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Self-Organized Criticality and Percolation

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Abstract. Self-organized criticality (SOC) is interpreted in terms of a second-order phase transition endowed with an order parameter/control parameter coupling. A simple argument for explaining the frequently observed similarity of SOC systems with percolation problems is presented. For a general subclass of systems a criterion for SOC is deduced. These ideas are applied to the sandpile model to approximately determine its critical point.

1 Transport on a Cluster

Many systems that show self-organized critical behavior, such as the sandpile model [1, 2] and mechanical models [3, 4], are composed of a large number of well distinguished parts. The state of the system may then be described by attributing to each component k a state variable $u(k)$ such its mass, momentum or strain.

If the system is in a stationary state, it is useful to examine its response to a perturbation. In order to discuss global transport properties, we first investigate the local ones: Let the permeability π_{ij} be the flow (e.g. of mass or momentum) from component i to site j, which is induced by a disturbance at i. [1] The directed bond between elements i and j is called open if $\pi_{ij} > 0$ and closed otherwise.

[1]Note that the local description is very crude: for instance no dependence of the characteristics of the perturbation is assumed. This simplification seems however to be admissible, since in SOC systems, much like in ordinary critical phenomena, only global properties matter.

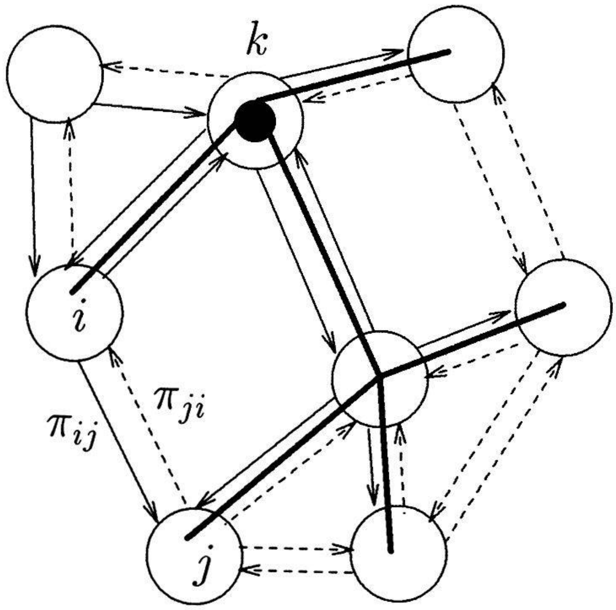

Figure 1: Propagation of a perturbation

A perturbation starting at component k propagates hence essentially [2] on the directed graph (cluster) consisting of open bonds emanating from k (Figure 1, where open (closed) bonds are denoted by (dotted) arrows, and the cluster is emphasized).

The statistical properties of the system in the long-time limit are therefore determined by the weights $\rho(\kappa)$ of the percolation clusters $\gamma(\kappa)$ associated with states κ on the attractor of the dynamics.

From the picture presented here it follows that the correlation length ξ may be identified with the linear dimension of the cluster. In contrast to a static percolation problem, where the density of open bonds acting as control parameter is fixed by the experimental setup, the bond density is here subject to a time evolution induced by the dynamics of the state variable u.

In order to discuss the the long-time behavior of the system, the permeabilities and dynamic equations for the state variable have to be specified.

It is now assumed that the invariant density $\rho(\kappa)$ can be expressed in terms of a density $\sigma(\kappa, p)$ at the percolation threshold p_c:

$$\begin{aligned} \rho(\kappa) &= \sigma(\kappa, p_c) \qquad (1.1) \\ p &= \frac{1}{|\langle ij \rangle|} \sum_{\langle ij \rangle} \Theta(\pi_{ij}) \end{aligned}$$

(Θ is the Heaviside function,$\langle ij \rangle$ denotes the set of ordered neighboring sites and $|\langle ij \rangle|$ its cardinality). The density σ in (1.1) is required to be such that the associated ensemble

[2]When loops are neglected.

averages $\langle\cdot\rangle_{\sigma(\kappa,p)}$ lead to consistent results, e.g. larger bond densities lead to larger correlation lengths:

$$\xi(p) \equiv \langle\xi\rangle_{\sigma(\kappa,p)} = \begin{cases} \frac{d\xi(p)}{dp} > 0 & (p < p_c) \\ \infty & (p > p_c) \end{cases} .$$

For the simplest case of a SOC system for instance, where the transport on the attractor takes place on critical Bernoulli clusters, and each configuration is visited with equal probability, the density σ may be expressed by:

$$\sigma^B(\kappa, p) \propto \delta_{p|\langle ij\rangle|, \sum_{\langle ij\rangle} \Theta(\pi_{ij})} \quad \text{(Bernoulli)}. \tag{1.2}$$

If the system is to show self-organized criticality, i.e. to develop infinite correlation lengths for long times, a sufficient condition is thus, that the probability for open bonds increases or diminishes for states in the sub- or supercritical phase respectively:

$$\frac{dp}{dt} = \begin{cases} > 0 & (p < p_c) \\ < 0 & (p > p_c) \end{cases} . \tag{1.3}$$

