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Autor: Frochaux, Etienne

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A Relativistic Quantum Equation for $N \geq 2$ Bosons in Two Space-Time Dimensions

By Etienne Frochaux

Département de Mathématiques, Ecole Polytechnique Fédérale,
CH-1015 Lausanne, Switzerland

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Abstract. We give a quantum and relativistic eigenvalue equation for $N \geq 2$ bosons in two space-time dimensions, which generalizes the corresponding Schrödinger equation. More precisely we find three self-adjoint operators P (momentum), H (Hamiltonian) and L (generator of the Lorentz transformations), acting on the Hilbert space of N free bosons in the Schrödinger picture, which satisfy the commutation rules of the Poincaré algebra. The eigenvalue equation for the mass operator $M^2 = H^2 - P^2$ leads to the above mentioned N-body equation. The possible existence of an eigenvalue assures that this model is non trivial.

1 Introduction

Quantum Field Theory (QFT) is until now the best framework to study theoretically the quantum and relativistic particle phenomena. This theory has been established initially to reproduce the experimental scattering results, in a relativistic and quantum framework. The bound states problem has been considered later and has been naturally thought in terms of scattering amplitudes. This has led to the famous Bethe-Salpeter method for finding the bound states, which treats the quantum and relativistic two-body problem in a satisfactory way, both from the physical and mathematical point of view, but which still carries important drawbacks (the bound state masses appear in a complicated way ; moreover the calculation is overloaded by the so-called 'relative time variables', which are variables without physical interpretation). Because of these difficulties, this method has not led to a clear and practicable theory of the relativistic corrections to the Schrödinger equation.

The difficulty in studying other particle phenomena than scattering, in QFT, comes from the difficulty of defining the notion of 'particles'. It is generally admitted that this notion can be given a clear meaning only in an asymptotic way. This statement, however, ignores that the particles can also be seen in the spectrum of the mass operator M. In fact the mass of each particle is an eigenvalue of M. In particular, once a bound state is known to exist in a given QFT model (by the Bethe-Salpeter method), its eigenspace can be approached by using a variational and perturbative method. Such a programme has been carried out for a class of QFT models, the $\mathcal{P}(\varphi)_2$ models [1], which describe a world of massive, identical bosons in a two-dimensional space-time. This study has the advantage of stating the quantum relativistic bound state problem in a completely new way. In particular, because the calculation can be restricted, without loss, to the so-called 'zero-time subspace', the appearance of the 'relative time variables' can be avoided. However, the main interest of this method is that it finally leads to an eigenvalue equation for a two-variable function. This equation, which has to play in QFT the same role as the Schrödinger equation in Quantum Mechanics (that is to give the discrete structure of the set of the bound states) can naturally be considered as the relativistic generalization of the two-body Schrödinger equation.

More precisely this method constructs step by step, by minimisation and perturbation arguments, a subspace which is parametrized by a two-variable function, and which contains the bound state eigenspace. The investigation of this subspace shows that, at first perturbation orders, it carries a representation of the Poincaré group. This perturbative result suggests that our QFT models contain some 'two-particle-like' representations, the bound states (if there are any) appearing as irreducible sub-representations. All these representations have the particularity that the time variable is not used (due to the restriction to the 'zero-time subspace'), even if a Lorentz transformation is performed. This mathematically advantageous property, which characterizes the representation of the physical observables called the 'Schrödinger picture', may however present difficulties of interpretation in the relativistic context.

It is natural to ask if such two-particle representations really exist, without using a perturbation approach, even leaving QFT. What we are looking for is a set of operators, in the Schrödinger picture, satisfying the commutation rules of the Lie algebra of the Poincaré group. Surprisingly this problem can be solved easily, and we have even found a general class of solutions [2]. Moreover, some of these representations appear to be unitary, strongly continuous and non-trivial [3]. In fact we have obtained the simplest quantum relativistic theory for two interacting particles.

In this paper we generalize this last results to the case of $N \geq 2$ (arbitrary large) number of massive, spinless, not necessary identical particles, moving in a two-dimensional space-time. Here QFT is no more involved. Three self-adjoint operators are constructed, P (momentum), H (Hamiltonian) and L (generator of the Lorentz transformations), which act on the same Hilbert space than the representation describing N free bosons in the Schrödinger picture. Moreover these operators satisfy the commutation rules of the Poincaré algebra on a dense subspace (which is a common core for the three operators). The eigenvalue equation for the square of the mass operator $M^2 = H^2 - P^2$ can be written down and leads to a simple N-body eigenvalue equation. The possible existence of eigenvalues assures that these models are non trivial.

We do not establish here the stronger mathematical statement that P, H, L are the generators of a unitary and strongly continuous representation of the Poincaré group. Such a result could be obtained by adaptating the methods of [3] (and by restricting the class of admissible P, H and L).

This paper is organized as follows. The operators P, H, L are given in Section 2 and are shown to be self-adjoint (Proposition 1). They satisfy the commutation rules of the Poincaré algebra (Proposition 2) on a dense subspace (which is a common core) provided the so-called 'interaction kernel' satisfies a 'fundamental equation'. In Section 3 we find a large class of solutions of this equation (Proposition 3). Then in Section 4 the associated eigenvalue equation for the bound states is given and is compared with the Schrödinger equation. It is shown in Proposition 4 that eigenvalues may occur, which proves that these models are non trivial.

2 The Operators P, H, L and the Fundamental Equation

The relativistic or Poincaré group of two space-time dimensions $\mathcal{P}_+^\uparrow$ is generated by the following action of $\mathbb{R}^3 \ni (\xi, \tau, \gamma)$ on $(x, t) \in \mathbb{R}^2$:

$$(\xi, \tau, \gamma) \cdot \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} \xi \\ \tau \end{pmatrix} .$$

In other words $\mathcal{P}_+^\uparrow$ is $(\mathbb{R}^3, \cdot)$ with the group law

$$(\xi, \tau, \gamma) \cdot (\xi', \tau', \gamma') = (\xi' \cosh\gamma + \tau' \sinh\gamma + \xi,\ \xi' \sinh\gamma + \tau' \cosh\gamma + \tau,\ \gamma' + \gamma)$$

for all $(\xi, \tau, \gamma), (\xi', \tau', \gamma') \in \mathbb{R}^3$. Note that $\mathcal{P}_+^\uparrow$ is connected (reflexions are not considered here).

The first representation which interest us is given by the following action on the one-variable functions f

$$(\xi, \tau, \gamma) \cdot f(p) = e^{i\xi p}\, e^{i\tau\omega(p)}\, f(p\cosh\gamma + \omega(p)\sinh\gamma)$$

where $\omega(p) = \sqrt{p^2 + m^2}$ and $m > 0$ is a parameter. It consists in a unitary, strongly continuous and irreducible representation of $\mathcal{P}_+^\uparrow$ in the function space $L^2(\mathbb{R}, d\sigma)$, the so-called 'invariant-measure' σ being given by $d\sigma(p) = dp/2\omega(p)$ (for a proof, see for instance [3]).

