

**Zeitschrift:** Helvetica Physica Acta  
**Band:** 68 (1995)  
**Heft:** 2

**Artikel:** An estimate regarding one-dimensional point interactions  
**Autor:** Fassari, Silvestro  
**DOI:** <https://doi.org/10.5169/seals-116731>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 07.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# An Estimate Regarding One-Dimensional Point Interactions

By Silvestro Fassari

Institut für Theoretische Physik der Universität Zürich,  
Winterthurerstr. 190, CH-8057 Zürich, Switzerland

*Abstract.* It is shown that the difference between the eigenfunction of the Hamiltonian of an attractive point interaction and the one of the Hamiltonian with the short-range potential converging to the point interaction goes to zero in the  $L^2$ -norm like a power of the scaling parameter with exponent less than  $1/2$ .

## 1 Introduction

As is well known, point interactions are used as an approximation for short-range potentials. In addition, Schroedinger and Dirac Hamiltonians with point interactions have the advantage of being exactly solvable models, that is to say one can explicitly compute eigenvalues, resonances and other important physical quantities. Most of the results regarding Hamiltonians with point interactions can be found in [1] and the related literature.

In this paper we are concerned with the estimate of the  $L^2$ -norm of the difference between the eigenvector of the Schroedinger Hamiltonian with a point interaction  $H_0 - \|V\|_1 \delta$  with  $V \geq 0$ ,  $\|e^{2a|x|}V\|_1 < \infty$  for some  $a > 0$ , i.e. the exactly solvable model, and that of the Hamiltonian  $H_\epsilon = -\frac{d^2}{dx^2} - V_\epsilon$ ,  $V_\epsilon(x) = \epsilon^{-1}V(\frac{x}{\epsilon})$ , that is to say the Hamiltonian with a short-range potential which converges to  $H_0$  in the norm resolvent sense and represents the more realistic model.

The estimate will show that such a norm goes to zero like a power of the scaling parameter

$\epsilon$  with exponent less than  $1/2$ .

The need for such an estimate has arisen in the context of the problem of the stabilisation of atoms in superintense laser fields for which we refer the reader to [4] and the literature cited therein.

## 2 The estimate

As we have anticipated in the introduction, we are going to find an estimate for  $\|\psi_\epsilon - \psi_0\|_2$ , the two functions inside the norm being defined by

$$H_\epsilon \psi_\epsilon = \left[-\frac{d^2}{dx^2} - V_\epsilon\right] \psi_\epsilon = E_\epsilon \psi_\epsilon \quad (2.1)$$

and

$$H_0 \psi_0 = \left[-\frac{d^2}{dx^2} - \|V\|_1 \delta\right] \psi_0 = E_0 \psi_0 \quad (2.2)$$

As is well known from [1, sect.I.3.2],  $H_\epsilon$  converges to  $H_0$  in the norm resolvent sense, which implies by means of Theor. 3.3.1 therein that the discrete spectrum of  $H_\epsilon$  consists of only one simple negative eigenvalue  $E_\epsilon$  given by an analytic function of the scaling parameter  $\epsilon$  in a neighbourhood of the origin.

Then we can state our result as follows:

**Theorem 2.1** *Let  $V, \psi_0, \psi_\epsilon$  be the functions defined above. Then,*

$$\frac{\|\psi_\epsilon - \psi_0\|_2}{\epsilon^{\frac{\gamma}{2}}} < \infty \quad (2.3)$$

for some  $\gamma < 1$ .

The main tool to derive the estimate will be the well-known Birman-Schwinger technique. Actually, as we are going to work in p-space, we shall use the integral operators

$$B(E_\epsilon) = \left(-\frac{d^2}{dx^2} - E_\epsilon\right)^{-\frac{1}{2}} V_\epsilon \left(-\frac{d^2}{dx^2} - E_\epsilon\right)^{-\frac{1}{2}} \quad (2.4)$$

and

$$B(E_0) = \left(-\frac{d^2}{dx^2} - E_0\right)^{-\frac{1}{2}} \delta \left(-\frac{d^2}{dx^2} - E_0\right)^{-\frac{1}{2}} \quad (2.5)$$

whose integral kernels are given by

$$B(p, p'; E_\epsilon) = (p^2 - E_\epsilon)^{-\frac{1}{2}} \frac{\hat{V}(\epsilon(p - p'))}{(2\pi)^{\frac{1}{2}}} (p'^2 - E_\epsilon)^{-\frac{1}{2}} \quad (2.6)$$

and

$$B(p, p'; E_0) = (p^2 - E_0)^{-\frac{1}{2}} \frac{\hat{V}(0)}{(2\pi)^{\frac{1}{2}}} (p'^2 - E_0)^{-\frac{1}{2}} \quad (2.7)$$

It is immediate to realise that  $B(E_0)$  is a rank one operator.

