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On the Configuration Space Topology in General Relativity

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Abstract The configuration-space topology in canonical General Relativity depends on the choice of the initial data 3-manifold. If the latter is represented as a connected sum of prime 3-manifolds, the topology receives contributions from all configuration spaces associated to each individual prime factor. There are by now strong results available concerning the diffeomorphism group of prime 3-manifolds which are exploited to examine the topology of the configuration spaces in terms of their homotopy groups. We explicitly show how to obtain these for the class of homogeneous spherical primes, and communicate the results for all other known primes except the non-sufficiently large ones of infinite fundamental group.

Section 1. Introduction

In recent years mathematicians have made progress in understanding the diffeomorphism group of 3-dimensional manifolds. The object of this paper is to show how this can be exploited to deepen our *topological* understanding of configuration spaces occurring in pure General Relativity. In particular, we shall investigate their homotopy groups and thus generalize already existing work on the fundamental group [Wi]. Besides for its intrinsic interest, a major motivation to study these topological structures stems from the canonical quantization programme for General Relativity. Here, general arguments suggest a topological origin of certain interesting features of quantum gravity (e.g. degenerate vacuum structure, absence of anomalies, superselection sectors), resembling those already familiar from other (successfully quantized) theories. Certainly, the arguments given in the context of quantum gravity are primarily meant to be of heuristic value, that is, they are believed to really give insight into some aspects of quantum gravity by using methods which are not necessarily believed to survive an eventual rigorous formulation. Amongst others, there are two reasons that entertain this belief: Firstly, arguments identical in structure have been successfully applied in other field theories (e.g. Yang-Mills) where there is a quantum theory that takes notice of the topological features in question. Secondly, all approaches to canonical quantum gravity need to implement the diffeomorphism group of the underlying 3-manifold as transformation group on the state space. Although being a difficult technical problem on its own, this implementation quite generally suggests itself as a mechanism for the quantum theory to encode information about the underlying 3-manifold. Note that topological invariants of the diffeomorphism group are also topological invariants of the underlying 3-manifold. In this respect a natural first step is therefore to study some obvious topological characteristics of the diffeomorphism group and see how accurately they determine the underlying 3-manifold. This particular question can now be studied to some degree by using the table presented in Section 4 together with formula (1.7).

In any generally covariant theory the topological structure of configuration space receives characteristic imprints from the diffeomorphism group, which is used to mutually identify physically equivalent points on an auxiliary space that labels physical states in a redundant way. If this auxiliary space is topologically trivial, as it is in the case of General Relativity, all the topological information in the homotopy groups of the quotient is determined by those of the diffeomorphism group. Generally, this holds whenever the configuration space is given as the base of a principal fiber-bundle with structure group the diffeomorphisms and contractible total space, as will be explained below. In this case the topology of the base is directly related to the topology of the fibres, and it is their topology which we are going to investigate. In theories where besides the diffeomorphisms there is an additional gauge group acting (which also occurs in the "connection" formulation of General Relativity [Ash]), additional topological structure is induced. In these cases our analysis can be used to provide the diffeomorphism contribution. In order to work within a fixed framework, we shall argue within standard General Relativity. But, as will become apparent, the investigation is really of a more general kind.

In the sequel of this introductory section and Section 2 we shall provide some basic material concerning the notion of configuration spaces in General Relativity, 3 - manifolds, and their diffeomorphism groups. In particular, the notion of a spinorial manifold is introduced. A more technical point is deferred to Appendix 1. Proofs of already existing results were only included when it seemed appropriate. Their setting given here might differ from the one originally given. This sets the stage for the derivations of some new results in Section 3. In Section 4 all the results next to some other useful information is combined in a table, and some first observations are made. This section should be accessible without going through the main body of the paper. Appendix 2 combines into five theorems some scattered results from the literature which we made essential use of.

Configuration Spaces, 3-Manifolds and Diffeomorphisms

The specification of initial data in General Relativity starts with the selection of a 3-manifold, Σ, on which initial data are constructed in the form of a Riemannian 3-metric and the extrinsic curvature. Together they satisfy an elliptic system of four differential equations, the constraints, which are separate from the evolution equations. As configuration space we address the quotient-space obtained from the space of all 3-metrics on Σ, where those metrics which label the same physical state are mutually identified. This reduces three (the so-called momentum constraints, which are linear in momenta) of the four constraint equations, the remaining one being the so called Hamiltonian constraint (which is quadratic in the momenta). The identification is generically given by the action of some normal subgroup (possibly the whole group) of the diffeomorphism group, which we choose to call its gauge part, since it connects redundant labels for the same physical state. General covariance then implies that the quotient of the full diffeomorphism group with respect to the gauge part acts on the configuration space as proper symmetries, which now connect different physical states. We shall refer to those simply as symmetries. In case the gauge part exhausts all of the diffeomorphism group, there will simply be no symmetries.

Exactly how much of all diffeomorphisms are considered to be gauge part depends on the physical situation one likes to describe and cannot be answered a priori within the formalism. In General Relativity two major situations arise: Firstly, Σ is closed and represents the whole universe, in which case there are no symmetries and the gauge part is the whole diffeomorphism group. This situation we shall refer to as the closed case. It is usually employed in classical- and quantum-cosmology. Secondly, Σ represents an isolated part of the universe, so that Σ is a manifold that outside some connected compact set is homeomorphic to the complement of a closed ball in R^3, i.e., to the cylinder $R \times S^2$. This cylinder can be thought of as the transition region between the system under study and the ambient universe relative to which the system is described. The gauge part is then given by those diffeomorphisms of Σ that asymptotically die-off as one moves along the cylinder in an outward direction. Slightly more precise, we may compactify Σ by a 2-sphere boundary at the outer end of the cylinder and take the gauge part as those diffeomorphisms that fix the boundary. They form a proper normal subgroup within the full diffeomorphism group of Σ. The action of symmetries is then interpreted as changing the relative positions of the system with respect to the ambient universe. This situation we shall refer to as the open case.

Mathematically the situations just described are surprisingly unique, in the sense that essentially (i.e. up to discrete groups) no other choice of a quotient symmetry group could have been made. This is due to two facts. Firstly, that the identity component of the diffeomorphism group of a closed manifold is simple (as group), and, secondly, that the identity component of the group of boundary-fixing diffeomorphisms of a manifold with connected boundary is simple and given by the unique non-trivial normal subgroup of the identity component of all diffeomorphisms [McD]. For more than one boundary component (i.e. a Σ with more than one asymptotic region, a case which we do not consider here), there will be more normal subgroups according to those diffeomorphisms that fix only some of the boundary components [McD]. For a closed manifold, a minimal non-trivial normal subgroup of the diffeomorphism group is given by its identity component, whereas in the open case a minimal choice is given by the identity component of asymptotically trivial diffeomorphisms (larger choices would be the identity component of all diffeomorphisms, or all asymptotically trivial diffeomorphisms). In any case, the only way in which continuous symmetry groups can arise is via asymptotic regions or boundaries. Clearly, selecting minimal normal subgroups as gauge part corresponds to a maximal choice of symmetries.

As it stands, the open and closed case do not seem to be intimately related. We will argue, however, that, in a sense explained below, our topological investigations cover both situations at the same time. Recall that in the closed case the configuration space, as defined above, namely as the quotient of the space of metrics modulo all diffeomorphisms, has a non-trivial singularity structure (described in [Fi 1]) due to the changing dimensions of isotropy groups at metrics with different isometries. Rather then working with this singular configuration space $\mathcal{S}(\Sigma)$ (for superspace), where e.g. the global dynamics is only defined by some regular dynamics on a singularity free resolution space (reflection conditions etc.), one may instead use the resolution space from the start. First arguments as to why canonical quantum gravity should also be formulated on a resolution space of superspace were already given in [DeW]. It turns out that there is a natural resolution space for $\mathcal{S}(\Sigma)$ which we call $\mathcal{Q}_R(\Sigma)$. Its construction is explained in lucid detail in [Fi 2] (see also [Sw]).

On the other hand, in the open case, an admissible and convenient way for our topological investigations is to consider the one-point compactification $\bar{\Sigma} := \Sigma \cup \{\infty\}$ of Σ by a point called ∞. In this case one requires Σ to be asymptotically regular in the sense that there exists a compact connected set $K \subset \Sigma$ so that $\Sigma - K$ is homeomorphic to the complement of a closed ball in R^3. This also ensures that the one-point compactification $\bar{\Sigma}$ is a manifold. The configuration space, $\mathcal{Q}(\Sigma)$, is then defined by the space of all metrics on $\bar{\Sigma}$, $Riem(\bar{\Sigma})$, modulo the diffeomorphisms that fix the frames at ∞. To answer topological questions we neither need to specify fall-off conditions nor the precise function space for the metric. To start the construction, we fix an oriented frame u at ∞. A general linear transformation of the tangent space $T_\infty(\bar{\Sigma})$ is said to be $\in SO(3)$, if its matrix representative with respect to u is $\in SO(3)$. Clearly, all conclusions to follow are independent of the choice of u. Let us now define:

$$\begin{aligned} D(\bar{\Sigma}) &: = \{\text{orientation preserving } C^\infty\text{-diffeomorphisms on } \bar{\Sigma}\} \\ D_\infty(\bar{\Sigma}) &: = \{\phi \in D \ / \ \phi(\infty) = \infty, \quad \phi_*|_\infty \in SO(3)\} \\ D_F(\bar{\Sigma}) &: = \{\phi \in D_\infty \ / \ \phi_*|_\infty = \text{id}\} \end{aligned} \tag{1.1}$$