To discuss equation (1.3) we get more specific and regard the simplest case, where the permeability π_{ij} is a function $\pi(v, w)$ of the state variables $u(i)$ and $u(j)$ at neighboring parts i and j: $\pi_{ij} = \pi(u(i), u(j))$. For the time dependence of the bond density one obtains [3]

$$\begin{aligned} \frac{dp}{dt} &= \frac{1}{|\langle ij\rangle|} \sum_i \dot{u}(i) \qquad (1.4) \\ &\times \sum_{\text{neighbors } j \text{ of } i} \left(\delta(\pi(u(i), u(j))) \frac{\partial \pi(u(i), u(j))}{\partial v} + \delta(\pi(u(j), u(i))) \frac{\partial \pi(u(j), u(i))}{\partial w} \right) . \end{aligned}$$

An important quantity, that can be easily accessed by numerical simulation or by experiment is the average of the state variable $y \equiv \frac{1}{N} \sum_i u(i)$. By approximating in (1.4) averages of products by products of averages, an equation for the temporal evolution of y is deduced:

$$\begin{aligned} &\langle \frac{dp}{dt} \rangle \approx c(y) \langle \frac{dy}{dt} \rangle \\ &c(y) \equiv \\ &\langle \frac{N}{|\langle ij\rangle|} \sum_{\text{neighbors } j \text{ of } i} \left(\delta(\pi(u(i), u(j))) \frac{\partial \pi(u(i), u(j))}{\partial v} + \delta(\pi(u(j), u(i))) \frac{\partial \pi(u(j), u(i))}{\partial w} \right) \rangle . \end{aligned} \tag{1.5}$$

In systems with a local conservation law [4] driven by the force K that releases the flow J through the boundary, the average system parameter evolves according to $\langle \frac{dy}{dt} \rangle \propto \langle K\rangle - \langle J\rangle$.

[3] Continuum notation is used here and later for conciseness.

[4] The lack of a local conservation law does not influence our line of thought, it just renders the criteria for SOC more complicated.

Inserting this into equation (1.5) yields

$$\langle\frac{dp}{dt}\rangle \propto c(y)(\langle K\rangle - \langle J\rangle). \tag{1.6}$$

It can be demonstrated that the flow $\langle J\rangle$ vanishes only if $p < p_c$ and hence essentially corresponds to the order parameter $\mathcal{M}$ [5] of a percolation problem:

$$J \sim \mathcal{M}. \tag{1.7}$$

From this we conclude that equation (1.6) describes a coupling of the control parameter to the order parameter.

Typically self-organized critical systems are weakly driven. In the limit of infinitesimal driving force $\langle K\rangle = 0^+$ condition (1.3), which guarantees self-organized criticality, holds for

$$c(y) > 0. \tag{1.8}$$

The starting point of our discussion was a dynamic percolation picture we heuristically introduced to describe the transport on a composite system. It was then argued, that the dynamics of SOC systems may be described in terms of a control parameter/ order parameter coupling. We now present an argument that further justifies this approach.

Departing from systems that show second-order phase transitions, self-organized systems are constructed, and it is investigated what restrictions are imposed to the dynamics:

Let μ and $\mathcal{M}$ be the control parameter and the order parameter of a system showing a second-order phase transition. In order to contruct a self-organized critical system [6], the control parameter is coupled to the order parameter

$$\frac{d\mu}{dt} = f(\mathcal{M}). \tag{1.9}$$

such that

$$\lim_{t\to\infty} \mu(t) = \mu_c. \tag{1.10}$$

As the system is to be self-organized, the critical value μ_c must not explicitly enter the equations, rather its implicit definition

$$\mu_c = \sup_{|\langle\mathcal{M}\rangle(\mu)|=/} \mu \tag{1.11}$$

has to be used.

From (1.10) we conclude, that

$$\begin{aligned} f(\mathcal{M}) &= \Xi - g(\mathcal{M}) \\ g(\mathcal{M}) &= \begin{cases} 0 & \text{for } \mathcal{M} = / \\ > 0 & \text{for other values of } \mathcal{M} \end{cases} \\ \langle\Xi\rangle &\to 0^+ \end{aligned} \tag{1.12}$$

[5] I.e. the probability P_∞ for an infinite cluster.

[6] See also [5] for related ideas.

is a sufficient condition for (1.11). The quantity Ξ plays the rôle of a driving force and may be a stochastic variable.

Since equation (1.9) is required to correspond to the time-evolution of the state variable u, its both sides are assumed to be local functionals of u:

$$\begin{aligned} \mu &= \int a(u, \partial_k u, \ldots)\, d^d x \\ \mathcal{M} &= \int b(u, \partial_k u, \ldots)\, d^d x. \end{aligned} \tag{1.13}$$

The time evolution of the state variable is required to be **local**, which is the case for

$$g(x) = x, \quad \Xi = \int \zeta(x)\, d^d x$$

in (1.12) and (1.13), corresponding to

$$\frac{\partial a(u, \partial_k u, \ldots)}{\partial t} = \zeta - b(u, \partial_k u, \ldots), \quad \mathcal{M} \geq /. \tag{1.14}$$

These criteria are observed by percolation problems, but not for instance by a Ginzburg-Landau model, where the control parameter cannot be expressed as functional of the field u, and the order parameter is not positive.

