \text{if } i \ne j \\ 0, & \text{if } i = j. \end{cases}$$

Remark 1. In fact, we can obtain the same conclusion for high order Little network, we consider 2-order case here only for the simplicity of notation.

The partition function and free energy are defined respectively by

$$\begin{aligned} \overline{Z}_N(\xi) &= \sum_{S\in\{-1,1\}^N} \exp(-\beta\overline{H}_N(\xi,S)) \\ &= \sum_{S\in\{-1,1\}^N} \exp\{\sum_{i=1}^N \log[\cosh(\beta\sum_{j=1}^N T_{ij}S_j)]\} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\overline{f}_N(\xi) &= -\frac{1}{\beta N}\log \overline{Z}_N(\xi)\\ &= -\frac{1}{\beta N}\log \sum_{S\in\{-1,1\}^N} \exp\{\sum_{i=1}^N \log[\cosh(\beta \sum_{j=1}^N T_{ij}S_j)]\}.\end{aligned} \tag{4.7}$$

As in the case of Hopfield model, we have the following self-average property in the a.s. sense for the free energy in the Little network.

Theorem 4 *i)If* $E|\xi_i^\mu|^4 \le M < \infty, \forall i, \mu$, *we have*

$$\lim_{N\to\infty} |\overline{f}_N(\xi) - E\overline{f}_N(\xi)| = 0, \qquad a.e.$$

ii)If $E|\xi_i^\mu|^{2q} \le M < \infty,\ i = 1,\cdots,N, \mu = 1,\cdots,p$ *and* q *an integer, we have*

$$E|\overline{f}_N(\xi) - E\overline{f}_N(\xi)|^{2q} \le \frac{b_q 2^{4q} p^{2q} M^2}{N^q}$$

where $b_q = 18(2q)^{3/2}(2q-1)^{1/2}$.

5 Proof of the Theorem 3 and Theorem 4

The main tool we emploit here to prove the Theorems is similar to that in section 3. So we only sketch the proof here. Rewrite $f_N(\xi) - Ef_N(\xi)$ in the sum of a sequence of martingale difference.

$$\begin{aligned}Ef_N(\xi) - f_N(\xi) &= Ef_N(\xi) - E(f_N(\xi)|\mathcal{F}_1) + E(f_N(\xi)|\mathcal{F}_1) + \\ &\qquad \cdots + E(f_N(\xi)|\mathcal{F}_{N-1}) - f_N(\xi)\\ &= \sum_{k=1}^N [E(f_N(\xi)|\mathcal{F}_{k-1}) - E(f_N(\xi)|\mathcal{F}_k)]\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}\mathcal{F}_0 &= \{\Phi, \Omega\},\\ \mathcal{F}_k &= \{\xi_j^\mu, j\le k, \mu = 1,\cdots,p\}\end{aligned} \tag{5.2}$$

for $k = 1,\cdots,N$.

Let us consider each term of the equation (5.1)

$$\begin{aligned}&E\ \ (f_N(\xi)|\mathcal{F}_{k-1}) - E(f_N(\xi)|\mathcal{F}_k)\\ &= \frac{1}{N}[-\frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_{k-1}) + \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_k)]\\ &= \frac{1}{N}\psi_k(N)\end{aligned} \tag{5.3}$$

where $\psi_k(N) = 1/\beta E(\log Z_N(\xi)|\mathcal{F}_k) - 1/\beta E(\log Z_N(\xi)|\mathcal{F}_{k-1})$.