Here, $D_\infty(\bar{\Sigma})$ represents those diffeomorphisms of Σ which induce "rigid" rotations on the 2-sphere at the end of the cylinder. It is easy to see that D_F is a normal (invariant) subgroup of D_∞, whereas neither of them is a normal subgroup of D. Since the space of orientation-preserving diffeomorphisms of the 2-sphere is homotopy equivalent to its isometries $SO(3)$ [Sm], we may for our topological purposes represent $D(\bar{\Sigma})$ by $D_\infty(\bar{\Sigma})$. As an important example let us consider the particular class of open cases, where the system under consideration is represented by asymptotically flat metrics. Here the requirement on the gauge part to lie within D_F is well motivated since we certainly want to include states with non-vanishing angular momentum at spatial infinity (∞). These are included in the phase space over $Riem(\bar{\Sigma})/D_F(\bar{\Sigma})$, but not in the smaller one over $Riem(\bar{\Sigma})/D_\infty(\bar{\Sigma})$. Diffeomorphisms on Σ must not disturb the fixed asymptotically euclidean structure and are therefore faithfully represented by $D_\infty(\bar{\Sigma})$. But the Hamiltonian theory does not tell us to regard $D_F(\bar{\Sigma})$ as its gauge part, rather, it only requires the constraints (the diffeomorphism- or gauge constraints) to generate gauge transformations. The group generated by them is the identity component $D_F^0(\bar{\Sigma})$ of $D_F(\bar{\Sigma})$. Declaring no further invariances as gauge than what is really required by the formalism leaves us with the (maximal) symmetry group $\mathcal{G}(\Sigma)$, given by:

$$\mathcal{G}(\Sigma) := D_\infty(\bar{\Sigma})/D_F^0(\bar{\Sigma}) \,. \tag{1.2}$$

It is now true that $\mathcal{Q}(\Sigma)$ is the basis of a D_F- principal fibre bundle with total space $Riem(\bar{\Sigma})$ [Bou][Fi 2]:

$$\begin{array}{ccc} D_F(\bar{\Sigma}) & \longrightarrow & Riem(\bar{\Sigma}) \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathcal{Q}(\Sigma) \end{array} \tag{1.3}$$

The action of D_F on $Riem$ is free since D_F cannot contain non-trivial isometries. A simple proof for this is obtained by using the exponential map at ∞. We can now compare this to the closed case by considering the space $\mathcal{Q}_R(\bar{\Sigma})$ for the closed manifold $\bar{\Sigma}$. If we denote by $F(\bar{\Sigma})$ the bundle of oriented frames over $\bar{\Sigma}$, the resolved configuration space is defined as the base of the following principal fibre bundle [Fi2]:

$$\begin{array}{ccc} D(\bar{\Sigma}) & \longrightarrow & Riem(\bar{\Sigma}) \times F(\bar{\Sigma}) \\ & & \downarrow \\ & & \mathcal{Q}_R(\bar{\Sigma}) \end{array} \tag{1.4}$$

Here, $D(\bar{\Sigma})$ acts on $F(\bar{\Sigma})$ by its standard lift, and the action is free by the same argument as above. We now have the following result which is part of Theorem 6.1 in [Fi 2]:

Theorem 1. *The spaces $\mathcal{Q}(\Sigma)$ and $\mathcal{Q}_R(\bar{\Sigma})$ are diffeomorphic (as ILH-manifolds).*

For us this implies that we can focus attention to $\mathcal{Q}(\Sigma)$. All the statements we are going to make about the abstract topology of $\mathcal{Q}(\Sigma)$ hold equally well for $\mathcal{Q}_R(\bar{\Sigma})$. Keeping this in mind we shall never mention $\mathcal{Q}_R(\bar{\Sigma})$ again. In [Sw] a smaller (in fact minimal) resolution space is considered which is obtained from (1.4) by restricting the total space to the subset $\{(g,u)\,|\,u \in O(g,\bar{\Sigma})\}$, where $O(g,\bar{\Sigma})$ is the bundle of oriented frames which are orthogonal with respect to the metric g.

$Riem(\bar{\Sigma})$ is a convex open cone in the topological vector space of smooth (0,2)-tensor fields. D_F is topologized to make it a topological group (i.e. at least as fine as compact open) with topological action on $Riem(\bar{\Sigma})$. $\mathcal{Q}$ is given the quotient topology which is the finest topology for which π is continuous (for more information on topologies of mapping spaces see [Mic]). There are two important points to make.

1. The map π is open and hence $\mathcal{Q}$'s topology unique.

Proof: take an open set $\mathcal{O}$ in $Riem(\bar{\Sigma})$. Since D_F acts as a topological group, its orbit, given by $\pi^{-1}(\pi(\mathcal{O}))$, is also open. Hence $\pi(\mathcal{O})$ is an open set and therefore π an open map. Conversely, let π be an open map and $U \subset \mathcal{Q}(\Sigma)$ an arbitrary set such that $\pi^{-1}(U)$ is open. Then $U = \pi(\pi^{-1}(U))$ is open and therefore $\mathcal{Q}(\Sigma)$'s topology equal or stronger than the quotient topology. But the quotient topology is already the strongest one compatible with the continuity of π. Hence it is unique. With respect to Q it therefore makes sense to refer to *its* topology.

2. Under the hypothesis of the validity of the Smale conjecture, Cerf proved that the diffeomorphism and homeomorphism groups of $\bar{\Sigma}$ are weakly homotopy equivalent spaces, i.e., there exists a map from one space to the other inducing isomorphisms on all fundamental groups. But the Smale conjecture is now proven [Ha 4]. As far as we are interested in the homotopy groups only, these results allows us to be imprecise about the degree of differentiability we work with. In particular, it allows us to use interchangeably $D_F(\bar{\Sigma})$ and $H_D(\bar{\Sigma})$, where the latter denotes the space of homeomorphisms fixing a disc containing ∞, which is more often employed in the literature (e.g.[FW],[HL]).

From the contractibility of Riem$(\bar{\Sigma})$ and the homotopy exact sequence associated with the bundle (1.3), we immediately obtain for all $n \geq 0$ the isomorphism:

$$\pi_n(D_F(\bar{\Sigma})) \cong \pi_{n+1}(\mathcal{Q}(\Sigma)). \tag{1.5}$$

The investigation of the homotopy groups of $\mathcal{Q}(\Sigma)$ is thus reduced to those of $D_F(\bar{\Sigma})$. No reference to the space of metrics is made anymore. It has dropped out of the homotopy exact sequence due to its contractibility ($\mathcal{Q}(\Sigma)$ is a classifying space for the group $D_F(\bar{\Sigma})$) and the only topological features are those of D_F. This is why investigations of this type bear a high degree of generality. For example, in so-called higher-derivative theories of gravity, $Riem(\bar{\Sigma})$ is replaced by $Riem(\bar{\Sigma}) \times K$, where K is the linear (and hence contractible) space of sections in some tensor bundle. The D_F action is then still free and the total space still contractible. The corresponding configuration spaces therefore still satisfy (1.5).

Let us now decompose $\bar{\Sigma}$ into its prime factors, $\bar{\Sigma}_i$, explicitly separating the irreducible primes P_i from the non-irreducible handle $S^2 \times S^1$. (A prime $\bar{\Sigma}_i$ is irreducible, $\Leftrightarrow$ every 2-sphere in $\bar{\Sigma}_i$ bounds a disc. The only closed, orientable, non-irreducible prime is the handle $S^2 \times S^1$.)

We write:

$$\bar{\Sigma} = \biguplus_{i=1}^{n+l} \bar{\Sigma}_i = \left(\biguplus_{i=1}^{n} P_i\right) \uplus \left(\biguplus_{i=1}^{l} S^2 \times S^1\right), \quad P_i \ncong S^2 \times S^1, \tag{1.6}$$

where we used $\uplus$ to denote the operation of taking the connected sum. It is known (see e.g. [McCu]) that there is a homotopy equivalence ($\sim$):

$$D_F(\bar{\Sigma}) \sim \left(\prod_{i=1}^{n} D_F(P_i)\right) \times \left(\prod_{i=1}^{l} \Omega SO(3)\right) \times \Omega\mathcal{C}(\bar{\Sigma}), \tag{1.7}$$

where the symbol $\Omega(\cdot)$ stands for the loop-space of $(\cdot)$, and where $\mathcal{C}(\bar{\Sigma})$ is a space that (vaguely speaking) labels and topologizes the relative configurations (sites) of the prime-manifolds $\bar{\Sigma}_i$ in the connected sum $\bar{\Sigma}$. In general its determination seems to be difficult and we refer to [McCu][HL] for more elaborate treatments. What interests us here is that consequently the homotopy groups of $D_F(\bar{\Sigma})$ satisfy:

$$\pi_k(D_F(\bar{\Sigma})) = \left(\prod_{i=1}^{n} \pi_k(D_F(P_i)) \times \prod_{j=1}^{l} \pi_{k+1}(SO(3))\right) \times \pi_{k+1}(\mathcal{C}(\bar{\Sigma})) \tag{1.8a}$$

$$\pi_0(D_F(\bar{\Sigma})) = \left(\prod_{i=1}^{n} \pi_0(D_F(P_i)) \times \prod_{j=i}^{l} Z_2\right) \tilde{\times}\ \pi_1(\mathcal{C}(\bar{\Sigma}))\,, \tag{1.8b}$$

where we used the general relation $\pi_k(\Omega X) = \pi_{k+1}(X)$ We had to separate the case $k = 0$ from all others for the reason that the homotopy equivalence (1.7) does not imply a direct product structure for the zeroth homotopy groups, as it does for all the higher ones (expressed by (1.8a)). Note also that the zeroth homotopies, which generally do not form groups, are indeed groups in the present case due to the fact that the spaces they are taken from are themselves topological groups. We write $\tilde{\times}$ to indicate that the product structure holds only on the level of sets but not groups. It is possible to obtain more information about the group structure of (1.8b) [Gi1]. At this point we only remark that the factor in brackets in (1.8b) forms a subgroup, though generally not a normal one.

This structure (1.7) rests on the restriction to asymptotically trivial diffeomorphisms D_F, for only in this case there is no topological intertwinement of the diffeomorphisms with support inside the prime factors $\bar{\Sigma}_i$ (called internal diffeomorphisms) with those of general support (called external diffeomorphisms). This would fail to hold if one considered D instead of D_F, as a simple counterexample shows (see e.g. [Gi1]). As far as the homotopy groups are concerned, we now see how the determination of $\mathcal{Q}(\Sigma)$'s topology is directly related to the determination of those for its prime factors $\mathcal{Q}(\Sigma_i)$. Following ref. [McCu], a more elaborate discussion of of formula (1.7) is given in ref. [Gi1], which also contains an application of (1.8b) to the case of where $\bar{\Sigma}$ is given by the connected sum of l handles.