Let us denote this representation by Π_1. We introduce the generators of the one-parameter groups

$$P \text{ (momentum) defined by } Pf = -i\partial_\xi(\xi,0,0)\cdot f|_{\xi=0}$$

$$H \text{ (Hamiltonian) defined by } Hf = -i\partial_\tau(0,\tau,0)\cdot f|_{\tau=0}$$

$$L \text{ (Lorentz generator) defined by } Lf = -i\partial_\gamma(0,0,\gamma)\cdot f|_{\gamma=0}$$

and an elementar calculation gives their action

$$\left.\begin{aligned} Pf(p) &= p\, f(p) \\ Hf(p) &= \omega(p)f(p) \\ Lf(p) &= -i\omega(p)f'(p) \end{aligned}\right\} \Pi_1$$

for suitable f (these operators are unbounded). They satisfy the commutation rules of the Poincaré algebra

$$[P,H]=0$$
$$[P,L]=iH$$
$$[H,L]=iP.$$

From these rules follows that the square of the mass operator $M^2 = H^2 - P^2$ commutes with all generators. The representation Π_1 is characterized by the fact that M^2 is just the identity times m^2. So Π_1 is called the 'one-particle representation of mass m'.

Let Π_N be the tensor product of N copies of Π_1. It describes a world of N particles of mass m without interaction. If the particles are identical, the symmetrical tensor product must be taken. To treat all cases (symmetrical or not) we neglect henceforth to mention this question (we could also consider the case of particles of different masses, which would lengthen all formulas without real interest). The action of the generators in the function space $L^2(\mathbb{R}^N, d\sigma_N)$, where the invariant-measure is $\sigma_N = \sigma\otimes\cdots\otimes\sigma$ (N times), are now given by

$$\left.\begin{aligned} Pf(p_1,\ldots,p_N) &= (p_1+\cdots+p_N)\, f(p_1,\ldots,p_N) \\ H_0 f(p_1,\ldots,p_N) &= (\omega(p_1)+\cdots+\omega(p_N))f(p_1,\ldots,p_N) \\ L_0 f(p_1,\ldots,p_N) &= -i(\omega(p_1)\partial_{p_1}+\cdots+\omega(p_N)\partial_{p_N})\, f(p_1,\ldots,p_N) \end{aligned}\right\} \Pi_N$$

for suitable f (the operators being unbounded). We have put an index 0 at H_0 and at L_0 to distinguish these operators from those defined below. It follows from the properties of the tensor product that Π_N is a unitary and strongly continuous representation of $\mathcal{P}_+^\uparrow$. In consequence, according to the Stone theorem, the operators P, H_0 and L_0 are self-adjoint and they satisfy the commutation rules of the Poincaré algebra.

Let us perform the change of variables $(p_1,\ldots,p_N)\to(P,q_1,\ldots,q_{N-1})$ given by

$$\begin{aligned} P &= p_1 + \cdots + p_N \\ q_j^2 &= \frac{1}{4}\left[(p_1-p_{j+1})^2 - (\omega(p_1)-\omega(p_{j+1}))^2\right] \\ \text{sign of } q_j &= \text{sign of } p_1 - p_{j+1} \end{aligned}$$

for all $1 \leq j \leq N-1$ (Appendix A gives all the technical formulas we need about it). In these variables the invariant-measure is written $d\mu$:

$$d\sigma_N(p_1,\ldots,p_N) = d\mu(P,q_1,\ldots,q_{N-1}) = 2^{N-2}\,\frac{dP\;d\sigma_{N-1}(q_1,\ldots,q_{N-1})}{\Omega(P,q_1,\ldots,q_{N-1})}$$

where we have put

$$\Omega(P,q_1,\ldots,q_{N-1}) = \sqrt{N^2m^2 + P^2 + 4\sum_{j=1}^{N-1} q_j^2 + \frac{4}{m^2}\sum_{i<j,=1}^{N-1}\left(q_i\omega(q_j) - q_j\omega(q_i)\right)^2}$$

(for $N=2$ the last sum must be removed). Note that $\Omega(P,q_1,\ldots,q_{N-1}) = \omega(p_1)+\cdots+\omega(p_N)$ (see Appendix A). In these variables the action of the generators becomes

$$\left.\begin{aligned} Pf(P,q_1,\ldots,q_{N-1}) &= P\,f(P,q_1,\ldots,q_{N-1}) \\ H_0 f(P,q_1,\ldots,q_{N-1}) &= \Omega(P,q_1,\ldots,q_{N-1})\,f(P,q_1,\ldots,q_{N-1}) \\ L_0 f(P,q_1,\ldots,q_{N-1}) &= -i\Omega(P,q_1,\ldots,q_{N-1})\,\partial_P\,f(P,q_1,\ldots,q_{N-1}) \end{aligned}\right\}\ \Pi_N$$

for suitable f (the operators being unbounded).

The aim now is to modify the representation Π_N by changing slightly the operators H_0 and L_0, but not P, without leaving $L^2(\mathbb{R}^N, d\mu)$ and without breaking the commutation rules. As in [2] we introduce the *interaction operator* $\mathcal{O}$ given by

$$\mathcal{O}f(P,\vec{q}) = \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')}\, f(P,\vec{q}')\,\frac{h(P,\vec{q},\vec{q}')}{\Omega(P,\vec{q}) + \Omega(P,\vec{q}')}$$

for all P and $\vec{q} = (q_1,\ldots,q_{N-1})$, where h is a kernel satisfying the symmetry condition

$$h(P,\vec{q},\vec{q}') = h(P,\vec{q}',\vec{q})^*$$

and which will be precised later (here * means complex conjugaison). The *interaction representation*, denoted by Π_N^h, on the same Hilbert space $L^2(\mathbb{R}^N, d\mu)$, is defined by

$$\left.\begin{aligned} &P \text{ as in } \Pi_N \\ &H = H_0 + \{H_0,\mathcal{O}\} \\ &L = L_0 + \{L_0,\mathcal{O}\} \end{aligned}\right\}\ \Pi_N^h$$

where we have used the notation $\{A,B\} = AB + BA$. We have to show that these operators make sense, by imposing appropriate conditions on h. We need some definitions. We denote by Dh the function

$$Dh(P,\vec{q},\vec{q}') = \frac{\Omega(P,\vec{q})\;\Omega(P,\vec{q}')}{\Omega(P,\vec{q}) + \Omega(P,\vec{q}')}\;\partial_P\,h(P,\vec{q},\vec{q}')$$

and let $\mathcal{B}$ be the Banach space made of the bounded and continuous functions $h(P,\vec{q},\vec{q}')$ on $\mathbb{R}^{2N-1}$ for which $Dh(P,\vec{q},\vec{q}')$ exist and are also bounded and continuous on $\mathbb{R}^{2N-1}$, equipped with the norm $|h| = \|h\|_\infty + \|Dh\|_\infty$. The mathematical sense of our operators is assured by the following result.