2 Example: The Sandpile Model

The sandpile model was originally introduced by Bak et al. [1, 2] to describe the temporal evolution of the local heights $u(i)$ of a sandpile by a d-dimensional cellular automaton. Using continuum notation, the time evolution is given by

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \Delta\Theta(u(x,t) - u_c) + K(x,t) \qquad (u_c \equiv 2d) \\ &\text{where} \\ K(x,t) &= \begin{cases} \delta(x - \xi(t)) & \text{if } u < u_c \quad \forall x \\ 0 & \text{else} \end{cases} \\ &\text{and} \\ u(x,t) &= 0 \qquad \text{on the boundary} \end{aligned}$$

$\xi(t)$ form a set of independent random variables with uniform distribution on the grid.

A comparison with (1.14) shows that

$$\begin{aligned} a(u, \partial_k u, \ldots) &= u \\ b(u, \partial_k u, \ldots) &= -\Delta\Theta(u(x,t) - u_c) \\ \zeta &= K, \end{aligned}$$

which yields for the contol parameter and the order parameter μ and $\mathcal{M}$

$$\begin{aligned} \mu \propto y &= \frac{1}{V} \int_V u(x)\, d^d x \\ \mathcal{M} \propto \mathcal{J} &= -\oint_{\partial V} \nabla \Theta(u(x) - u_c)\, dS \quad , \end{aligned}$$

agreeing with (1.7).

When disregarding loops of the cluster, a perturbation propagates if $u(i) = u_c - 1$, and the permeability is thus approximatively $\pi_{ij} = \Theta(u(i) - u_c - 1)$. Criterion (1.8) for SOC is hence fulfilled as expected.

The statistical properties of the percolation-cluster are then approximated[7] by a Bernoulli cluster with density $\sigma \approx \sigma^B$ from (1.2) which leads to an approximate bond-density $p(y)$ determined by

$$\begin{aligned} p(y) &\approx P_{(u_c-1)}(y) \\ P_u(y) &= \langle \delta_{u,u(x)} \rangle_{\sigma^B(\kappa, p(y))}. \end{aligned}$$

Using standard combinatorial enumeration techniques [7] and a saddlepoint approximation [8], one shows that

$$\begin{aligned} P_u(y) &= \zeta(y)^u \frac{1 - \zeta(y)}{1 - \zeta(y)^{u_c}} \\ \zeta(y) &: \sum_{k=0}^{u_c - 1} (k - y) \zeta(y)^k = 0. \end{aligned} \tag{2.1}$$

The critical average height y_c is then approximately determined in terms of the corresponding threshold value for Bernoulli percolation p_c^B:

$$p_c \approx p_c^B = P_{(u_c-1)}(y_c),$$

which is in good agreement with the numerical results [9] (Table 1). For $d = 1$ the trivial result $y_c = 1$ is reproduced. For large d, the critical value converges toward the mean-field value with a correction which originates from the fact that the mean-field critical bond density is $\frac{1}{2d-1}$ and not $\frac{1}{2d}$, resulting in a breaking of the equipartition property of the probabilities for u.

In spite of the good agreement of the values obtained by the method presented here with the exact ones, it must be kept in mind that percolation and the sandpile model do not belong to the same universality class (e.g., their critical dimensions disagree). This may be understood as a consequence of the fact that the distribution σ^B used in the approximation does neither take into account forbidden configurations nor does it deal with correlations between neighboring bonds.

[7]Dhar [6] actually proved, that all configurations on the attractor are visited with equal probability.

d		Simulation	Percolation	Mean-field
1	y_c	1.000	1.000	0.500
	$P_0(y_c)$	0.000	0.000	0.500
	$P_1(y_c)$	1.000	1.000	0.500
2	y_c	2.125	2.191	1.500
	$P_0(y_c)$	0.074	0.080	0.250
	$P_1(y_c)$	0.174	0.148	0.250
	$P_2(y_c)$	0.306	0.272	0.250
	$P_3(y_c)$	0.446	0.500	0.250
3	y_c	3.135	3.077	2.500
	$P_0(y_c)$	0.054	0.102	0.167
	$P_1(y_c)$	0.117	0.122	0.167
	$P_2(y_c)$	0.166	0.145	0.167
	$P_3(y_c)$	0.201	0.174	0.167
	$P_4(y_c)$	0.223	0.208	0.167
	$P_5(y_c)$	0.238	0.249	0.167
$d \to \infty$	y_c	—	$d - \frac{1}{2} + O(1/d)$	$d - \frac{1}{2}$
	$P_u(y_c)$	—	$\frac{1}{2d} + O(1/d)$	$\frac{1}{2d}$

Table 1: Comparison of numerical [9], percolation and mean-field [10] results.

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