Section 2. Some Preparatory Material

At the end of this section and the whole of Section 3 we will perform explicit calculations for $\pi_0(D_F(\bar{\Sigma}))$ and $\pi_k(D_F(\bar{\Sigma}))$ $(k \geq 1)$ respectively, where $\bar{\Sigma}$ is taken from the subclass of homogeneous spherical primes. In order not to overload the actual proofs, we shall establish some preparatory material first.

A Closer Look at Diffeomorphisms

The strategy for our calculations is very simple: there are hard theorems available concerning the topological structure of $D(\bar{\Sigma})$ for $\bar{\Sigma}$ prime. For convenience we collect somé of them in Appendix 2. Most interesting for us is the question of validity of a conjecture made by Hatcher [Ha 2] (abbreviated HC, in the literature also referred to as the generalized Smale conjecture). It asserts that for spherical primes the spaces of diffeomorphisms and isometries are homotopy equivalent. Restricted to the 3-sphere, this is known as the Smale conjecture which has been shown to hold in [Ha 4]. Some of the proofs presented in this paper require the validity of HC, the status of which has been indicated in the table presented in Section 4.

From now on we shall sometimes drop the explicit reference to the manifold $\bar{\Sigma}$ by just writing D_F, D_∞, etc., without any argument. We obtain information about D_F by relating it to D by some standard fibrations which we shall now describe. First note that orientable 3-manifolds are always parallelizable. Their bundle of oriented frames, $F(\bar{\Sigma})$, is thus given by the product $\bar{\Sigma} \times GL^+(3,R)$. The different diffeomorphism groups in (1.1) are topologically related by the following three principal fibre bundles (u still denotes the fixed frame at ∞):

$$\begin{array}{ccc} D_F & \stackrel{\hat{i}}{\longrightarrow} & D \\ & & \big\downarrow \hat{p} \\ & & F(\bar{\Sigma}) \end{array} \qquad \hat{p}(\phi) := T\phi(u) = (\phi(\infty), \phi_*|_\infty(u)) \tag{2.1}$$

$$\begin{array}{ccc} D_\infty & \stackrel{i}{\longrightarrow} & D \\ & & \big\downarrow p \\ & & \bar{\Sigma} \end{array} \qquad p(\phi) := \phi(\infty) \tag{2.2}$$

$$\begin{array}{ccc} D_F & \stackrel{\tilde{i}}{\longrightarrow} & D_\infty \\ & & \big\downarrow \tilde{p} \\ & & SO(3) \end{array} \qquad \tilde{p}(\phi) := \phi_*|_\infty \tag{2.3}$$

As a first application we introduce the concept of spinoriality of a manifold $\bar{\Sigma}$. Associated with (2.3) is the fibration

$$\begin{array}{ccc} D_F(\bar{\Sigma})/D_F^0(\bar{\Sigma}) & \stackrel{\tilde{i}}{\longrightarrow} & \mathcal{G}(\Sigma) \\ & & \big\downarrow \tilde{p} \\ & & SO(3) \end{array} \tag{2.4}$$

where $\mathcal{G}(\Sigma)$ is the symmetry group defined in (1.2). Note that since D_F is normal in D_∞, D_F/D_F^0 is normal in $\mathcal{G}$. Discreteness of D_F/D_F^0 then implies that its centralizer in $\mathcal{G}$ contains the identity component $\mathcal{G}^0$, so that $\mathcal{G}^0 \cap D_F/D_F^0 \subset centre(D_F/D_F^0)$.

From the exact homotopy sequence associated with (2.4) we infer (unless stated otherwise, we use the multiplicative notation for abelian groups, so that the neutral element is then denoted by 1):

$$1 \rightarrow \pi_1(\mathcal{G}) \xrightarrow{\tilde{p}_*} Z_2 \xrightarrow{\partial_*} \pi_0(D_F) \xrightarrow{\tilde{\imath}_*} \pi_0(\mathcal{G}) \rightarrow 1 \tag{2.5}$$

which, by injectivity of $\tilde{p}_*$, gives us two possibilities:

1. $\pi_1(\mathcal{G}) = 0$ and $\pi_0(D_F)$ is a Z_2 extension of $\pi_0(\mathcal{G})$. In this case (2.4) is non-trivial and $\mathcal{G}$ is given by $\{\pi_0(D_F) \times SU(2)\}/Z_2$. Here the Z_2 is generated by $(-1,-1)$ where the -1 in the left factor generates the image of ∂_* in $\pi_0(D_F)$ and the -1 in the right factor generates the centre of $SU(2)$. Note that the image of ∂_* is mapped to the identity in $\pi_0(\mathcal{G})$ via $\tilde{\imath}_*$. If we identify D_F/D_F^0 with its image in $\mathcal{G}$ (via $\tilde{\imath}$), this says that the image of ∂_* is in $\mathcal{G}^0$ and therefore in the centre of D_F/D_F^0.

2. $\pi_1(\mathcal{G}) = Z_2$ and $\pi_0(D_F) = \pi_0(\mathcal{G})$, in which case (2.3) is trivial and $\mathcal{G}(\Sigma)$ a direct product $\pi_0(D_F) \times SO(3)$.

In the first case, $\mathcal{G}(\Sigma)$ contains $SU(2)$ but not $SO(3)$ as a subgroup. Let a manifold, Σ, for which this is the case, be for obvious reasons called *spinorial*. Whether a manifold is spinorial is a purely topological question and has been decided for all known prime manifolds (see Theorem 2 below). In view of Corollary 1 (below) this provides sufficient information for the general case. From (2.3) and the discussion above one obtains the following exact sequence and criteria for spinoriality:

$$1 \rightarrow \pi_1(D_F) \rightarrow \pi_1(D_\infty) \xrightarrow{\tilde{p}_*} Z_2 \xrightarrow{\partial_*} \pi_0(D_F) \rightarrow \pi_0(D_\infty) \rightarrow 1 \tag{2.6}$$

$$\Sigma \text{ is spinorial} \Leftrightarrow \begin{cases} \partial_* \text{ into, or} \\ \pi_1(D_F(\bar{\Sigma})) \cong \pi_1(D_\infty(\bar{\Sigma})), \text{ or} \\ \pi_0(D_F) \text{ is a central extension of } \pi_0(D_\infty) \text{ by } Z_2, \end{cases} \tag{2.7a}$$

$$\Sigma \text{ is not spinorial} \Leftrightarrow \begin{cases} \tilde{p}_* \text{ onto, or} \\ \pi_0(D_F(\bar{\Sigma})) \cong \pi_0(D_\infty(\bar{\Sigma})). \end{cases} \tag{2.7b}$$

For later application let us cite the following known results in form of two lemmas, a theorem and a corollary.

Lemma 1. *If $\bar{\Sigma}$ is an irreducible prime manifold listed in our table (i.e. different from $S^1 \times S^2$), then the bundle projection p of (2.2) has the property that p_* maps $\pi_1(D)$ onto the centre C of $\pi_1(\bar{\Sigma})$.*

Proof. That $p_*(\pi_1(D(\bar{\Sigma}))) \subset C$ follows from Corollary 5.22 of ref. [McCa] for arbitrary $\bar{\Sigma}$ (his/her remark 5.24). A proof for surjectivity onto C is contained in Section V of ref. [Wi] under the hypothesis that homotopy implies isotopy of diffeomorphisms. According to Theorem A1 in Appendix 2 this is now known to hold for all primes in our table •

Lemma 2. *If $\bar{\Sigma}$ is not spinorial then the bundle projection $\hat{p}$ of (2.1) has the property that $\hat{p}_*(\pi_1(D))$ contains $(1,-1) \in \pi_1(F(\bar{\Sigma})) \cong \pi_1(\bar{\Sigma}) \times Z_2$.*

Proof. This is proven in Lemma 2.1 of ref. [FW]. The simple and instructive proof is worth a look at this point. Let $s \mapsto R_s \in D^0_\infty$ (the identity component of D_∞) be the rotation of a 3-disc D_1 against a slightly larger, concentric 2-sphere S_2 (compare Appendix 1). Non-spinoriality implies the existence of a path $s \mapsto \phi_s \in D^0_F$ so that $\phi_0 = R_1$ and $\phi_1 = id$. The product path, γ, defined by

$$\gamma := \begin{cases} R_{2s} & \text{for } s \in [0, \frac{1}{2}] \\ \phi_{(2s-1)} & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

defines a loop at id in D^0_∞ satisfying $\tilde{p}_*([\gamma]) = -1$. Moreover, from the end of the exact sequence for (2.2) (shown in (3.3)), we have $p_*([\gamma]) = p_* \circ i_*([\gamma]) = 1$, so that $\hat{p}_*([\gamma]) = (p_*([\gamma]), \tilde{p}_*([\gamma])) = (1,-1) \in \pi_1(\bar{\Sigma}) \times Z_2$ •

Theorem 2. *Amongst the known prime manifolds the only non spinorial ones are the the lens spaces, $L(p,q)$, and the handle $S^1 \times S^2$.*

Proof. Non-spinoriality for $L(p,q)$ and $S^1 \times S^2$ can actually be visualized. The demonstration of this fact is deferred to the Appendix 1. In ref. [He], chapter 4.3, Theorem 1 implies spinoriality for the following prime manifolds: those with infinite fundamental group different from $S^1 \times S^2$ (given by the so-called $K(\pi,1)$'s), and those with finite fundamental group which have a non-cyclic 2-Sylow subgroup. An alternative and somewhat simplified proof of this theorem is given in [P]. The remaining cases consist of some S^3/G with non cyclic G for which HC is known to hold. Using this fact, they were shown to be spinorial in ref. [FW], Theorem 2.2, and the remark following Corollary 2.2. For the latter one needs to add that in the meantime the validity of HC for the spaces S^3/D^*_{4m}, $m \geq 2$, has been shown in ref. [MR].