Proposition 1. *There exists $K_1 \in (0,\infty)$, depending only on $m > 0$ and $N \geq 2$, such that, for all $h \in \mathcal{B}$ with $\|h\|_\infty < K_1$, the operators P, H, L are self-adjoint.*

Proof. It is clear for P. From the symmetry of h follows that $\mathcal{O}$ is a symmetric operator. Now if A, B are symmetric operators, so is $\{A, B\}$. Thus H and L are symmetric. To see that H is self-adjoint we note that

$$\{H_0, \mathcal{O}\}f(P,\vec{q}) = \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')}\, f(P,\vec{q}')\, h(P,\vec{q},\vec{q}')$$

defines a bounded operator because

$$\begin{aligned}
\|\{H_0,\mathcal{O}\}f\|^2 &= \int d\mu(P,\vec{q}) \left| \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')}\, f(P,\vec{q}')\, h(P,\vec{q},\vec{q}') \right|^2 \\
&\leq \|h\|_\infty^2 \int d\mu(P,\vec{q}) \left| \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')}\, \frac{f(P,\vec{q}')}{\Omega(P,\vec{q}')} \right|^2 \\
&\leq \|h\|_\infty^2\, k \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q})}\, d\mu(P,\vec{q}')\, |f(P,\vec{q}')|^2 \\
&\leq (\|h\|_\infty\, k\, \|f\|)^2
\end{aligned}$$

where we have used successively the boundedness of the function $|h(\cdot)|$, the Cauchy-Schwarz inequality and the Fubini theorem, and put

$$\begin{aligned}
k &= \sup_{P\in\mathbb{R}} \int \frac{d\sigma_{N-1}(\vec{q})}{\Omega(P,\vec{q})} = \int \frac{d\sigma_{N-1}(\vec{q})}{\Omega(0,\vec{q})} \\
&\leq \int \left(\prod_{j=1}^{N-1} \frac{dq_j}{2\omega(q_j)} \right) \frac{1}{2\sqrt{\|\vec{q}\|^2 + m^2}} < \frac{1}{2^N m} \left(\int \frac{dq}{(1+q^2)^{\frac{N}{2(N-1)}}} \right)^{N-1}
\end{aligned}$$

which gives a well defined constant. We have used the inequality $\Omega(P,\vec{q}) \geq \Omega(0,\vec{q}) \geq 2\sqrt{\|\vec{q}\|^2 + m^2}$ which is easily established. (A similar calculation shows that $\mathcal{O}$ is a bounded operator). So $H = H_0 + \{H_0, \mathcal{O}\}$ is self-adjoint on the domain of self-adjointness of H_0. To study the domain of L we write the new term (after some simple algebra) as follows

$$\begin{aligned}
\{L_0,\mathcal{O}\}f(P,\vec{q}) &= \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')}\, (L_0 f)(P,\vec{q}')\, \frac{h(P,\vec{q},\vec{q}')}{\Omega(P,\vec{q}')} \\
&- i \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')}\, f(P,\vec{q}') \left[\frac{Dh(P,\vec{q},\vec{q}')}{\Omega(P,\vec{q}')} - \frac{P\, h(P,\vec{q},\vec{q}')}{\Omega(P,\vec{q}')^2} \right].
\end{aligned}$$

The second term defines a bounded operator (by using the same technics as for $\{H_0, \mathcal{O}\}$, because the factor in brackets is bounded). For the first one we observe that the square of his norm can be written as

$$\int d\mu(P,\vec{q}) \left| \int \frac{d\sigma_{N-1}(\vec{q}\,')}{\Omega(P,\vec{q}\,')} \, (L_0 f)(P,\vec{q}\,') \, \frac{h(P,\vec{q},\vec{q}\,')}{\Omega(P,\vec{q}\,')} \right|^2$$

$$\leq \left(k \left\| \frac{h}{\Omega} \right\|_\infty \|L_0 f\| \right)^2 \leq \left(\frac{k}{Nm} \, \|h\|_\infty \, \|L_0 f\| \right)^2$$

by taking the result for $\|\{H_0, \mathcal{O}\}f\|^2$ (and by using $\Omega \geq Nm$). From the Kato-Rellich Theorem ([4] Section X.2) follows that $L = L_0 + \{L_0, \mathcal{O}\}$ is self-adjoint on the domain of L_0 provided $\|h\|_\infty < K_1 = Nm/k$. $\Delta\Delta$

It remains to check the commutation rules. We will see that the first two rules are automatically satisfied (for all h). But the last one imposes the following equation for h, that we call our fundamental equation

$$0 = Dh(P,\vec{q},\vec{q}\,') \; + \; \frac{1}{2} \int \frac{d\sigma_{N-1}(\vec{q}\,'')}{\Omega(P,\vec{q}\,'')^2} \left\{ -\frac{P}{\Omega(P,\vec{q}\,'')} h(P,\vec{q},\vec{q}\,'')h(P,\vec{q}\,'',\vec{q}\,') \right.$$
$$\left. + \; Dh(P,\vec{q},\vec{q}\,'')h(P,\vec{q}\,'',\vec{q}\,') \; + \; h(P,\vec{q},\vec{q}\,'')Dh(P,\vec{q}\,'',\vec{q}\,') \right\}$$

for all $(P,\vec{q},\vec{q}\,') \in I\!R^{2N-1}$. This is this equation which guarantees the relativistic structure of the theory, as announced by the following result.

Proposition 2. *Let $h \in \mathcal{B}$ with $\|h\|_\infty < K_1$ satisfying the fundamental equation. Let us suppose moreover that*

$$\big(\Omega(0,\vec{q}) + \Omega(0,\vec{q}\,')\big) \; \big(|h(P,\vec{q},\vec{q}\,')| + |Dh(P,\vec{q},\vec{q}\,')|\big)$$

is bounded for all $(P,\vec{q},\vec{q}\,') \in I\!R^{2N-1}$. Then there exists a dense domain in $L^2(I\!R^N, d\mu)$ which is a common core for P, H, L and on which these operators satisfy the commutation rules of the Poincaré algebra.

Proof. We have already mentioned that P, H_0, L_0 are the generators of a unitary and strongly continuous representation of the Poincaré group in $L^2(I\!R^N, d\mu)$. Then it follows from Theorem 3 of [5] that there exists a dense invariant domain for the three generators which is also a common core for them. Because the largest invariant domain is

$$\mathcal{D} \; = \; \left\{ f \;\middle|\; P^\ell \, \Omega(P,\vec{q})^n \, \partial_P^m \, f(P,\vec{q}) \in L^2(I\!R^N, d\mu) \quad \text{for all} \;\; \ell, m, n \in I\!N \right\}$$

it must be a common core for P, H_0, L_0 and then, by the Kato-Rellich Theorem ([4] Section X.2), a common core for P, H, L.

Let us show that the products of operators PH, HP, PL, LP, HL, LH are well defined on $\mathcal{D}$. Because $\mathcal{D}$ is invariant under P, H_0, L_0 and because $\{H_0, \mathcal{O}\}$ is bounded we have only to verify this statement for $P\{H_0,\mathcal{O}\}$, $P\{L_0,\mathcal{O}\}$, $\{L_0,\mathcal{O}\}P$, $H_0\{L_0,\mathcal{O}\}$, $L_0\{H_0,\mathcal{O}\}$, $\{L_0,\mathcal{O}\}H_0$, $\{L_0,\mathcal{O}\}\{H_0,\mathcal{O}\}$. From the calculation of the proof of Proposition 1 follows that

$$\int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')} (XYf)(P,\vec{q}') \left(|h(P,\vec{q},\vec{q}')| + |Dh(P,\vec{q},\vec{q}')|\right)$$

where X and Y are any choice of P, H_0, L_0, defines a vector of $L^2(I\!R^N, d\mu)$ provided $f \in \mathcal{D}$. This argument suffices for $P\{H_0,\mathcal{O}\}$, $P\{L_0,\mathcal{O}\}$, $\{L_0,\mathcal{O}\}P$ and $\{L_0,\mathcal{O}\}H_0$. Now $H_0\{L_0,\mathcal{O}\}$ introduces a factor $\Omega(P,\vec{q})$ which needs, to be treated as before, that $\Omega(P,\vec{q})|h(P,\vec{q},\vec{q}')|$ and $\Omega(P,\vec{q})|Dh(P,\vec{q},\vec{q}')|$ are bounded, which indeed holds, imposed by our hypothesis. Now

$$L_0\{H_0,\mathcal{O}\}f \;=\; \int \frac{d\sigma'}{\Omega'}(L_0 f)'\frac{\Omega h}{\Omega'} \;-i\; \int \frac{d\sigma'}{\Omega'} f' \left[\Omega\partial_P h - \frac{\Omega P h}{\Omega'^2}\right]$$