Hence we have that

$$Ef_N(\xi) - f_N(\xi) = \frac{1}{N}\sum_{k=1}^{N}\psi_k(N) \tag{5.4}$$

and note that only for *fixed* N, the sequence $\{\psi_k(N), \mathcal{F}_k, k = 1, \cdots, N\}$ is a sequence of martingale difference due to the fact that

$$\begin{aligned} E(\psi_k(N)|\mathcal{F}_{k-1}) &= \frac{1}{\beta}E(E(\log Z_N(\xi)|\mathcal{F}_k)|\mathcal{F}_{k-1}) - \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_{k-1}) \\ &= 0. \end{aligned} \tag{5.5}$$

Lemma 2 *If* $E|\xi_i^\mu|^q \le M < \infty, \forall i, \mu$, *we have that*

$$E|\psi_k|^q \le 2^q p^q M^r, \ k = 1, \cdots, N.$$

Proof Define the Hamiltonian

$$H_N^{(k)}(\xi, S) = -\frac{1}{N^{r-1}}\sum_{\substack{i_1\neq i_2\neq\cdots\neq i_r\\ i_1,i_2,\cdots,i_r\neq k}}\sum_{\mu=1}^{p}\xi_{i_1}^\mu\xi_{i_2}^\mu\cdots\xi_{i_r}^\mu S_{i_1}S_{i_2}\cdots S_{i_r} \tag{5.6}$$

for $k = 1, \cdots, N$. Note that $H_N^{(k)}$ is obtained by cancelling all terms ralated to $\xi_k^\mu, \mu = 1, \cdots, p$ in $H_N(\xi, S)$, i.e.

$$H_N(\xi, S) = H_N^{(k)}(\xi, S) + R_N^{(k)}(\xi, S) \tag{5.7}$$

for

$$R_N^{(k)}(\xi, S) = -\frac{1}{N^{r-1}}\sum_{\substack{i_2\neq\cdots\neq i_r\\ i_2,\cdots,i_r\neq k}}\sum_{\mu=1}^{p}\xi_k^\mu\xi_{i_2}^\mu\cdots\xi_{i_r}^\mu S_k S_{i_2}\cdots S_{i_r}.$$

Let

$$f^{(k)}(\xi) = -\frac{1}{\beta}\log\frac{\sum_S \exp(-\beta H_N(\xi, S))}{\sum_S \exp(-\beta H_N^{(k)}(\xi, S))}, \tag{5.8}$$

then we have the following expression for $\psi_k(N), k = 1, \cdots, N$

$$\begin{aligned} \psi_k(N) &= \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_k) - \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_{k-1}) \\ &= \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_k) - \frac{1}{\beta}E(\log Z_N(\xi)|\mathcal{F}_{k-1}) \\ &\quad -\frac{1}{\beta}E[\log\sum_S \exp(-\beta H_N^{(k)}(\xi, S))|\mathcal{F}_k] \\ &\quad +\frac{1}{\beta}E[\log\sum_S \exp(-\beta H_N^{(k)}(\xi, S))|\mathcal{F}_{k-1}] \\ &= \frac{1}{\beta}E[\log\frac{\sum_S \exp(-\beta H_N(\xi, S))}{\sum_S \exp(-\beta H_N^{(k)}(\xi, S))}|\mathcal{F}_k] \\ &\quad -\frac{1}{\beta}E[\log\frac{\sum_S \exp(-\beta H_N(\xi, S))}{\sum_S \exp(-\beta H_N^{(k)}(\xi, S))}|\mathcal{F}_{k-1}] \\ &= \frac{1}{\beta}E(f^{(k)}(\xi)|\mathcal{F}_k) - \frac{1}{\beta}E(f^{(k)}(\xi)|\mathcal{F}_{k-1}) \end{aligned} \tag{5.9}$$

where in the second equality, we use the independent of $\{\xi_k^\mu, \mu = 1, \cdots, p\}$ for different k. Therefore, after processing the same as in inequality (3.10), we have that