The integral operator  $B(E_\epsilon)$  is shown to be trace class by using a well-known result (see [2]), since the operator is positive and its kernel is continuous due to the fact that  $\hat{V}$  is a continuous function vanishing at infinity because of the  $L^1$ -decay of  $V$ . We can easily compute its trace by using the lemma invoked:

$$\|B(E_\epsilon)\|_1 = \frac{\hat{V}(0)}{(2\pi)^{\frac{1}{2}}} \int \frac{dp}{p^2 + |E_\epsilon|} = \frac{\pi \hat{V}(0)}{(2\pi |E_\epsilon|)^{\frac{1}{2}}} = \frac{\|V\|_1}{2|E_\epsilon|^{\frac{1}{2}}} \quad (2.8)$$

As a consequence of the Birman-Schwinger principle,  $B(E_\epsilon)$  has an eigenvalue equal to one. Let  $\hat{\chi}_\epsilon^{(1)}$  be the associated normalised eigenfunction. We recall that 1 is also the only eigenvalue of  $B_0$  and therefore we are allowed to apply the Kato-Rellich theorem (Theor. XII.8 in [3]) at  $\lambda_1(0) = 1 = \lambda_1(\epsilon)$  with  $\epsilon \in [0, 1]$  to obtain that the analytic eigenfunction can be written in terms of  $\hat{\chi}_0$ , the normalised eigenfunction of  $B_0$  explicitly given by  $(\frac{|E_0|^{\frac{1}{2}}}{\pi})^{\frac{1}{2}}(p^2 + |E_0|)^{-\frac{1}{2}}$ , as follows:

$$P(\epsilon)\hat{\chi}_0 = |\hat{\chi}_\epsilon^{(1)}\rangle\langle\hat{\chi}_\epsilon^{(1)}|\hat{\chi}_0 \quad (2.9)$$

Let us therefore estimate the norm of the difference of the two eigenvectors:

$$\begin{aligned} \| [1 - P(\epsilon)]\hat{\chi}_0 \|_2^2 &= (\hat{\chi}_0, [1 - P(\epsilon)]\hat{\chi}_0) = \\ &= (\hat{\chi}_0, [B(0) - |\hat{\chi}_\epsilon^{(1)}\rangle\langle\hat{\chi}_\epsilon^{(1)}| - \sum_{l>1} \lambda^{(l)}(\epsilon) |\hat{\chi}_\epsilon^{(l)}\rangle\langle\hat{\chi}_\epsilon^{(l)}|] \hat{\chi}_0) + \\ &+ (\hat{\chi}_0, [\sum_{l>1} \lambda^{(l)}(\epsilon) |\hat{\chi}_\epsilon^{(l)}\rangle\langle\hat{\chi}_\epsilon^{(l)}|] \hat{\chi}_0) = (\hat{\chi}_0, [B(0) - B(E_\epsilon)] \hat{\chi}_0) + \\ &+ (\hat{\chi}_0, [B(E_\epsilon)(1 - P(\epsilon))] \hat{\chi}_0) \end{aligned} \quad (2.10)$$

Let us first consider the second summand. It can easily be seen that it is bounded by

$$\|B(E_\epsilon)\| - 1 = \frac{\|V\|_1}{2|E_\epsilon|^{\frac{1}{2}}} - 1 = \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}} \quad (2.11)$$

exploiting the fact that  $\frac{\|V\|_1}{2|E_0|^{\frac{1}{2}}} = 1$ .

Going back to the first summand, its positivity can be shown as follows:

$$(\hat{\chi}_0, B(E_\epsilon)\hat{\chi}_0) = \alpha_0^2(\epsilon)(\hat{\chi}_\epsilon^{(1)}, B(E_\epsilon)\hat{\chi}_\epsilon^{(1)}) + (\tilde{\chi}_\epsilon, B(E_\epsilon)\tilde{\chi}_\epsilon) \quad (2.12)$$

having set  $\hat{\chi}_0 = \alpha_0(\epsilon)\hat{\chi}_\epsilon^{(1)} + \tilde{\chi}_\epsilon$  with  $\tilde{\chi}_\epsilon$  belonging to the orthogonal set of  $\hat{\chi}_\epsilon^{(1)}$  and exploited the definition of  $\hat{\chi}_\epsilon^{(1)}$  as well as the self-adjointness of  $B(E_\epsilon)$ . The quantity written above is

bounded by  $\alpha_0^2(\epsilon) + \|\tilde{\chi}_\epsilon\|_2^2 = \|\hat{\chi}_0\|_2^2$  provided  $B(E_\epsilon)$  has no eigenvalues greater than 1 on the orthogonal set of  $\hat{\chi}_\epsilon^{(1)}$ . This is actually the case since, as we have already seen, the trace of  $B(E_\epsilon)$  restricted to the orthogonal set of  $\hat{\chi}_\epsilon^{(1)}$  is equal to  $\frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}}$ , which is certainly less than 1 due to the fact that  $|E_0| < 4|E_\epsilon|$  for any  $\epsilon$  sufficiently small.