(in symbolic obvious notation) is well defined because Ωh is bounded (note that $\Omega\partial_P h$ is bounded because $\Omega D h$ is bounded). For the last term we have only to consider the unbounded part of $\{L_0,\mathcal{O}\}$ (see the proof of Proposition 1), that we denote by $\{L_0,\mathcal{O}\}_{u.b.}$, which gives the contribution

$$\begin{aligned}\{L_0,\mathcal{O}\}_{u.b.}\{H_0,\mathcal{O}\}f \;&=\; \int \frac{d\sigma'}{\Omega'}(L_0 f)' \left[\int \frac{d\sigma''}{\Omega''}\frac{h^{q,q''}h^{q'',q'}}{\Omega'}\right] \\ &\quad - i\int \frac{d\sigma'}{\Omega'} f \left[\int \frac{d\sigma''}{\Omega''} h^{q,q''}\left(\partial_P h^{q'',q'} - \frac{P h^{q'',q'}}{\Omega''^2}\right)\right]\end{aligned}$$

(in symbolic notation again) which is well defined because the factors in brackets [...] are bounded.

Now the commutation rules can be checked on $\mathcal{D}$ without care of validity domain. By using $[\mathcal{O}, P] = 0$ (by construction) we get $[P,H] = [P,H_0] + \{[P,H_0],\mathcal{O}\}$, which is 0 because $[P,H_0] = 0$. By the same reason $[P,L] = [P,L_0] + \{[P,L_0],\mathcal{O}\}$, which, by using $[P,L_0] = iH_0$ becomes $[P,L] = iH_0 + i\{H_0,\mathcal{O}\} = iH$. Thus the two first commutation rules are satisfied, whatever h is. The third one $[H,L] = iP$ holds if and only if

$$[\{H_0,\mathcal{O}\},L_0] \;+\; [H_0,\{L_0,\mathcal{O}\}] \;+\; [\{H_0,\mathcal{O}\},\{L_0,\mathcal{O}\}] \;=\; 0.$$

The linear part $A = [\{H_0,\mathcal{O}\},L_0] + [H_0,\{L_0,\mathcal{O}\}]$, applied to a function $f \in \mathcal{D}$, gives simply

$$Af(P,\vec{q}) \;=\; 2i\int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')} f(P,\vec{q}')\; Dh(P,\vec{q},\vec{q}')$$

(see the calculation in Appendix B). Note that A is a bounded operator. The bilinear part $B = [\{H_0,\mathcal{O}\},\{L_0,\mathcal{O}\}]$ leads to

$$\begin{aligned}Bf(P,q) \;&=\; i\int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P,\vec{q}')} f(P,\vec{q}') \int \frac{d\sigma_{N-1}(\vec{q}'')}{\Omega(P,\vec{q}'')^2} \left\{-\frac{P}{\Omega(P,\vec{q}'')}h(P,\vec{q},\vec{q}'')h(P,\vec{q}'',\vec{q}') \right. \\ &\quad \left. +\; Dh(P,\vec{q},\vec{q}'')h(P,\vec{q}'',\vec{q}) \;+\; h(P,\vec{q},\vec{q}'')Dh(P,\vec{q}'',\vec{q})\right\}\end{aligned}$$

(see Appendix B). Note that B is also bounded. The condition $A+B=0$, which must hold on all $L^2(\mathbb{R}^N, d\mu)$, leads to the fundamental equation. $\Delta\Delta$

Remark. Proposition 2 does not establish that P, H, L are the generators of a unitary and strongly continuous representation of the Poincaré group, which would require the existence of a common invariant domain for these operators (see the counter-examples of Section VIII.5 of [4] and of [5]). Such a result has been obtained in [3] for the case $N=2$, but with stronger assumptions on h. However the methods used in [3] could easily be applied to more general cases.

3 Existence of Solutions of the Fundamental Equation

We come now to the cruxial point of the paper, the solution of the fundamental equation in the Banach space $\mathcal{B}$ (introduced just before Proposition 1).

Proposition 3. *There exists $K_2 \in (0,\infty)$ such that, for all $c \in C^0(\mathbb{R}^{2N-2})$ with $\|c\|_\infty < K_2$, there exists one and only one solution $h \in \mathcal{B}$ of the fundamental equation which satisfies $h(0,\vec{q},\vec{q'}) = c(\vec{q},\vec{q'})$ for all $(\vec{q},\vec{q'}) \in \mathbb{R}^{2N-2}$ and $|h| < 2K_2$.*

Moreover, there exists $K_3 \in (0,\infty)$ such that, if c satisfies also $\Omega(0,\vec{q})\Omega(0,\vec{q'})|c(\vec{q},\vec{q'})| < K_3$ for all $(\vec{q},\vec{q'}) \in \mathbb{R}^{2N-2}$, then the solution h satisfies all the hypothesis of Proposition 2.

Proof. Let us introduce the bilinear operator F given by

$$\begin{aligned} &F(g,h)(P,\vec{q},\vec{q'}) = \\ &\frac{1}{2}\int_0^P d\xi \left(\frac{1}{\Omega(\xi,\vec{q})} + \frac{1}{\Omega(\xi,\vec{q'})}\right) \int \frac{d\sigma_{N-1}(\vec{q''})}{\Omega(\xi,\vec{q''})^2} \Big\{ - Dg(\xi,\vec{q},\vec{q''})h(\xi,\vec{q''},\vec{q'}) \\ &\qquad - g(\xi,\vec{q},\vec{q''})Dh(\xi,\vec{q''},\vec{q'}) + \frac{\xi}{\Omega(\xi,\vec{q''})} g(\xi,\vec{q},\vec{q''})h(\xi,\vec{q''},\vec{q'}) \Big\}. \end{aligned}$$

By integration the fundamental equation becomes

$$h(P,\vec{q},\vec{q'}) = c(\vec{q},\vec{q'}) + F(h,h)(P,\vec{q},\vec{q'})$$

where $c(\vec{q},\vec{q'})$ is an arbitrary function (the integration 'constant'). Note that we have automatically $h(0,\vec{q},\vec{q'}) = c(\vec{q},\vec{q'})$. We will obtain the solution h of this equation, for all suitable c, by applying the Banach fixed-point Theorem [6, Sec 1.1]. The crucial remark is that F satisfies the two properties

1) $(P,\vec{q},\vec{q'}) \mapsto F(g,h)(P,\vec{q},\vec{q'})$ is bounded and continuous on $\mathbb{R}^{2N-1}$ provided g, Dg, h,