Due to our strategy to be explicit in the case of homogeneous S^3/G's, and anticipating some results from the next subsection, we want to give a simple proof of spinoriality for non-cyclic G under the hypothesis of validity of HC. (More precisely, it only relies on one of its implications, namely that $\pi_1(D)$ and $\pi_1(Isom)$ have the same number of generators. We are, however, not aware of a single case where this but not the full HC is known to hold.) By discreteness of $S(G)$ we learn from (2.12) that $\bar{p}_*$ maps $Z_2 \cong \pi_1(Isom(S^3/G))$ injectively into $G = \pi_1(S^3/G)$. Since $\bar{p}$ is just the restriction of p in (2.2), which also appears as the first component of $\hat{p}$ in (2.1), we learn that $\hat{p}_*(\pi_1(D))$ contains the element $(-1,1)$ or $(-1,-1)$ [by Lemma 1, $\hat{p}_*$ maps $\pi_1(D)$ into $Z_2 \times Z_2 = centre\, G \times \pi_1(SO(3))$]. By HC, $\pi_1(D) \cong \pi_1(Isom) = Z_2$ so that $\hat{p}_*(\pi_1(D))$ cannot in addition contain $(1,-1)$ by mapping only one generator. Lemma 2 then implies spinoriality •

Corollary 1. *A closed, oriented, connected 3-manifold $\bar{\Sigma}$ is non-spinorial, if and only if its prime decomposition consists entirely of lens spaces and handles.*

Proof. In view of Theorem 2 we need to prove that such a manifold is non-spinorial, if and only if none of its prime factors is spinorial. For this we assume that $\bar{\Sigma}$ is the connected sum of n prime factors which build up $\bar{\Sigma}$ as follows (we partially follow ref. [McCu]): Take a 3-sphere and on it n+1 closed, disjoint 3-discs D_i, $0 \leq i \leq n$, with 2-sphere boundaries $\partial D_i = S_i$. We take D_0 as a neighbourhood of ∞, remove the interiors of D_i for $i \geq 1$, and construct $\bar{\Sigma}$ by identifying the n 2-sphere boundaries S_i $(1 \leq i \leq n)$ with the corresponding boundaries that one obtains by removing an open 3-disk on each prime factor respectively. In $\bar{\Sigma}$ we now connect S_i with S_{i+1} for each $1 \leq i \leq n-1$ by a thin cylindrical tube (topology $I \times S^1$) and obtain a single new 2-sphere, S (the connected sum of all S_i for $i \geq 1$), which is isotopic to S_0. A 2π-rotation parallel to S_0 (i.e. a rotation parallel to S_0 and an arbitrarily close concentric one, as explained in Appendix 1) is also isotopic to such a rotation parallel to S. The latter one may be chosen to have the connecting tubes as axis (i.e. the rotation acts only on the S^1 part of $I \times S^1$). Shrinking the tubes to zero diameter defines an isotopy to rotations parallel to each S_i for $1 \leq i \leq n$ (compare the remark after Theorem 1, chapter 4.3, of reference [He]). From (1.8b) and Theorem 2 we infer that this diffeomorphism corresponds to the identity element of $\pi_0(D_F(\bar{\Sigma}))$, if and only if the connected sum does not contain a single spinorial prime •

Corollary 1 suggests that non-spinorial manifolds are somewhat more special than spinorial ones. Note, however, that the non-spinorial primes still suffice to build up (in a non-unique fashion) manifolds of any given homology by taking connected sums. Non-spinorial manifolds are, therefore, not *homologically* special. Note also that the lens spaces $L(p,q)$ are homogeneous, if and only if $q = \pm 1 \bmod p$. These are therefore the only non-spinorial primes in the class of homogeneous space forms to which we specialize in the following subsection. Finally, we note that the existence of spinorial manifolds has been already used in the literature to speculate on the existence of certain spinorial states in quantum gravity [FS]. We will come back to this point in Observation 1 of Section 4.

Homogeneous Spherical Primes

In this subsection we consider a special class of elliptic spaces, that is, spaces of the form S^3/G, where G is a finite subgroup of $SO(4)$ with free action on S^3. We identify S^3 with the group manifold of $SU(2)$ with its standard bi-invariant (round) metric and use the isomorphism $SO(4) \cong (SU(2) \times SU(2))/Z_2$, where the Z_2 is generated by $(-1,-1)$. Elements of $SO(4)$ are then written as Z_2-equivalence classes $[g,h]$ with $g,h \in SU(2)$, and elements of S^3/G as G-equivalence classes $[x]$ with $x \in S^3$. We can now write down the (left) $SO(4)$-action on S^3

$$SO(4) \times S^3 \rightarrow S^3\,, \qquad ([g,h],x) \mapsto g \cdot x \cdot h^{-1}\,. \tag{2.8}$$

$SO(4)$ has a left- and right- $SU(2)$ subgroup, given by the sets $SU(2)_L := [SU(2),1]$ and $SU(2)_R := [1, SU(2)]$ respectively. It also has an obvious diagonal- $SO(3)$ subgroup given by all elements of the form $[g,g]$. We call it $SO(3)_D$. If $N_{SO(4)}(G)$ denotes the normalizer of G in $SO(4)$, it is easy to see that the residual orientation preserving isometry group acting on S^3/G is given by $Isom(S^3/G) = N_{SO(4)}(G)/G$:

$$N_{SO(4)}(G) \times S^3/G \rightarrow S^3/G\,, \qquad ([g,h],[x]) \mapsto [g \cdot x \cdot h^{-1}]\,. \tag{2.9}$$

It acts transitively on S^3/G, if and only if G is a finite, freely acting subgroup of either $SU(2)_L$ or $SU(2)_R$. As subgroups of $SU(2)$ these are given by Z_p, D^*_{4n} for $n \geq 2$, T^*, O^*

and I^* which denote the cyclic group of order p and the $SU(2)$-double covers (denoted by *) of the symmetry groups of the n-prism, tetrahedron, octahedron and icosahedron of orders $4n$, 24, 48 and 120 respectively. We shall now restrict to theses cases and may choose $G = G_R$, that is, we may choose G to sit inside $SU(2)_R$, for the manifold obtained by using $SU(2)_L$ would certainly be homeomorphic.

For our topological purposes we may indeed restrict ourselves to the right action for identifying S^3 to S^3/G. But as prime manifolds in the oriented category one needs to distinguish those obtained by right and left identifications. This follows from the fact that a homogeneous S^3/G does not allow for any orientation reversing diffeomorphism if $G \neq Z_i$ for $i \leq 2$ [Wi], and uniqueness of the prime decomposition in the oriented category. For those G this implies that $S^3/G \uplus S^3/G$ is not homeomorphic to (the minus sign indicates reversed orientation) $S^3/G \uplus (-S^3/G)$. On the other hand, the diffeomorphism ϕ that relates the two manifolds obtained by right and left identifications is just given by $[p]_R \mapsto \phi([p]_R) := [p^{-1}]_L$, where $p \in S^3 \cong SU(2)$ and $[\cdot]_R, [\cdot]_L$ denote the right and left cosets respectively. But this is an orientation reversing diffeomorphism.

For all but the cyclic groups Z_p of odd order, G contains the centre Z_2 of $SU(2)$. Standard properties of groups and their quotients imply the following homomorphism equivalences:

$$Isom(S^3/G) \cong SO(3) \times \frac{N_{SU(2)}(G)}{G} \cong SO(3) \times \frac{N_{SO(3)}(G/Z_2)}{G/Z_2}\,. \tag{2.10}$$

Let $S(G)$ denote the (unique) conjugacy class of stabilizer subgroups for $Isom(S^3/G)$'s action on S^3/G. We shall usually identify it with the stabilizer subgroup at $[e] \in S^3/G$, where e is the identity element of $S^3 \cong SU(2)$. One easily shows that $S(G) \cong N_{SU(2)}(G)/Z_2 \subset SO(3)_D$, where Z_2 is the centre of $SU(2)$. For $G \neq Z_p$, where p odd, this can be written as $S(G) \cong N_{SO(3)}(G/Z_2)$. As a closed subgroup of $SO(3)$ it must be either discrete, or a finite disjoint union of circles, or the whole of $SO(3)$. The last case is realized for $G = Z_2$ and the second case for $G = Z_p$ $(p > 2)$, where $S(G)$ is given by two circles in two perpendicular planes (viewed as subset of unit quaternions) one of which is the unique 1-parameter subgroup containing Z_p. In the first case we have explicitly (note: D_{4n} is the projection of D^*_{8n} into $SO(3)$):

$$S(G) = \begin{cases} O & \text{for } G = D^*_8, T^* \text{ and } O^* \\ D_{4n} & \text{for } G = D^*_{4n} \\ I & \text{for } G = I^* \end{cases} \tag{2.11}$$

By restricting the total space of the bundle (2.2) to the isometries, and, accordingly, the fibre D_∞ to $S(G)$, we obtain the principal-bundle

$$\begin{array}{ccc} S(G) & \xrightarrow{i} & Isom(S^3/G) \\ & & \downarrow{\scriptstyle \bar{p}} \\ & & S^3/G \end{array} \qquad \bar{p}([g,h]) := [g \cdot x \cdot h^{-1}]\,, \tag{2.12}$$

where $[g,h] \in N_{SO(4)}(G)$ and $x \in S^3$ represents the point $\infty = [x]$ (e.g. $x = e$) on S^3/G. The projection $\bar{p}$ is just the restriction of p in (2.2) to $Isom(S^3/G)$. Next we prove a lemma which we shall use for the calculation of $\pi_k(D_F)$, $k \geq 2$, later on.

Lemma 3. *If $G \neq Z_2$ is a non-trivial finite subgroup of $SU(2)$ which acts freely on S^3 via the standard orthogonal action, and if HC holds for S^3/G, then the projection map p from bundle (2.2) induces isomorphisms*

$$p_* : \pi_k(D(S^3/G)) \longrightarrow \pi_k(S^3/G) \qquad \forall k \geq 2 .$$

Proof. The validity of HC implies that the inclusion $I : Isom(S^3/G) \hookrightarrow D(S^3/G)$ induces isomorphisms $I_* : \pi_k(Isom(S^3/G)) \rightarrow \pi_k(D(S^3/G)) \forall k$. Since for $G \neq Z_2$ we have $\pi_k(S(G)) = 0 \forall k \geq 2$, the homotopy exact sequence for (2.12) implies that $\bar{p}_* : \pi_k(Isom(S^3/G)) \rightarrow \pi_k(S^3/G)$ is an isomorphism $\forall k \geq 2$. On the other hand, $\bar{p} = p \circ I$, and hence $\bar{p}_* = p_* \circ I_*$, which proves the claim •

Calculation of $\pi_0(D_F(S^3/G))$

For the spherical primes S^3/G, the zeroth homotopy group of D_F has already been calculated in [Wi]. Here we give a separate calculation in the homogeneous subclass.