Since  $|E_\epsilon| < |E_0|$  we have:

$$\begin{aligned} (\hat{\chi}_0, [B(0) - B(E_\epsilon)]\hat{\chi}_0) &\leq \frac{|E_0|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\pi} \times \int \int (p^2 + |E_0|)^{-\frac{1}{2}} \times \\ &\times (p^2 + |E_\epsilon|)^{-\frac{1}{2}} [\hat{V}(0) - \hat{V}(\epsilon(p - p'))] (p'^2 + |E_\epsilon|)^{-\frac{1}{2}} (p'^2 + |E_0|)^{-\frac{1}{2}} dp dp' \end{aligned} \quad (2.13)$$

which goes to 0 as  $\epsilon \rightarrow 0$  by dominated convergence. Hence,

$$\|[1 - P(\epsilon)]\hat{\chi}_0\|_2^2 \leq F(\epsilon) + \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}}, \quad (2.14)$$

$F(\epsilon)$  denoting the right hand side of (13).

Since  $P(\epsilon)\hat{\chi}_0$  is eigenvector of  $B(E_\epsilon)$  with eigenvalue 1, it follows that

$$\hat{\psi}_\epsilon = (p^2 + |E_\epsilon|)^{-\frac{1}{2}} P(\epsilon)\hat{\chi}_0 \quad (2.15)$$

is the solution of the Schroedinger equation with the  $V_\epsilon$  potential in p-space. Similarly,

$$\hat{\psi}_0 = (p^2 + |E_0|)^{-\frac{1}{2}} \hat{\chi}_0 \quad (2.16)$$

is the eigenfunction of  $H_0$  in p-space.

Therefore,

$$\begin{aligned} \|\hat{\psi}_\epsilon - \hat{\psi}_0\|_2 &\leq \|(p^2 + |E_\epsilon|)^{-\frac{1}{2}} [1 - P(\epsilon)]\hat{\chi}_0\|_2 + \|[(p^2 + |E_\epsilon|)^{-\frac{1}{2}} - (p^2 + |E_0|)^{-\frac{1}{2}}]\hat{\chi}_0\|_2 \leq \\ &\leq \|(p^2 + |E_\epsilon|)^{-\frac{1}{2}}\|_\infty \| [1 - P(\epsilon)]\hat{\chi}_0\|_2 + \|(p^2 + |E_\epsilon|)^{-\frac{1}{2}} - (p^2 + |E_0|)^{-\frac{1}{2}}\|_\infty \leq \\ &\leq \frac{1}{|E_\epsilon|^{\frac{1}{2}}} [F(\epsilon) + \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}}]^{\frac{1}{2}} + \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_0|^{\frac{1}{2}}|E_\epsilon|^{\frac{1}{2}}} \end{aligned} \quad (2.17)$$

By using the assumption on the potential  $V$  we can now estimate the quantity  $F(\epsilon)$ .

As a consequence of Lemma 1.XIII.11 in [3], we have:

$$\hat{V}(0) - \hat{V}(\epsilon(p - p')) \leq 2^{1-\gamma} (2\pi)^{-\frac{1}{2}} \epsilon^\gamma |p - p'|^\gamma \|(1 + |x|)^\gamma V\|_1 \quad (2.18)$$

for some  $\gamma < 1$ .

Furthermore, by noting that  $1 + |p - p'| \leq 1 + |p| + |p'| \leq (1 + |p|)(1 + |p'|) \leq 2(1 + p^2)^{\frac{1}{2}}(1 + p'^2)^{\frac{1}{2}}$  we get:

$$F(\epsilon) \leq 2\epsilon^\gamma (2\pi)^{-\frac{1}{2}} \|(1 + |x|)^\gamma V\|_1 \left( \int \frac{(1 + p^2)^{\frac{\gamma}{2}}}{p^2 + |E_\epsilon|} dp \right)^2, \quad (2.19)$$

which implies (3) after being inserted into (17).

## Acknowledgements

It is my pleasure to thank Dr. K. Sonnenmoser for having requested the estimate given in this paper and discussed the main points of the proof. Support by the Swiss National Science Foundation is gratefully acknowledged.

## References

- [1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, Solvable Models in Quantum Mechanics, Springer-Verlag, New York-Berlin-Heidelberg, 1988.
- [2] M. Reed, B. Simon, Methods of Modern Mathematical Physics III, Scattering Theory, Academic Press, New York-San Francisco-London, 1979.
- [3] M. Reed, B. Simon, Methods of Modern Mathematical Physics IV, Analysis of Operators, Academic Press, New York-San Francisco-London, 1978.
- [4] K. Sonnenmoser, Stabilisation of atoms in superintense laser fields, J. Phys. B: At. Mol. Opt. Phys. 26 (1993), 457-475.