Dh are also bounded and continuous on $\mathbb{R}^{2N-1}$, and we obtain the estimation

$$\|F(g,h)\|_\infty \;\leq\; k_1\, (\|g\|_\infty + \|Dg\|_\infty)\, (\|h\|_\infty + \|Dh\|_\infty)$$

where k_1 is the well defined constant given by

$$k_1 \;=\; \sup_{(P,\vec{q},\vec{q'})\in\mathbb{R}^{2N-1}} \;\frac{1}{2}\int_0^P d\xi\, \left(\frac{1}{\Omega(\xi,\vec{q})}+\frac{1}{\Omega(\xi,\vec{q'})}\right)\int \frac{d\sigma_{N-1}(\vec{q''})}{\Omega(\xi,\vec{q''})^2}.$$

2) $DF(g,h)$ is easily calculated and leads to

$$\begin{aligned} DF(g,h)(P,\vec{q},\vec{q'}) \;&=\; \frac{1}{2}\int \frac{d\sigma_{N-1}(\vec{q''})}{\Omega(P,\vec{q''})^2}\Big\{-\frac{P}{\Omega(P,\vec{q''})} g(P,\vec{q},\vec{q''})h(P,\vec{q''},\vec{q'}) \\ &\quad +\; Dg(P,\vec{q},\vec{q''})h(P,\vec{q''},\vec{q'}) \;+\; g(P,\vec{q},\vec{q''})Dh(P,\vec{q''},\vec{q'})\Big\}. \end{aligned}$$

It is also continuous and bounded on $\mathbb{R}^{2N-1}$ provided g, Dg, h, Dh are so too, and we get now the estimation

$$\|DF(g,h)\|_\infty \;\leq\; k_2\, (\|g\|_\infty + \|Dg\|_\infty)\, (\|h\|_\infty + \|Dh\|_\infty)$$

where $k_2 = \sup_{P\in\mathbb{R}} \int d\sigma_{N-1}(\vec{q})\Omega(P,\vec{q})^{-2} < \infty$.

We resume these inequalities by saying that F is a bilinear operator on $\mathcal{B}$ satisfying

$$|F(g,h)| \;\leq\; k_3\, |g|\, |h|$$

where $k_3 = k_1 + k_2$ is a well defined constant.

The fixed-points of the non-linear operator $A(h) = c + F(h,h)$ are the solutions of the fundamental equation. Let us check the hypothesis of the Banach Theorem. For that we need a closed ball $B_\lambda \subset \mathcal{B}$ of radius $\lambda > 0$, which will be delimited later.

First hypothesis: $A \;:\; B_\lambda \longrightarrow B_\lambda$. This imposes $c \in B_\lambda$ and leads to the following condition

$$\begin{aligned} |A(h)| \;=\; |c+F(h,h)| \;\leq\; |c| + |F(h,h)| \;&\leq\; |c| + k_3|h|^2 \\ &\leq\; \|c\|_\infty + k_3\lambda^2 \;<\; \lambda. \end{aligned}$$

We can choice for instance $\|c\|_\infty < \lambda/2$ and $k_3\lambda^2 < \lambda/2$, which leads to $\lambda < (2k_3)^{-1}$ and $\|c\|_\infty < (4k_3)^{-1}$.

Second hypothesis: $|A(g) - A(h)| \;\leq k\; |g-h|$ for some $0 < k < 1$, for all $g,h \in B_\lambda$. In our situation we find

$$\begin{aligned} |A(g)-A(h)| \;=\; |F(g,g)-F(h,h)| \;&=\; |F(g,g-h) - F(h-g,h)| \\ &\leq\; k_3\, (|g|+|h|)\, |g-h| \;\leq\; 2k_3\, \lambda\, |g-h|. \end{aligned}$$

Let us put $k = 2k_3\lambda$. The condition $k < 1$ leads to the same bound for λ as before.

In conclusion, for all $0 < \lambda < (2k_3)^{-1}$ and $c \in C^0(\mathbb{R}^{2N-2})$ satisfying $\|c\|_\infty < \lambda/2$, the Banach fixed-point Theorem assures the existence of a unique solution h in B_λ, given by

$$h = \lim_{n\to\infty} A^n(h_0)$$

whatever the initial function $h_0 \in B_\lambda$ is. Thus we can choice $h_0 = 0$ and we get

$$h = \lim_{n\to\infty} A^n(0).$$

(See the first terms of this sequence in the Remark just after the proof.) Note that this limit is not 0 in general, because it is continuous and satisfies $h(0, \vec{q}, \vec{q'}) = c(\vec{q}, \vec{q'})$.

By taking different values of λ (such as $\lambda = 2\|c\|_\infty + \varepsilon$ or $\lambda = (2k_3)^{-1} + \varepsilon$, for arbitrary small ε) the conclusion of the Banach fixed-point Theorem can be reformulated as follows. Let $K_2 = (4k_3)^{-1}$. For all $c \in C^0(\mathbb{R}^{2N-2})$ satisfying $\|c\|_\infty < K_2$, the limit $h = \lim_{n\to\infty} A^n(0)$ converges in $\mathcal{B}$ and satisfies $|h| < 2\|c\|_\infty$; moreover h satisfies $h = A(h)$ and this equation admits no other solution in $\{g \in \mathcal{B} \mid |g| < 2K_2\}$.

It remains to check the hypothesis of Proposition 2 for suitable c. Let

$$K_3 = \min\left\{\frac{K_1 N^2 m^2}{2}\ ,\ K_2 N^2 m^2\right\}.$$

Let $c \in C^0(\mathbb{R}^{2N-2})$ satisfying $\Omega(0,\vec{q})\Omega(0,\vec{q'})|c(\vec{q},\vec{q'})| < K_3$ for all $(\vec{q},\vec{q'}) \in \mathbb{R}^{2N-2}$. Then $\|c\|_\infty < K_3(Nm)^{-2} \leq K_2$, as required by the fixed-point Theorem. Moreover we get $\|h\|_\infty < |h| < 2\|c\|_\infty < 2K_3(Nm)^{-2} \leq K_1$, in agreement with Proposition 2.

On the other hand let $g \in \mathcal{B}$ such that $|g(P,\vec{q},\vec{q'})| + |Dg(P,\vec{q},\vec{q'})| \leq K\Omega(0,\vec{q})^{-1}\Omega(0,\vec{q'})^{-1}$ for some $K \in (0,\infty)$ for all $(P,\vec{q},\vec{q'}) \in \mathbb{R}^{2N-1}$. By using this inequality in the integrals defining F and DF we get the estimation

$$|F(g,g)(P,\vec{q},\vec{q'})| + |DF(g,g)(P,\vec{q},\vec{q'})| \leq k_3 \left(\frac{K}{Nm}\right)^2 \frac{1}{\Omega(0,\vec{q})\ \Omega(0,\vec{q'})}.$$

Let us take a function c as above. Let us suppose that for some $n \in \mathbb{N}^*$ we know that $|A(0)^n(P,\vec{q},\vec{q'})| + |DA(0)^n(P,\vec{q},\vec{q'})| < C_n\Omega(0,\vec{q})^{-1}\Omega(0,\vec{q'})^{-1}$ for some constant $C_n \leq N^2m^2/(2k_3)$ (this is true for $n = 1$). Then

$$\begin{aligned}|A^{n+1}(0)(P,\vec{q},\vec{q'})| &+ |DA^{n+1}(0)(P,\vec{q},\vec{q'})| \\ &= |c(\vec{q},\vec{q'}) + F(A^n(0),A^n(0))(P,\vec{q},\vec{q'})| + |DF(A^n(0),A^n(0))(P,\vec{q},\vec{q'})| \\ &\leq \left(K_3 + k_3\frac{C_n^2}{N^2m^2}\right) \frac{1}{\Omega(0,\vec{q})\ \Omega(0,\vec{q'})} = \frac{C_{n+1}}{\Omega(0,\vec{q})\ \Omega(0,\vec{q'})}\end{aligned}$$

with $C_{n+1} = K_3 + k_3C_n^2/(Nm)^2 \leq N^2m^2/(2k_3)$. Thus for such c all terms of the sequence $\{A^n(0)(P,\vec{q},\vec{q'})\}_{n=1}^{\infty}$ is bounded by $N^2m^2(2k_3)^{-1}\left[\Omega(0,\vec{q})\Omega(0,\vec{q'})\right]^{-1}$, and so are their limit h.