We start with the lens spaces $L(p,1)$. Lemma 1 applied to the exact homotopy sequence for (2.2) implies that its very last two group entries are isomorphic: $\pi_0(D_\infty) \cong \pi_0(D)$. Theorem 2 and (2.7*b*) then imply $\pi_0(D_F) \cong \pi_0(D) \cong \pi_0(Isom)$ (Theorem A2, Appendix 2). This is isomorphic to Z_2 for $p > 2$, and to the trivial group for $p \leq 2$.

For G non-cyclic, we have that $Z_2 := \{1, -1\} \subset SU(2)$ is the centre and the centralizer of G in $SU(2)$. This implies that the action of $S(G)$ on S^3/G, $[p] \mapsto [gpg^{-1}]$, is effective on the coset $[e] \subset S^3$ and therefore on the fundamental group $\pi_1(S^3/G, [e])$. Thus no nontrivial element in $S(G)$ lies in the identity component in D_∞ (we take $\infty = [e]$), so that $\pi_0(D_\infty)$ contains the subgroup $S(G)$. We wish to show that it actually saturates all of $\pi_0(D_\infty)$. This can be done if $\pi_0(D) \cong \pi_0(Isom)$ holds (see Theorem A2), for then we infer from Lemma 1 that the last part of the sequence for (2.2) just says that $\pi_0(D_\infty)$ is a G/Z_2 - extension of $\pi_0(D)$, and therefore ($|\cdot|$ denotes the order of the group $\cdot$):

$$|\pi_0(D_\infty)| = \frac{|G|}{2}|\pi_0(D)| = \frac{|G|}{2}\frac{|N_{SO(3)}(G/Z_2)|}{|G/Z_2|} = \left|N_{SO(3)}(G/Z_2)\right| . \tag{2.13}$$

Next, by spinoriality (Theorem 1), we know from (2.6) and (2.7*a*) that $\pi_0(D_F)$ is a Z_2 - extension of $\pi_0(D_\infty)$. We wish to show that it must be the $SU(2)$-double cover $N_{SU(2)}(G)$. To do this in detail requires some notation.

Let $N_{SU(2)}(G) = \{g_1, \cdots, g_N\}$, $G = \{g_1, \cdots, g_K\}$, $K \leq N$, where $g_1 = e$, and $\{\theta_1, \cdots, \theta_N\} \subset su(2)$ (the Lie-algebra of $SU(2)$) such that $\exp(\theta_i) = g_i$ (we take indices $i, \cdots$ to run from 1 to N, and $a, \cdots$ from 1 to K). There is a homomorphism $\sigma : N_{SU(2)}(G) \rightarrow P_K$ into the permutation group of K objects, $g_i \mapsto \sigma(g_i) =: \sigma_i$, defined by $g_a g_i^{-1} = g_i^{-1} g_{\sigma_i(a)}$. Let $r : S^3 \rightarrow R$ be the distance function from e (with respect to the bi-invariant metric) and ρ a C^∞ step-function, such that $\rho(r) = 0$ for $r \leq \epsilon$ and $= 1$ for $r \geq 2\epsilon$. We then define $\lambda := \rho \circ r$. Further, we let B_1^2 and B_1^1 be the closed 2ϵ- and ϵ- balls around $e = g_1$ respectively. We right-translate them to 2ϵ- respectively ϵ- balls centered at g_a, i.e., $B_a^2 = B_1^2 \cdot g_a$ and $B_a^1 = B_1^1 \cdot g_a$ for each a. We further take ϵ small enough for B_a^2 to lie within a regular neighbourhood with respect to the covering $S^3 \to S^3/G$. In particular, they are all disjoint. Their projections into S^3/G are called B^2 and B^1.

To each element of g_i of $N_{SU(2)}(G)$ we now assign a diffeomorphism T_i of $S^3 - \cup_{a=2}^K B_a^2$, defined by

$$T_i: \quad p \mapsto \exp(\lambda(p)\theta_i) \cdot p \cdot \exp(-\lambda(p)\theta_i) . \tag{2.14}$$

It is easy to see that: (i) T_i leaves B_1^1 pointwise fixed, (ii) T_i leaves $B_1^2 - B_1^1$ invariant, (iii) T_i leaves $S^3 - \cup_{a=1}^K B_a^2$ invariant and (iv) T_i maps ∂B_a^2 onto $\partial B_{\sigma_i(a)}^2$. In fact, any point in a small closed outer collar-neighbourhood of ∂B_a^2 can be written as pg_a, where p is from such a neighbourhood of ∂B_1^2. Under T_i it is mapped according to

$$pg_a \mapsto g_i p g_a g_i^{-1} = T_i(p) g_{\sigma_i(a)} \, . \tag{2.15}$$

We now use (2.15) to smoothly extend the maps to all of S^3, that is, for any point $pg_a \in B_a^2$, where $p \in B_1^2$, we set

$$T_i(pg_a) = T_i(p) g_{\sigma_i(a)} \, , \tag{2.16}$$

where the $T_i(p)$ are defined in (2.14). By construction, this defines diffeomorphisms of S^3/G whose action on the fundamental group of S^3/G is simple conjugation: $p \mapsto g_i p g_i^{-1}$; for, as before, a path on S^3 from e to the g_a is mapped to a path from e to $g_{\sigma_i(a)}$, so that $T_i * (g_a) = g_i g_a g_i^{-1}$. Only the centre Z_2 of $N_{SU(2)}(G)$ acts trivially on the fundamental group. However, the diffeomorphism corresponding to the generator of this central Z_2 is easily seen from (2.14) to be just the relative 2π-rotation of the spheres ∂B_1^2 and ∂B_1^1, which, by spinoriality, is not isotopic to the identity keeping a frame at $[e]$ fixed. By regarding each element T_i as representing a class in $\pi_0(D_F)$ (denoted by $[T_i]$), the map $g_i \mapsto [T_i]$ defines an injective homomorphism of $N_{SU(2)}(G)$ into $\pi_0(D_F(S^3/G))$, as required.

Section 3. Calculation of $\pi_k(D_F(S^3/G))$ for $k \geq 1$

Within the class of spaces considered here, $\bar{\Sigma} = S^3$ $(G = Z_1)$ and $\bar{\Sigma} = RP^3$ $(G = Z_2)$ receive a special status due to them being group manifolds. This allows us to make a somewhat more concise statement than in the other cases. We have the following

Lemma 4. *a): If $\bar{\Sigma}$ is a topological group, (2.2) is trivial.*
b): For $\bar{\Sigma} \cong RP^3$ or $\cong S^3$, (2.3) is trivial.

Proof. a): Define a global section $\sigma : \bar{\Sigma} \to D$, $x \mapsto L_x$ (=left translations). We have $p \circ \sigma = \mathrm{id}|_{\bar{\Sigma}}$. A global trivialization is given by

$$\phi^{-1} : \bar{\Sigma} \times D_\infty \to D; \quad (x, h) \mapsto L_x \circ h$$
$$\phi : D \to \bar{\Sigma} \times D_\infty; \quad g \mapsto \left(p(g), L_{p(g)}^{-1} \circ g\right)$$

b): Define a global section $\sigma : SO(3) \to D_\infty$, $\alpha \mapsto \mathrm{Ad}(\alpha)$, where $\mathrm{Ad}(\alpha)(p) = \alpha p \alpha^{-1} \,\forall p \in \bar{\Sigma}$. We have that $\mathrm{Ad}(\alpha)_*|_e = \alpha$, and hence $\tilde{p} \circ \sigma = \mathrm{id}|_{SO(3)}$. A corresponding trivialization is given by

$$\begin{aligned} \phi^{-1} : \, SO(3) \times D_F \to D_\infty \,; &\quad (\alpha, h) \mapsto \mathrm{Ad}(\alpha) \circ h \\ \phi : \, D_\infty \to SO(3) \times D_F \,; &\quad g \mapsto \big(\tilde{p}(g), \mathrm{Ad}^{-1}(\tilde{p} \circ g)\big) \ \bullet \end{aligned}$$

By HC we know that $D(S^3)$ and $D(RP^3)$ have the homotopy type of $SU(2) \times SO(3)$ and $SO(3) \times SO(3)$ respectively. Hence we have the

Theorem 3. *$\mathcal{Q}(\Sigma)$ has altogether trivial homotopy groups for $\bar{\Sigma} = S^3$ or $\bar{\Sigma} = RP^3$.*

Proof: By the previous lemma, the bundles (2.2) and (2.3) are product bundles. We thus have for all $k \geq 0$:

$$\pi_k(D) \cong \pi_k(\bar{\Sigma}) \times \pi_k(D_\infty) \tag{3.1}$$

$$\pi_k(D_\infty) \cong \pi_k(D_F) \times \pi_k(RP^3) \tag{3.2}$$

Setting $\bar{\Sigma}$ either equal to S^3 or RP^3, and inserting the space $Isom$ for D proves the claim $\bullet$

Calculation of $\pi_1(D_F(\bar{\Sigma}))$

Since for $\bar{\Sigma} = S^3/G$ we have $\pi_2(\bar{\Sigma}) = 0$, the exact sequences for (2.2) and (2.1) end as follows:

$$1 \to \pi_1(D_\infty) \xrightarrow{i_*} \pi_1(D) \xrightarrow{p_*} G \to \pi_0(D_\infty) \to \pi_0(D) \to 1 \tag{3.3}$$

$$1 \to \pi_1(D_F) \xrightarrow{\hat{i}_*} \pi_1(D) \xrightarrow{\hat{p}_*} G \times Z_2 \to \pi_0(D_F) \to \pi_0(D) \to 1 \tag{3.4}$$

We shall first deal with G non-cyclic. For non-cyclic G we have $centreG = Z_2$, and by HC $\pi_1(D) = \pi_1(Isom) = Z_2$, so that Lemma 1 tells us that p_* in (3.3) is an isomorphism, and, therefore, $\pi_1(D_\infty) = 0$. Spinoriality together with (2.7a) then imply $\pi_1(D_F) = 0$.