$$E|\psi_k(N)|^q \le 2^q E|f^{(k)}(\xi)|^q \tag{5.10}$$

On the other hand, we note that

$$\begin{aligned}
|f^{(k)}(\xi)| &= \frac{1}{\beta}|\log \frac{\sum_S \exp(-\beta H_N(\xi,S))}{\sum_S \exp(-\beta H_N^{(k)}(\xi,S))}| \\
&= \frac{1}{\beta}|\log \frac{\sum_S \exp(-\beta H_N^{(k)}(\xi,S)) \cdot \exp(-\beta R_N^{(k)}(\xi,S))}{\sum_S \exp(-\beta H_N^{(k)}(\xi,S))}| \\
&\le \frac{1}{\beta}|\log \frac{\sum_S \exp(-\beta H_N^{(k)}(\xi,S)) \cdot \exp[\frac{\beta}{N^{r-1}} \sum\limits_{\substack{i_2 \neq \cdots \neq i_r \\ i_2,\cdots,i_r \neq k}} \sum\limits_{\mu=1}^{p} |\xi_k^\mu||\xi_{i_2}^\mu| \cdots |\xi_{i_r}^\mu|]}{\sum_S \exp(-\beta H_N^{(k)}(\xi,S))}| \\
&\le \frac{1}{N^{r-1}} \sum_{\substack{i_2 \neq \cdots \neq i_r \\ i_2,\cdots,i_r \neq k}} \sum_{\mu=1}^{p} |\xi_k^\mu||\xi_{i_2}^\mu| \cdots |\xi_{i_r}^\mu|.
\end{aligned} \tag{5.11}$$

Hence we derive that

$$\begin{aligned}
E|\psi_k(N)|^q &\le 2^q (\frac{1}{N^{r-1}})^q p^{q-1} \sum_{\mu=1}^{p} E(\sum_{\substack{i_2 \neq \cdots \neq i_r \\ i_2,\cdots,i_r \neq k}} |\xi_k^\mu||\xi_{i_2}^\mu| \cdots |\xi_{i_r}^\mu|)^q \\
&\le 2^q (\frac{1}{N^{r-1}})^q p^{q-1} \sum_{\mu=1}^{p} (N^{r-1})^{q-1} \sum_{\substack{i_2 \neq \cdots \neq i_r \\ i_2,\cdots,i_r \neq k}} E|\xi_k^\mu|^q |\xi_{i_2}^\mu|^q \cdots |\xi_{i_r}^\mu|^q \\
&= 2^q p^{q-1} \frac{1}{N^{r-1}} \sum_{\mu} \sum_{\substack{i_2 \neq \cdots \neq i_r \\ i_2,\cdots,i_r \neq k}} M^r \\
&\le 2^q p^q M^r.
\end{aligned} \tag{5.12}$$

Now we are ready to prove Theorem 3.

Proof of Theorem 3

i). We omit it here since it is the same as in the proof of Theorem 1.

ii). From equation (5.4), we see that

$$E(Ef_N(\xi) - f_N(\xi))^{2q} = E\frac{1}{N^{2q}}(\sum_{k=1}^{N} \psi_k(N))^{2q}$$

After repeating the similar procedure of the proof of i) and by using of Burkholder's inequality again, we have

$$\begin{aligned} E(Ef_N(\xi)-f_N(\xi))^{2q} &\le E\frac{1}{N^{2q}}b_q(\sum_{k=1}^{N}\psi_k(N)^{2q})N^{q-1} \\ &\le \frac{b_q 2^{2q}p^{2q}M^r}{N^q} \end{aligned} \tag{5.13}$$

where b_q is the constant in the Burkhold's inequality.

Proof of Corollary 1 We only need to check that the conditions in Theorem 1 is fulfilled. In fact, it is straightforward.