Finally we get

$$\left(\Omega(0,\vec{q})+\Omega(0,\vec{q'})\right)\ |h(P,\vec{q},\vec{q'})|\ \leq\ \frac{N^2m^2}{2k_3}\ \frac{\Omega(0,\vec{q})+\Omega(0,\vec{q'})}{\Omega(0,\vec{q})\ \Omega(0,\vec{q'})}\ \leq\ \frac{Nm}{k_3}$$

which is bounded for all $(P,\vec{q},\vec{q'}) \in \mathbb{R}^{2N-1}$, as required by Proposition 2. $\Delta\Delta$

Remark. The proof gives more than what is stated in Proposition 3. It gives how to compute h from the arbitrary function c. h is given by a sequence which converges in $\mathcal{B}$ and which begins as follows :

$$\begin{aligned} A(0) &= c \\ A^2(0) &= A(c) \ = \ c \ + \ F(c,c) \\ A^3(0) &= A(c+F(c,c)) \\ &= c \ + \ F(c,c) \ + F(F(c,c),c) \ + \ F(c,F(c,c)) \ + \ F(F(c,c),F(c,c)) \end{aligned}$$

etc...

4 The Bound States Equation

The operator $M^2 = H^2 - P^2$ commutes with all generators. Thus any eigenspace of M^2 is a sub-representation of $\mathcal{P}_+^\uparrow$. If a sub-representation is irreducible, it decribes a one-particle world, that is a bound state, the mass of which is given by the square-root of the eigenvalue.

The operator $M_0^2 = H_0^2 - P^2$ is just the multiplication operator by the function

$$M_0(\vec{q})^2 \ = \ N^2m^2 \ + \ 4\sum_{j=1}^{N-1} q_j^2 \ + \ \frac{4}{m^2}\sum_{i<j,=1}^{N-1} \left(q_i\omega(q_j) - q_j\omega(q_i)\right)^2$$

for all $\vec{q} \in \mathbb{R}^{N-1}$ (for $N = 2$ the last sum must be removed). Its spectrum is absolutely continuous and covers the complete intervalle $[N^2m^2, \infty)$.

With our operators H and P, the operator M^2 takes the form

$$M^2 f(P,\vec{q}) \ = \ M_0(\vec{q})^2\ f(P,\vec{q}) \ + \ \int \frac{d\sigma_{N-1}(\vec{q'})}{\Omega(P,\vec{q'})}\ f(P,\vec{q'})\ K(P,\vec{q},\vec{q'})$$

for suitable f, where the kernel K is given by

$$K(P,\vec{q},\vec{q'}) \ = ¿\ \left(\Omega(P,\vec{q})+\Omega(P,\vec{q'})\right)\ h(P,\vec{q},\vec{q'}) \ + \ \int \frac{d\sigma_{N-1}(\vec{q''})}{\Omega(P,\vec{q''})}\ h(P,\vec{q},\vec{q''})\ h(P,\vec{q''},\vec{q'}).$$

Thus the bound states are obtained by solving the eigenvalue equation

$$m_B^2 f(P,\vec{q}) \ = \ M_0(\vec{q})^2\ f(P,\vec{q}) \ + \ \int \frac{d\sigma_{N-1}(\vec{q'})}{\Omega(P,q')}\ f(P,\vec{q'})\ K(P,\vec{q},\vec{q'})$$

where $m_B > 0$ is the bound state mass. Here m_B is expected to be smaller than Nm and f to belong to $D(M_0^2) \subset L^2(\mathbb{R}^N, d\mu)$ where $D(M_0^2)$ is the domain of the multiplication operator M_0^2. This equation appears to be the relativistic counterpart of the Schrödinger equation because it plays the same role : it gives the discrete structure of the set of the bound states. Note that it has the same general form (made of a 'kinetic term' plus a 'potential term') but with three important differences:

1) the 'kinetic term' is the multiplication operator by the function $M_0(\vec{q})^2$ which is not a degree-two polynomial in the momenta as soon as $N \geq 3$. However it remains 'elliptic' in the sense that $M_0(\vec{q})^2 \geq M_0(0)^2$, the equality holding only if $\vec{q} = 0$.

2) the interaction part is *non-local* which means that $k(P, \vec{q}, \vec{q'})$ is not a function of P and $\vec{q} - \vec{q'}$ only. A look at the begining of the sequence for h suffices to convince ourselves that it is not the case, even if we choose as initial function $c(\vec{q} - \vec{q'})$.

3) the interaction part depends on P, the total momentum. In fact the only solution h of the fundamental equation which is independent on P, i.e. which satisfies $Dh = 0$, is $h = 0$ (which requires $c = 0$). Thus this dependence is necessary to get a non-free theory. Note that it is not arbitrary, but imposed by the fundamental equation.

In a relativistic description of a bound state, it is not possible to eliminate the variable P like in the non-relativistic case. A simple argument pleads for it, which is the following: by a change of inertial frame, the bound state must be subject to the Lorentz contraction. The simplification due to the centre of mass separation in Classical Mechanics has to be obtained in the relativistic case by taking the centre of mass frame. By putting $P = 0$ the eigenvalue equation becomes simply

$$\Omega(0, \vec{q})\ f(\vec{q})\ +\ \int \frac{d\sigma_{N-1}(\vec{q'})}{\Omega(0, \vec{q'})}\ f(\vec{q'})\ c(\vec{q}, \vec{q'})\ =\ m_B f(\vec{q})$$

(because $M^2|_{P=0} = H^2$ we have written the eigenvalue equation for H) where f must belong to $D(\Omega(0, \cdot)) \subset L^2\left(\mathbb{R}^{N-1}, d\sigma_{N-1}/\Omega(0, \cdot)\right)$. We recall that c is an arbitrary continuous function on $\mathbb{R}^{2N-2}$ satisfying $\Omega(0, \vec{q})\Omega(0, \vec{q'})|c(\vec{q}, \vec{q'})| < K_3$.

The existence of a solution of this equation, for some c, can be considered as a proof that these models are non-trivial (which means that they describe two particles which effectively interact). So we conclude by showing that an eigenvalue may effectively occur.