For $G = Z_p$, $\bar{\Sigma} \cong L(p, 1)$ is non-spinorial so that the Lemmas 1 and 2 imply ($G = centreG$) that $\hat{p}_*$ in (3.4) is onto. HC then implies that (3.4) reduces to

$$1 \longrightarrow \pi_1(D_F) \xrightarrow{\hat{i}_*} Z \times Z_2 \xrightarrow{\hat{p}_*} Z_p \times Z_2 \longrightarrow 1\,, \tag{3.5}$$

hence $\pi_1(D_F) \cong ker\, \hat{p}_*$. For even p it is immediate that $ker\, \hat{p}_*$ cannot contain the $Z_2 \subset Z \times Z_2$, for, otherwise, $\hat{p}_*$ cannot be onto. Hence, in this case, $\pi_1(D_F) \cong Z$. This result is generally true, as one can see using a slightly more explicit argument which we briefly sketch. $N_{SU(2)}(Z_p)$ contains the (unique) circle group in which Z_p lies. If we parameterize this circle subgroup in by $s \in [0, 4\pi]$ then the Z factor in $\pi_1(D)$ is generated by the loop $s \in [0, \frac{4\pi}{p}]$. Visualizing the corresponding transformation on the lens representing $L(p,1)$, one infers that this generator is mapped via $\hat{p}_*$ onto the generator of Z_p. Since $\hat{p}_*$ is onto, $Z_2 \subset Z \times Z_2$ cannot lie in its kernel. Hence $\pi_1(D_F) \cong Z$.

Calculation of $\pi_k(D_F)$ for $k \geq 2$

A typical piece of the exact sequence for (2.2) looks like:

$$\cdots \longrightarrow \pi_{k+1}(D) \overset{p_*^{(k+1)}}{\longrightarrow} \pi_{k+1}(\bar{\Sigma}) \longrightarrow \pi_k(D_\infty) \longrightarrow \pi_k(D) \overset{p_*^{(k)}}{\longrightarrow} \pi_k(\bar{\Sigma}) \longrightarrow \cdots \tag{3.6}$$

By Lemma 3, $p_*^{(k)}$ is an isomorphism for $k \geq 2$, so that $\pi_k(D_\infty) = 0 \quad \forall k \geq 2$. The exact sequence for (2.3) then implies $\pi_k(D_F) \cong \pi_{k+1}(SO(3)) \quad \forall k \geq 2$.

Summary

According to (1.5) we can summarize the homotopy groups of the (connected) configuration space $\mathcal{Q}(S^3/G)$ for S^3/G homogeneous:

$$\pi_k(\mathcal{Q}(RP^3)) \cong 0 \quad \forall k \geq 1 \tag{3.7}$$

$$\pi_k(\mathcal{Q}(L(p,1))) \cong \begin{cases} Z_2 & \text{for } k = 1 \\ Z & \text{for } k = 2 \\ \pi_k(S^3) & \text{for } k \geq 3 \end{cases} \tag{3.8}$$

$$\pi_k(\mathcal{Q}(S^3/G)) \cong \pi_k\left(S^3/N_{SU(2)}(G)\right) \quad \forall k \geq 1, \quad \text{and } G \text{ non-cyclic} \tag{3.9}$$

In particular the last equation is very suggestive. We stress, however, that we did not establish a homotopy equivalence $\mathcal{Q}(S^3/G) \sim S^3/N_{SU(2)}(G)$.

Section 4. Results and Some Observations

Prime $\bar{\Sigma}$	HC	$\pi_1(\mathcal{Q}(\Sigma))$	$\pi_2(\mathcal{Q}(\Sigma))$	$\pi_k(\mathcal{Q}(\Sigma))$
S^3/D_8^*	$*$	O^*	1	$\pi_k(S^3)$
S^3/D_{4n}^*	$*$	D_{8n}^*	1	$\pi_k(S^3)$
S^3/T^*	?	O^*	1	$\pi_k(S^3)$
S^3/O^*	w	O^*	1	$\pi_k(S^3)$
S^3/I^*	?	I^*	1	$\pi_k(S^3)$
$S^3/D_8^* \times Z_p$	$*$	$Z_2 \times O^*$	Z	$\pi_k(S^3 \times S^3)$
$S^3/D_{4n}^* \times Z_p$	$*$	$Z_2 \times D_{8n}^*$	Z	$\pi_k(S^3 \times S^3)$
$S^3/T^* \times Z_p$	?	$Z_2 \times O^*$	Z	$\pi_k(S^3 \times S^3)$
$S^3/O^* \times Z_p$	w	$Z_2 \times O^*$	Z	$\pi_k(S^3 \times S^3)$
$S^3/I^* \times Z_p$	?	$Z_2 \times I^*$	Z	$\pi_k(S^3 \times S^3)$
$S^3/D'_{2^k(2n+1)} \times Z_p$	$*$	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3 \times S^3)$
$S^3/T'_{8\cdot 3^k} \times Z_p$	?	O^*	Z	$\pi_k(S^3 \times S^3)$
$L(p,q_1)$	w	Z_2	Z	$\pi_k(S^3)$
$L(p,q_2)$	$w, *$	$Z_2 \times Z_2$	$Z \times Z$	$\pi_k(S^3 \times S^3)$
$L(p,q_3)$	w	Z_2	$Z \times Z$	$\pi_k(S^3 \times S^3)$
RP^3	$*$	1	1	1
S^3	$*$	1	1	1
$S^2 \times S^1$		$Z_2 \times Z_2$	Z	$\pi_k(S^3 \times S^2)$
$K(\pi,1)_{\mathrm{sl}}$		$\mathrm{Aut}_+^{Z_2}(\pi_1)$	1	$\pi_k(S^3)$

The table summarizes the results for the homotopy groups of the configuration spaces $\mathcal{Q}(\Sigma)$. The last column comprises all homotopy groups higher than the second, i.e., $k \geq 3$. The body of the table is divided into five horizontal blocks of which the first three represent the spherical space forms, the fourth the single handle-manifold (wormhole), and the fifth the sufficiently large $K(\pi,1)$ primes. We recall that a space is called $K(\pi,1)$ if its only non-trivial homotopy group, being isomorphic to π, is the first. Also, a 3-manifold is called sufficiently large if it contains a surface of genus ≥ 1 whose fundamental group is mapped injectively into the fundamental group of the ambient 3-manifold by the inclusion map. In other words, non-contractible loops on the surface (with fixed base point) are still non-contractible even when allowed to move off the surface.

The calculation for the last two blocks is almost trivial in view of the strong results obtained in [Ha 3] and [Ha 1] respectively. $\mathrm{Aut}_+^{Z_2}$ in the last block denotes a central Z_2 extension (according to (2.6) imposed by spinoriality) of the "orientation preserving" automorphisms of π_1. For the 3-torus it is just given by the Steinberg group $St(3,Z)$ which is a central Z_2-extension of $SL(3,Z)$. (See paragraph 10 of [Mil] for an instructive presentation of $St(3,Z)$.)

The calculations for the spherical space forms depend on the validity of the Hatcher conjecture (HC), which is not known to hold in all cases. The calculations of $\pi_1(\mathcal{Q})$, however, depend only on a weak implication thereof, namely $\pi_0(D_F) \cong \pi_0(Isom)$. Since there are cases where this weak form of HC but not HC itself is known to hold, we indicated their status of validity separately in the second column (compare Appendix 2, Theorems A1 and A2). Here, an asterisk, $*$, denotes the validity of HC, a w the validity of the weak form only, and the question mark, (?), that no such result is known to us. Assuming (w), the groups $\pi_1(\mathcal{Q}(S^3/G))$ were first calculated in [Wi]. Subsequently the table was completed to the present form in [Gi 2].

The first block contains the homogeneous spherical primes with non-cyclic fundamental group, the second the non-homogeneous ones. Here, the order of the additional cyclic group, p, has to be coprime to the order of the group the Z_p is multiplied with, and ≥ 2 in the first five, and ≥ 1 in the remaining two cases. The third block contains all the spherical primes with cyclic fundamental group Z_p, otherwise known as the lens spaces $L(p,q)$, where q has to be coprime to $p \geq 2$. Here, q_1 stands for $q = \pm 1$ mod p, q_2 for $q \neq \pm 1$ mod p and $q^2 = 1$ mod p, and q_3 for the remaining cases. Amongst all $L(p,q_2)$ are those of the form $L(4n, 2n-1), n \geq 2$. For those the $*$ is valid in the second column and (w) for all others. Finally, RP^3 and S^3 are listed separately. Together with $L(p,q_1)$ they comprise the homogeneous spaces in the third block, and, taken together with the first block, all the homogeneous spherical primes in the list.

Amongst others, the last block contains all closed orientable 3-manifolds that can support a flat metric. These are of the form R^3/Γ, where Γ is a discrete infinite group that acts properly discontinuously on R^3. There are six such groups, $Z \times Z \times Z$ yielding the 3-torus, and five extensions by this group of Z_2, Z_3, Z_4, Z_6 and $Z_2 \times Z_2$ respectively. Another infinite class of manifolds contained in the last block is given by manifolds of the form $S^1 \times R_g$, where R_g denotes a closed Riemannian surface of genus $g \geq 1$. A more detailed table is given in [Gi 1].

Some Observations

In standard canonical quantization one regards the wave function as a section of a possibly non-trivial complex line bundle over configuration space. For not simply connected configuration spaces, this induces an action of the fundamental group on sections via some 1-dimensional representations and thus also on the space of quantum states. In the conventional Schrödinger-representation based canonical quantization approach to quantum gravity one tries to repeat this construction, using the classical configuration space as domain space for the state functional.

Since this space is infinite dimensional, the attempts to construct a Hilbert space of states are overshadowed by the lack of appropriate measures. This might be taken to point towards the necessity to use an appropriate distributional-dual as domain space on which a wider class of measures are available, as was repeatedly emphasized in [Is]. No general construction has, however, been found yet. This difficulty is not particular to theories of gravity and renders the availability and construction of the Schrödinger representation a non-trivial technical problem in any field theory. In our case it is easy to see that the obvious action of D_F on the distributional dual of the space of Riemannian metrics is not free, thus rendering an analogous structure to (1.3) and hence the given derivation of (1.5) impossible. As long as these technical and related interpretational issues are not settled the presently given standard arguments concerning the significance of the classical configuration space topology for quantum theory are therefore necessarily heuristic in nature, as already stated in the Introduction. Present day investigations into possible implications of quantum gravity (in the canonical formulation) seem almost exclusively to pretend the classical configuration space as domain for the state functional. Notable exceptions are [IK] and the loop representation discussed in [Ash] (and references therein).