Proof of Theorem 4 After taking similar procedure as the the proof of Theorem 1, we arrive at that

$$E\overline{f}_N(\xi)-\overline{f}_N(\xi)=\frac{1}{N}\sum_{k=1}^{N}\overline{\psi}_k(N) \tag{5.14}$$

where

$$\begin{aligned} \overline{\psi}_k(N) &= \frac{1}{\beta}E(\log\overline{Z}_N(\xi)|\mathcal{F}_k)-\frac{1}{\beta}E(\log\overline{Z}_N(\xi)|\mathcal{F}_{k-1}) \\ &= \frac{1}{\beta}E(\overline{f}^{(k)}(\xi)|\mathcal{F}_k)-\frac{1}{\beta}E(\overline{f}^{(k)}(\xi)|\mathcal{F}_{k-1}) \end{aligned} \tag{5.15}$$

for

$$\overline{f}^{(k)}(\xi)=-\frac{1}{\beta}\log\frac{\sum_{S\in\{1,-1\}^N}\exp[\sum_{i=1}^{N}\log(\cosh(\beta\sum_{j=1}^{N}T_{ij}S_j))]}{\sum_{S\in\{1,-1\}^N}\exp[\sum_{i\neq k}\log(\cosh(\beta\sum_{j\neq k}T_{ij}S_j))]}. \tag{5.16}$$

Therefore, again after processing the same as in the inequality (3.10) we have that

$$E|\overline{\psi}_k(N)|^q\le 2^qE|\overline{f}^{(k)}(\xi)|^q \tag{5.17}$$

and

$$\begin{aligned} \overline{f}^{(k)}(\xi) &= -\frac{1}{\beta}\log\{\sum_{S\in\{1,-1\}^N}\exp[\sum_{i\neq k}\log(\cosh(\beta\sum_{j=1}^{N}T_{ij}S_j))] \\ &\quad\cdot\exp[\log(\cosh(\beta\sum_{j\neq k}T_{kj}S_j))]\} \\ &\quad+\frac{1}{\beta}\log\{\sum_{S\in\{1,-1\}^N}\exp[\sum_{i\neq k}\log(\cosh(\beta\sum_{j\neq k}T_{ij}S_j))]\}. \end{aligned} \tag{5.18}$$

Furthermore, by noting that

$$\begin{aligned} \log\cosh(\beta\textstyle\sum_{j=1}^{N}T_{ij}S_j) &= \log\frac{\exp(-\beta\sum_{j=1}^{N}T_{ij}S_j)+\exp(\beta\sum_{j=1}^{N}T_{ij}S_j)}{2} \\ &\le \log\frac{\exp(-\beta\sum_{j\neq k}T_{ij}S_j)+\exp(\beta\sum_{j\neq k}T_{ij}S_j)}{2}+(\beta|T_{ik}|) \end{aligned} \tag{5.19}$$

and

$$\log\cosh(\beta\sum_{j=1}^{N}T_{ij}S_j)\ge\log\frac{\exp(-\beta\sum_{j\neq k}T_{ij}S_j)+\exp(\beta\sum_{j\neq k}T_{ij}S_j)}{2}-(\beta|T_{ik}|), \tag{5.20}$$

we obtain

$$
\begin{aligned}
|\overline{f}^{(k)}(\xi)| &\le \frac{1}{\beta}\log\{\sum_{S\in\{1,-1\}^N}\exp[\sum_{i\neq k}\log(\cosh(\beta\sum_{j\neq k}T_{ij}S_j))] \\
&\quad \cdot\exp[\sum_{i\neq k}\beta|T_{ik}|]\exp[\sum_{j\neq k}\beta|T_{kj}|]\} \\
&\quad -\frac{1}{\beta}\log\{\sum_{S\in\{1,-1\}^N}\exp[\sum_{i\neq k}\log(\cosh(\beta\sum_{j\neq k}T_{ij}S_j))]\} \\
&\le \sum_{j\neq k}|T_{kj}|+\sum_{i\neq k}|T_{ik}| \\
&\le \frac{2}{N}\sum_j\sum_\mu|\xi_k^\mu||\xi_j^\mu|.
\end{aligned}
\tag{5.21}
$$

Hence we obtain Theorem 4 after treating similarly as in the proof of Theorem 3.

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