Proposition 4. *Let $N = 2$ or 3. Let $c(\vec{q}, \vec{q'}) = -K_3\Omega(0, \vec{q})^{-1}\Omega(0, \vec{q'})^{-1}$ for all $(\vec{q}, \vec{q'}) \in \mathbb{R}^{2N-2}$, where K_3 is the constant introduced in Proposition 3. Then the above eigenvalue equation has a solution.*

Proof. The operator

$$Vf(\vec{q}) = \int \frac{d\sigma_{N-1}(\vec{q'})}{\Omega(0,\vec{q'})} f(\vec{q'})\, c(\vec{q},\vec{q'}) = -\frac{K_3}{\Omega(0,\vec{q})} \int \frac{d\sigma_{N-1}(\vec{q'})}{\Omega(0,\vec{q'})^2} f(\vec{q'})$$

on $L^2\left(I\!R^{N-1}, d\sigma_{N-1}/\Omega(0,\cdot)\right)$ is compact, its range being one-dimensional. Thus the operator $\Omega(0,\cdot)+V$ is self-adjoint (on the domain of $\Omega(0,\cdot)$). In particular its essential spectrum is that one of $\Omega(0,\cdot)$, that is the complete intervalle $[Nm,\infty)$. The discrete spectrum is obtained by solving the eigenvalue equation. Because of the simple range of V it is easily solved and gives the only solution

$$f(\vec{q}) = \frac{K}{\Omega(0,\vec{q})\,(\Omega(0,\vec{q}) - Nm + \mathcal{E})} \qquad \text{and} \qquad m_B = Nm - \mathcal{E}$$

where K is a normalisation constant (note that f belongs to the domain of $\Omega(0,\cdot)$) and where $\mathcal{E} > 0$ is the only solution of the equation

$$1 = K_3 \int \frac{d\sigma_{N-1}(\vec{q})}{\Omega(0,\vec{q})^3}\, \frac{1}{\Omega(0,\vec{q}) - Nm + \mathcal{E}} .$$

(For small $\|\vec{q}\|$ we have $\Omega(0,\vec{q}) - Nm = 2\|\vec{q}\|^2 + O(\|\vec{q}\|^3)$; the existence of an unique solution for $N = 2$ or 3 follows because the function $\lambda \mapsto \int d\sigma_{N-1}(\vec{q})\, [\Omega(0,\vec{q})^3(\Omega(0,\vec{q}) - Nm + \lambda)]^{-1}$ decreases monotonously from ∞ to 0 when λ varies from 0 to ∞.) $\Delta\Delta$

Remark. We have not proved here that the operator M^2 itself (acting on $L^2(I\!R^N, d\mu)$) has an eigenvalue. This question is discussed in [3] (for $N = 2$), where it is shown that the existence of eigenvalues of $M^2|_{P=P_0}$ for fixed P_0 leads to a gap in the spectrum of M^2.

Appendix A. The N Free Particles Model

Let us consider $N \geq 2$ particles of the same mass $m > 0$, of momentum $p_1,\ldots,p_N$ and energy $\omega(p_1),\ldots,\omega(p_N)$ respectively, where $\omega(p_j) = \sqrt{p_j^2 + m^2}$.

The change of variables $(p_1,\ldots,p_N) \to (P, q_1\ldots,q_{N-1})$ introduced in Section 2 is made in two steps. First we perform $(p_1,\ldots,p_N) \to (\alpha,\chi_1,\ldots,\chi_{N-1})$, given by

$$\begin{aligned} p_1 &= m\,\sinh(\alpha + \chi_1 + \chi_2 + \cdots + \chi_{N-1}) \\ p_2 &= m\,\sinh(\alpha - \chi_1 + \chi_2 + \cdots + \chi_{N-1}) \\ \cdots &\quad \cdots \\ p_N &= m\,\sinh(\alpha + \chi_1 + \cdots + \chi_{N-2} - \chi_{N-1}). \end{aligned}$$

We get immediately

$$\omega(p_1) = m\,\cosh(\alpha + \chi_1 + \chi_2 + \cdots + \chi_{N-1})$$

$$\omega(p_2) = m \cosh(\alpha - \chi_1 + \chi_2 + \cdots + \chi_{N-1})$$
$$\cdots \quad \cdots$$
$$\omega(p_N) = m \cosh(\alpha + \chi_1 + \cdots + \chi_{N-2} - \chi_{N-1}).$$

Note that $\partial_\alpha p_i = \omega(p_i)$ and $\partial_{\chi_j} p_i = \omega(p_i)$ for all $1 \le i \le N$ and $1 \le j \le N-1$ except for $i = j+1$ in which case $\partial_{\chi_j} p_{j+1} = -\omega(p_{j+1})$. Thus the Jacobian of the transformation is

$$J = \left| \det \begin{pmatrix} \omega(p_1) & \omega(p_1) & \cdots & \omega(p_1) \\ \omega(p_2) & -\omega(p_2) & \cdots & \omega(p_2) \\ \vdots & \vdots & & \vdots \\ \omega(p_N) & \omega(p_N) & \cdots & -\omega(p_N) \end{pmatrix} \right| = \omega(p_1)\cdots\omega(p_N)\; j_N$$

where

$$j_N = \left| \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & -1 \end{pmatrix} \right| = 2^{N-1}.$$

Thus $J = 2^{N-1}\omega(p_1)\cdots\omega(p_N)$. The invariant measure becomes

$$d\sigma_N(p_1,\ldots,p_N) = \left(\prod_{i=1}^{N} \frac{dp_i}{2\omega(p_i)} \right) = \frac{1}{2}\, d\alpha\, d\chi_1 \cdots d\chi_{N-1}.$$

The variable α is related to the Lorentz transformation because $\partial_\alpha f(p_1,\ldots,p_N) = \sum \omega(p_i)\partial_{p_i} f(p_1,\ldots,p_N)$, thus $L_0 = -i\partial_\alpha$. The variables χ_j, for $1 \le j \le N-1$, are related to the Lorentz-invariants

$$q_j^2 = \frac{1}{4}\left[(p_1 - p_{j+1})^2 - (\omega(p_1) - \omega(p_{j+1}))^2 \right]$$

as follows

$$\begin{aligned} 4\frac{q_j^2}{m^2} &= (\sinh a - \sinh b)^2 - (\cosh a - \cosh b)^2 = -2 + 2(\sinh a \sinh b - \cosh a \cosh b) \\ &= -2 + 2\cosh(a-b) = -2 + 2\cosh(2\chi_j) = 4\sinh^2 \chi_j \end{aligned}$$

where we have put $a = \alpha + \chi_1 + \ldots + \chi_{N-1}$ and $b = \alpha + \chi_1 + \ldots - \chi_j + \ldots + \chi_{N-1}$. Now we perform the second change of variables $(\alpha, \chi_1, \ldots, \chi_{N-1}) \to (P, q_1, \ldots, q_{N-1})$, given by

$$\begin{aligned} P &= p_1 + \ldots + p_N \\ q_i &= m \sinh \chi_i \end{aligned}$$

for $1 \le i \le N-1$. The Jacobian of the transformation is

$$\begin{aligned} J^{-1} &= \left| \det \begin{pmatrix} \omega_1 + \ldots + \omega_N & \omega_1 - \omega_2 + \ldots + \omega_N & \cdots & \omega_1 + \ldots - \omega_N \\ 0 & m\cosh\chi_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m\cosh\chi_{N-1} \end{pmatrix} \right| \\ &= (\omega(p_1) + \ldots + \omega(p_N))\, m^{N-1} \cosh\chi_1 \cdots \cosh\chi_{N-1} \end{aligned}$$

where $\omega_j = \omega(p_j)$ for all $1 \leq j \leq N$, and the invariant measure becomes

$$d\sigma_N(p_1,\ldots,p_N) = \frac{2^{-1}\,dP}{\omega_1+\ldots+\omega_N}\left(\prod_{i=1}^{N-1}\frac{dq_i}{\sqrt{q_i^2+m^2}}\right) = 2^{N-2}\,\frac{dP\,d\sigma_{N-1}(q_1,\ldots,q_{N-1})}{\Omega(P,q_1,\ldots,q_{N-1})}$$

where we have put $\Omega(P,q_1,\ldots,q_{N-1}) = \omega(p_1)+\ldots+\omega(p_N)$. Because $\partial_\alpha f(P,q_1,\ldots,q_{N-1}) = (\partial_\alpha P)\partial_P f(P,q_1,\ldots,q_{N-1})$, the operators of the free theory are now given by

$$\begin{aligned} P &= \text{multiplication by } P \\ H_0 &= \text{multiplication by } \Omega(P,q_1,\ldots,q_{N-1}) \\ L_0 &= -i\Omega(P,q_1,\ldots,q_{N-1})\partial_P \end{aligned}$$