So far topological investigations have focused on $D_F/D_F^0 \cong \pi_1(\mathcal{Q})$. Its significance is made plausible with the assumption of an action of D_F on some linear space of auxiliary states whose subspace annihilated by D_F^0 carries a residual action of D_F/D_F^0. A (pretended) Schrödinger representation is then employed in trying to visualize this action in terms of physical operations (e.g. exchanges, rotations etc.) on preferred (e.g. localized) states (see e.g. [ABBJRS]).

Applied to the case at hand, the possible fundamental groups are structured according to (1.8b), with those in the table above occurring in the bracket of (1.8b), i.e., as subgroups. In this context one would say that a spinorial manifold Σ admits *abelian* spinorial states, if $\pi_1(\mathcal{Q}(\bar{\Sigma}))$ allows for one dimensional representations which represent the Z_2 generated by a 2π-rotation parallel to an asymptotic sphere non-trivially. States transforming non-trivially under such a rotation were first considered in [FS]. Their assertion was that if a single state existed that was not left invariant by this Z_2, its antisymmetric combination with the Z_2 transformed one would be a state that changed sign under that Z_2. But in order to correspond to a pure state it must also be a member of an irreducible representation subspace. The simplest possibility would be to carry a one-dimensional representation. Here we make the following

Observation 1. *No 3-manifold, $\bar{\Sigma}$, whose prime decomposition consists entirely of primes taken from the upper four horizontal blocks in our table allows for abelian spinorial states.*

Proof. Clearly we assume the presence of at least one spinorial prime, since the statement is trivial otherwise. As shown in the proof for Corollary 1, the rotation parallel to the sphere at ∞ is isotopic to rotations parallel to each connecting sphere for the primes. This generates an extending Z_2 subgroup (see 2.6) in each factor of the first term in (1.8b) that corresponds to a spinorial prime. $\bar{\Sigma}$ thus allows for abelian spinorial states, if and only if at least one of the spinorial $\bar{\Sigma}_i$ does. However, none of the groups $O^*, D^*_{8n}\, n \geq 1, I^*$ (see table) has a 1-dimensional representation that represent the extending Z_2 (the centre) non-trivially.

The latter fact can be proven from the presentations

$$\begin{aligned} D^*_{4m} &= \{x, y \mid x^2 = (xy)^2 = y^m\} \\ O^* &= \{x, y \mid x^2 = (xy)^3 = y^4;\ x^4 = 1\} \\ I^* &= \{x, y \mid x^2 = (xy)^3 = y^5;\ x^4 = 1\}\,, \end{aligned}$$

from which the corresponding abelianized groups are readily determined. In the first case one has to distinguish between m odd and m even. For m even one obtains the group $Z_2 \times Z_2$ (generated by x and

y, taken as abelianized generators), and for m odd the group Z_4 (generated by x, where $x^2 = y$). In the second and third case one obtains the groups Z_2 (generated by x) and the trivial group respectively. Since in each case it is x^2 that generates the extending Z_2 (the centre), only D^*_{4m} for m odd has the desired representation. But as a $\pi_1(Q)$ only even m occur. •

We do not know whether the result can easily be extended to include general $K(\pi, 1)$-primes as well. We explicitly checked, however, that e.g. the three-torus as well does not allow for abelian spinorial states. In that case $Aut_+(Z \times Z \times Z)$ is $SL(3, Z)$ and $Aut_+^{Z_2}(Z \times Z \times Z)$ is isomorphic to the Steinberg group $St(3, Z)$ (see e.g. paragraph 10 in [Mil] for more information about $St(n, Z)$). But $St(3, Z)$ is a perfect group, i.e. its own commutator subgroup, and has hence no non-trivial abelian representations.

It is instructive to compare this to the case of two spatial dimensions with two-torus topology, which has been investigated in [ABBJRS]. In two dimensions all closed genus $g \geq 1$ surfaces, R_g, are spinorial. This is in fact very easy to prove by looking at the action of the diffeomorphism on the fundamental group of $R_g - \infty$. $\pi_0(D_F)$ is then a central extension of $\pi_0(D_\infty)$ by Z. In [ABBJRS] it has been shown that for the two-torus $\pi_0(D_F)$ is isomorphic to $St(2, Z)$ (a central Z-extension of $SL(2, Z)$. It is not perfect.). $St(2, Z)$ can be presented with two generators, a and b, and single relation $ab^{-1}a = b^{-1}ab^{-1}$. From this a presentation for $SL(2, Z)$ is obtained by imposing the additional relation $R := (ab^{-1}a)^4 = E$ (E = identity), where in $SL(2, Z)$ a and b are realized as upper and lower triangular 2×2 matrices with unit entries. R is generated by a 2π-rotation. The abelianization of $St(2, Z)$ is Z, generated by a', the image of a in the abelianized group under the canonical quotient map. R', the image of R, is generated by a'^{12}. Hence there exist 12 inequivalent one-dimensional unitary representations for any given assignment $R \mapsto \exp(i\theta)$, $\theta \in [0, 2\pi)$.

In order to obtain (pure) spinorial states one has to go to higher dimensional representations. For example, the group O has five irreducible representations of dimensions 1, 1, 2, 3 and 3, whereas O^* has in addition three more of dimensions 2, 2 and 4, all of which represent the extending Z_2 non-trivially. Pure states now correspond to sections in higher dimensional (complex) vector bundles, and the inequivalent sectors, labeled by the inequivalent (unitary) representations ρ, still carry an action of the centre of ρ, but not of the full group. (We define the centre of a representation ρ of a group G by $C_\rho(G) := \{a \in G \,/\, \rho(ab) = \rho(ba)\, \forall b \in G\}$.) In particular, 2π-rotations are always represented. Let us also note at this point that higher dimensional representations automatically appear in carrying through the standard quantum-mechanical formalism in presence of a discrete gauge group. Excluding them means to a priori exclude potentially interesting sectors [Gi 3].

Pushing the original setting a bit further, we remark that the possible, inequivalent line bundles with connection over $\mathcal{Q}$ are classified by $H_1(\mathcal{Q}, Z) \oplus FH_2(\mathcal{Q}, Z)$ (see e.g.[Wo]), where F denotes the free part. It is also convenient to split H_1 into a free-, and a torsion part (denoted by T). The free part of H_1 alone then accounts for the different flat connections with unchanged bundle topology, whereas $TH_1(\mathcal{Q}, Z) \oplus FH_2(\mathcal{Q}, Z) \cong H^2(\mathcal{Q}, Z)$ labels the topologically inequivalent bundles. Only the latter define different sectors for the quantum theory, and may e.g. show up as a non-trivial spectral flow for the Dirac operator [AN]. Whereas for non-trivial FH_2 one cannot make any statements without explicitly analyzing the dynamics as to whether it actually makes use of a non-trivial class, a trivial FH_2 clearly excludes such possibilities from the beginning. It is therefore interesting to see whether there are $\bar{\Sigma}$'s whose associated $\mathcal{Q}(\bar{\Sigma})$'s have trivial $FH_2(\mathcal{Q}, Z)$. From the table and (1.8) we see that this is generally not the case. However, we have the

Observation 2. *Let $\bar{\Sigma}$ be a homogeneous spherical prime of non-cyclic fundamental group and $\mathcal{Q} = \mathcal{Q}(\bar{\Sigma})$ its associated configuration space; then $FH_2(\mathcal{Q}, Z) = 0$.*

Proof. From the table we infer that for the $\bar{\Sigma}$ in question the associated $\mathcal{Q}(\bar{\Sigma})$ have finite universal cover of which the first two homotopy groups are trivial. A standard spectral sequence argument, which we suppress at this point, then shows that $H_2(\mathcal{Q}, Q)$ (rational coefficients) is trivial, which is equivalent to the statement made •

Taken together with Observation 1 this excludes a possibly conjectured connection between spinoriality and the possibility of non-trivial spectral flows.

A very special case is that of $\bar{\Sigma} = RP^3$. Here we have immediately

Observation 3. *RP^3 is the unique, non-trivial prime 3-manifold on our table whose configuration space has altogether trivial homotopy (and hence homology) groups.*

From (1.8) one thus infers that a multi-RP^3 manifold, $\bar{\Sigma}$, therefore receives all the non-trivial topology of $\mathcal{Q}(\bar{\Sigma})$ from the factor $\Omega\mathcal{C}$. A more detailed study of $\mathcal{C}$ shows [HM] that in this case $\pi_*(\mathcal{Q}(\bar{\Sigma}))$ contains as subgroups those for the configuration space of n identical objects in R^3 (if $\bar{\Sigma}$ is the connected sum of n RP^3's). Its fundamental group is P_n, the permutation group of n objects, and the higher ones are given in [FN]. In [ABBJRS] the particular case of two RP^3's has been used to discuss, and rule out, within the present framework a spin-statistics relation of a general kind, as e.g. demonstrated in [So] for field theories admitting kinks (and, crucially, anti-kinks). However, that within the present framework a general correlation cannot exist is a consequence of the fact that rotations of spinorial primes generate a normal subgroup. In particular, any representation of the factor group obtained by dividing out the rotations gives rise to a representation of the full group with rotations represented trivially. In this sense the representations for rotations and exchanges are decoupled, irrespectively of the ambiguities in the definition of the latter [ABBJRS].

Appendix 1

In this appendix we show how to prove non-spinoriality for $\bar{\Sigma} = L(p,q)$ and $\bar{\Sigma} = S^1 \times S^2$. To this end we pick a curve $\alpha : [0,1] \to SO(3)$ that generates $Z_2 = \pi_1(SO(3), id)$, find a covering curve in D_∞, starting at the identity, and show that it ends in the identity component D_F^0 of D_F (we refer to (2.3)). We may choose $\alpha(s) = \exp(2\pi s\hat{z})$ and denote the corresponding linear map in R^3 by $R_z[2\pi s]$, or $R_z[\varphi]$ for general angles φ.