It remains to calculate the function $\Omega(P,q_1,\ldots,q_{N-1})$ or , which is equivalent, the function $M_0(q_1,\ldots,q_{N-1})^2 = \Omega(P,q_1,\ldots,q_{N-1})^2 - P^2$. We consider the Lorentz-invariants

$$q_{i,j}^2 = \frac{1}{4}\left[(p_i-p_j)^2 - (\omega(p_i)-\omega(p_j))^2\right]$$

for $1 \leq i < j \leq N$ (note that $q_{i,i} = 0$ for all i and $q_{1,j+1}^2 = q_j^2$). A elementar calculation gives $M_0^2 = N^2m^2 + 4\sum_{i<j,=1}^N q_{i,j}^2$. Let β_i, for $1 \leq i \leq N$, be the hyperbolic angles such that $p_i = m\sinh\beta_i$. By a calculation already made (for q_i) follows $4q_{i,j}^2/m^2 = -2+2\cosh(\beta_i-\beta_j)$. In the case $N > 2$, for all $2 \leq i < j \leq N$ we have $\beta_i - \beta_j = 2(\chi_{j-1}-\chi_{i-1})$, so that $q_{i,j}^2 = m^2\sinh(\chi_{j-1}-\chi_{i-1})^2 = m^{-2}(q_{j-1}\omega(q_{i-1}) - q_{i-1}\omega(q_{j-1}))^2$. Finally we get

$$\begin{aligned} M_0(q_1,\ldots,q_{N-1})^2 &= N^2m^2 + 4\sum_{i=1}^{N-1} q_i^2 + 4\sum_{i<j,=2}^{N} q_{i,j}^2 \\ &= N^2m^2 + 4\sum_{i=1}^{N-1} q_i^2 + \frac{4}{m^2}\sum_{1<j,=1}^{N-1}(q_j\omega(q_i) - q_i\omega(q_j))^2. \end{aligned}$$

Appendix B. The Fundamental Equation

We establish the fundamental equation. It is the condition on h for which the relation $[\{H_0,\mathcal{O}\},L_0]+[H_0,\{L_0,\mathcal{O}\}]+[\{H_0,\mathcal{O}\},\{L_0,\mathcal{O}\}] = 0$ holds. The linear part in $\mathcal{O}$ is simplified as follows

$$[\{H_0,\mathcal{O}\},L_0] + [H_0,\{L_0,\mathcal{O}\}] = 2(H_0\mathcal{O}L_0 - L_0\mathcal{O}H_0 + iP\mathcal{O})$$

where we have used $[H_0,L_0] = iP$ and the fact that P and $\mathcal{O}$ commute. Applying it to a suitable function f gives, in symbolic obvious notation

$$\begin{aligned} A &= 2(H_0\mathcal{O}L_0 - L_0\mathcal{O}H_0 + iP\mathcal{O})\,f(P,q) \\ &= -2i\Omega\int\frac{d\sigma'}{\Omega'}\,\frac{(\Omega'\partial f)h}{\Omega+\Omega'} + 2i\Omega\partial\int\frac{d\sigma'}{\Omega'}\,\frac{\Omega' fh}{\Omega+\Omega'} + 2iP\int\frac{d\sigma'}{\Omega'}\,\frac{fh}{\Omega+\Omega'} \\ &= 2i\int d\sigma'\,f\left(\Omega\partial\frac{h}{\Omega+\Omega'} + \frac{P}{\Omega'}\,\frac{h}{\Omega+\Omega'}\right) \\ &= 2i\int d\sigma_{N-1}(\vec{q}')\,f(P,\vec{q}')\,\frac{\Omega(P,\vec{q})\,\partial h(P,\vec{q},\vec{q}')}{\Omega(P,\vec{q})+\Omega(P,\vec{q}')} \end{aligned}$$

To compute the bilinear part, we take the expressions for $\{H_0, \mathcal{O}\} f(P, \vec{q})$ and $\{L_0, \mathcal{O}\} f(P, \vec{q})$ given in the proof of Proposition 1, for $f \in \mathcal{D}$, and we get, in symbolic notation

$$
\begin{aligned}
B &= [\{H_0, \mathcal{O}\}, \{L_0, \mathcal{O}\}]\ f(P, q) \\
&= \int \frac{d\sigma''}{\Omega''} h^{q,q''}(-i) \int \frac{d\sigma'}{\Omega'} \left[(\partial f') h^{q'',q'} + f' \left(\frac{D h^{q'',q'}}{\Omega'} - \frac{P h^{q'',q'}}{\Omega'^2} \right) \right] \\
&\quad + i \int \frac{d\sigma''}{\Omega''} h^{q,q''} \partial \int \frac{d\sigma'}{\Omega'} f' h^{q'',q'} \ + \int \frac{d\sigma''}{\Omega''} \left[\frac{D h^{q,q''}}{\Omega''} - \frac{P h^{q,q''}}{\Omega''^2} \right] \int \frac{d\sigma'}{\Omega'} f' h^{q'',q'} \\
&= i \int \frac{d\sigma''}{\Omega''} \int \frac{d\sigma'}{\Omega'} f' \left[h^{q,q''} \left(-\frac{D h^{q'',q'}}{\Omega'} + \frac{P h^{q'',q'}}{\Omega'^2} + \Omega'' \partial \frac{h^{q'',q'}}{\Omega'} \right) \right. \\
&\qquad \left. + \left(\frac{D h^{q,q''}}{\Omega''} - \frac{P h^{q,q''}}{\Omega''^2} \right) h^{q'',q'} \right] \\
&= i \int \frac{d\sigma_{N-1}(\vec{q}')}{\Omega(P, \vec{q}')} f(P, \vec{q}')) \int \frac{d\sigma_{N-1}(\vec{q}'')}{\Omega(P, \vec{q}'')^2} \left\{ -\frac{P}{\Omega(P, \vec{q}'')} h(P, \vec{q}, \vec{q}'') h(P, \vec{q}'', \vec{q}') \right. \\
&\qquad \left. + D h(P, \vec{q}, \vec{q}'') h(P, \vec{q}'', \vec{q}) \ + \ h(P, \vec{q}, \vec{q}'') D h(P, \vec{q}'', \vec{q}) \right\}
\end{aligned}
$$

The condition $A + B = 0$, which must hold for all functions f, leads to the fundamental equation given at the end of Section 2.

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