Let now $\sigma : R^3 \supset B_2 \to \bar{\Sigma}$ be an embedding of $B_2 = \{x \in R^3 \,/\, \|x\| \leq 2\}$ into $\bar{\Sigma}$. We let r denote the distance from the origin of B_2 and set:

$$\begin{aligned} D_2 &= Image\,\sigma\big|_{r\leq 2}, \quad D_1 = Image\,\sigma\big|_{r\leq 1}, \quad T = Image\,\sigma\big|_{1\leq r\leq 2} \\ S_2 &= Image\,\sigma\big|_{r=2}, \quad S_1 = Image\,\sigma\big|_{r=1}, \quad \infty = Image\,\sigma\big|_{r=0} \end{aligned} \tag{A1}$$

On B_2 we can define a path of homeomorphisms, ρ_s, by

$$\rho_s = \begin{cases} R_z[2\pi s(2-r)] & \text{for } 2 \geq r \geq 1 \\ R_z[2\pi s] & \text{for } r \leq 1 \ , \end{cases} \tag{A2}$$

which then defines a path of homeomorphisms, R_s, of $\bar{\Sigma}$, by setting

$$R_s = \begin{cases} \sigma \circ \rho_s \circ \sigma^{-1} & \text{on } D_2 \\ id & \text{on } \bar{\Sigma} - \overset{\circ}{D}_2 \ . \end{cases} \tag{A3}$$

This map is not differentiable on S_2 and S_1, but it may easily be smoothed by modifying it in arbitrarily small collar-neighbourhoods of these two spheres. We imagine this being done but without giving the details here, since in order to calculate the projection map $\tilde{p}$ in (2.3) we only need differentiability in a neigbourhood of ∞. R_s "rigidly" rotates D_1 by an angle $2\pi s$ about the z-axis by progressing "rigid" rotations of the spheres $r = const.$ within T. For $s = 1$ we have $R_s\big|_{D_1} = id$ so that we call R_1 a rotation parallel to S_1 and S_2 (see e.g. [He][L]). Each R_s fixes ∞ and projects to $\alpha(s)$ via $\tilde{p}$. R_1 is in D_F since it fixes a disc (D_1) containing ∞. In order to show that R_1 is in D_F^0, we now explicitly construct a path, K_s, from id to R_1 within D_F. As above, it is sufficient to construct a path of homeomorphisms rather than diffeomorphisms that fix D_1 and which we then imagine to be smoothed appropriately. The details are irrelevant for us. For the construction it is convenient to represent the spaces $L(p,q)$ and $S^1 \times S^2$ by the following fundamental domains:

$L(p,q)$: Take a solid ball $\|x\| \leq 3$ in R^3, and identify the 2-dimensional sectors $s_k : \frac{2\pi}{p}(k-1) \leq \phi \leq \frac{2\pi}{p}k$ on the upper hemisphere with the sectors s_{k+q} on the lower hemisphere, by first reflecting them on the equatorial plane ($z = 0$), followed by a rotation about the z-axis. We take $T = \{x \in R^3 \,/\, 1 \leq \|x\| \leq 2\}$, so that ∞ corresponds to $x = 0$.

$S^1 \times S^2$: Take a solid spherical shell $1 \leq \|x\| \leq 6$ in R^3, and identify the inner and outer 2-sphere boundaries radially (i.e. points of equal polar angles are identified). We take $T = \{x \in R^3 \,/\, 1 \leq \|x - (0,0,3)\| \leq 2\}$ so that ∞ corresponds to $(0,0,3)$.

The crucial observation is that the $SO(3)$- rotation, $R_z[\varphi]$, applied to these domains is compatible with the boundary identifications, and, therefore, defines a homeomorphism of the manifolds in question. But then it is obvious how to reach R_1 by a path of homeomorphisms that fix D_1: instead of rotating D_1 against S_2, we rotate $\bar{\Sigma} - D_2$ and with it S_2 against S_1 by just the negative amount. That is (r still denotes the distance from ∞ and σ is the identity since we work within R^3)

$$K_s = \begin{cases} id & \text{on } D_1 : (r \leq 1) \\ R_z[-2\pi s(r-1)] & \text{on } D_2 - \overset{\circ}{D}_1 : (1 \leq r \leq 2) \\ R_z[-2\pi s] & \text{on } \bar{\Sigma} - \overset{\circ}{D}_2 : r \geq 2 \end{cases} \tag{A4}$$

so that

$$K_1 = \begin{cases} id & \text{on } D_1 \\ R_z[2\pi(1-r)] & \text{on } D_2 - \overset{\circ}{D}_1 \\ R_z[-2\pi] & \text{on } \bar{\Sigma} - \overset{\circ}{D}_2 \end{cases} \tag{A5}$$

which, by $R_z[\varphi + n\,2\pi] = R_z[\varphi]$, is equal to R_1.

Although we work with oriented manifolds only, let us note for completeness that from the fundamental domain we used to construct $S^1 \times S^2$, we can also construct $S^2 \tilde{\times} S^2$, the unique non-orientable 2-sphere bundle over S^1. For this we just identify the inner and outer 2-sphere boundaries via the antipodal map (in standard polar angles: $(\theta, \phi) \mapsto (\pi - \theta, \phi + \pi)$), rather than using the identity. The rotations $R_z[\varphi]$ are then still compatible with the boundary identifications, hence proving non-spinoriality for $S^1 \tilde{\times} S^2$.

By an obvious generalization of these geometric constructions we can visualize the "if" part of Corollary 1, i.e., that any connected sum of lens spaces and handles (also non orientable ones) are non spinorial. For example, a connected sum of n lens spaces can be represented by a solid ball with $n-1$ open balls removed from its interior with all n centers aligned along the, say, z axis. The typical lens space identifications are now performed on the one outer and the $n-1$ inner 2-sphere boundaries such that the rotation about the z axis is still compatible with these identifications (i.e. all lens edges lie in parallel planes perpendicular to the z axis. One can include l handles by removing $2l$ more open balls (also aligned along the z axis) whose inner 2-sphere boundaries are pairwise identified.

Appendix 2

In this appendix we collect some of the results on the diffeomorphism group of 3-manifolds which were of relevance in our investigations.

Theorem A1. *For spherical primes (S^3/G), the handle ($S^1 \times S^2$), sufficiently large $K(\pi,1)$'s ($K(\pi,1)_{sl}$) and most of the non-sufficiently large $K(\pi,1)$'s which are Seifert, two diffeomorphisms are isotopic if and only if they are homotopic.*

Proof. For the spherical primes these statements are proven in [HR] (Lens spaces $L(p,q)$), [BR] (octahedral spaces $S^3/O^* \times Z_p$), [Asa] or [R] (prism and generalized prism spaces $S^3/D^*_{4m} \times Z_p$, $S^3/D'_{2^k\cdot(2n+1)} \times Z_p$ and [BO] (icosahedral, tetrahedral and generalized tetrahedral $S^3/I^* \times Z_p$, $S^3/T^* \times Z_p$, $S^3/T'_{8\cdot 3^k} \times Z_p$). For the handle this is proven in [Gl], for the $K(\pi,1)_{sl}$'s in [Wa] and for the Seifert non-sufficiently large $K(\pi,1)$'s in [Sc]. •

Theorem A2. *For all spherical primes but the icosahedral, tetrahedral and generalized tetrahedral spaces one has the isomorphism $\pi_0(D) \cong \pi_0(Isom)$, where $Isom$ denotes the space of orientation preserving isometries.*

Proof. $\pi_0(D)$ has been calculated in the references just given, except [BO]. The calculations for $\pi_0(Isom)$ may be found in [Wi]. •

For the 3-sphere it is proven in [Ha 4] that there is a homotopy equivalence $D(S^3) \sim Isom(S^3)$. As generalization it has been conjectured in [Ha 2] that there is a homotopy equivalence for all spherical primes (Hatcher conjecture (HC)): $D(S^3/G) \sim Isom(S^3/G)$. The efforts to prove this are so far summarized in the following

Theorem A3. *HC holds for the real projective space RP^3, the lens spaces $L(4k, 2k-1)$,* *where $k > 1$, and the prism and generalized prism spaces $S^3/D_4^* \times Z_p$ and $S^3/D'_{2^k \cdot (2n+1)} \times Z_p$ respectively.*

Proof. The assertion for RP^3 is made in [Ha 2]. For $S^3/D_{4n}^* \times Z_p$, $p > 1$, and $S^3/D'_{2^k \cdot (2n+1)} \times Z_p$, $p \geq 1$, this was proven in [I 1][I 2] and for S^3/D_{4n}^* and $L(4k, 2k-1)$ $k > 1$ in [MR] •

Similar results are known for the handle and the $K(\pi, 1)_{sl}$'s:

Theorem A4. *The group of diffeomorphisms of $S^2 \times S^1$ has the homotopy type of $O(2) \times O(3) \times \Omega O(3)$, where $\Omega(\cdot)$ denotes the loop space of $(\cdot)$.*

Proof. This is proven in [Ha 3]. If one considers orientation preserving diffeomorphisms only, one may write: $D(S^2 \times S^1) \cong Z_2 \times S^1 \times SO(3) \times \Omega SO(3)$ •

Theorem A5. *Let $\bar{\Sigma}$ be an orientable, sufficiently large $K(\pi, 1)$ prime with fundamental group $\pi = G$. Its diffeomorphism group, Diff, has homotopy groups $\pi_0(Diff) \cong Out(G)$, $\pi_1(Diff) \cong centre\, G$ and $\pi_k(Diff) = 0 \quad \forall k \geq 2$.*

Proof. The proof is given in [Ha 1] for the larger class of P^2-irreducible (irreducible which contain no two-sided RP^2), sufficiently large $K(\pi, 1)$'s. If one restricts to orientation preserving diffeomorphisms, one has $\pi_0(D) \cong Out_+(G)$, the outer automorphisms which respect the orientation homomorphism (an index 2 subgroup of $Out(G)$ if $\bar{\Sigma}$ allows for orientation reversing diffeomorphisms, otherwise identical to $Out(G)$) •

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