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PARASUPERSYMMETRIC QUANTUM MECHANICS WITH AN ARBITRARY NUMBER OF PARASUPERCHARGES AND ORTHOGONAL LIE ALGEBRAS

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Abstract

The connection between parasupersymmetric quantum mechanics (generated by an arbitrary number of parasupercharges and characterized by an arbitrary order of paraquantization) and orthogonal Lie algebras is enhanced. In particular, the irreducible representations of such Lie algebras are exploited to specify the Hamiltonian.

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I. INTRODUCTION

The superposition of bosons and p = 2-parafermions[1], currently referred to as parasupersymmetric quantum mechanics, has essentially been developed through two non-equivalent approaches following the Rubakov-Spiridonov[2] or Beckers-Debergh[3] papers. Physically speaking, these approaches correspond to the so-called Ξ and Λ (or V)-types of three-level systems[4], respectively. Algebraically speaking, they lead to generalized Z_2-structures - the parasuperalgebras[5] - implied in particular by the Hamiltonian and the corresponding parasupercharges. More precisely, if we limit ourselves to the Beckers-Debergh developments (the only ones enhancing a spin-orbit term inside the Hamiltonian and, in a parallel way, having a meaning within a relativistic analysis[6]), we are dealing with two parasupercharges $Q_\pm$ satisfying the relations[3]

$$[H, Q_\pm] = 0, \tag{1.1a}$$

$$\left[Q_\pm, [Q_\mp, Q_\pm]\right] = 2\, Q_\pm H, \tag{1.1b}$$

$$(Q_\pm)^{p+1} = 0, \tag{1.1c}$$

where H is the Hamiltonian and p the so-called order of paraquantization[1]. Through the introduction of the following Hermitian operators

$$Q_1 = \frac{1}{2}(Q_+ + Q_-), \quad Q_2 = -\frac{i}{2}(Q_+ - Q_-), \tag{1.2}$$

the above relations (1.1) can be rewritten as

$$[H, Q_a] = 0, \tag{1.3a}$$

$$\left[Q_a, [Q_b, Q_c]\right] = \delta_{ab} Q_c H - \delta_{ac} Q_b H, \tag{1.3b}$$

$$(Q_1 \pm i\, Q_2)^{p+1} = 0, \tag{1.3c}$$

where $a, b, c = 1, 2$. As already noticed[7], we can restrict ourselves to subspaces of the original Hilbert space corresponding to specific (positive) eigenvalues E of the Hamiltonian and define

$$L_1 = \frac{Q_1}{\sqrt{E}} \ , \quad L_2 = \frac{Q_2}{\sqrt{E}} \ , \quad L_3 = -\frac{i\,[Q_1 \, , Q_2]}{E} \ , \tag{1.4}$$

in order to reduce the trilinear relations (1.3b) to bilinear ones generating an so (3)-algebra

$$[\,L_j \, , L_k\,] = i\,\varepsilon_{jkm}\,L_m \, . \tag{1.5}$$

The relation (1.3c) is nothing but the characterization of the subtended representation of so (3) , i.e. $D^{(p/2)}$. In particular, for $p = 1$, we recover the operators of the usual supersymmetric quantum mechanics[8], an expected result in connection with the $p = 1$-parafermions coinciding with ordinary fermions associated with Pauli matrices.

These developments are evidently typical of the consideration of <u>two</u> parasupercharges only. The question we want to address in this paper is the impact of an <u>arbitrary</u> number N of parasupercharges, by maintaining the order of paraquantization as an arbitrary integer. In Section 2, we consider a few examples, i.e. $N = 2$ (§2.A), $N = 3$ (§2.B) and $N = 4$ (§2.C) in connection with their respective interests in physics[9,10]. The generalization to arbitrary even (§3.A) and odd (§3.B) N is studied in Section 3 and we conclude with some comments in Section 4.

2. THE EXPLICIT EXAMPLES N = 2 , 3 AND 4

Let us present some explicit examples in order to show how the ideas can be developed.

2.A. The case N = 2

Some results connected to this context have already been recalled in the Introduction. We want here to complete the information by exploiting the link between the so-called $N = 2$-parasupersymmetric quantum mechanics and the Lie algebra so (3) . In other words, we propose to define

$$Q_a = \frac{1}{\sqrt{2}} S_a \, (p + i\eta \quad W(x)), \quad a = 1, 2, \tag{2.1}$$

where S_a are the ladder matrices of $so(3)$, f.i.

$$S_1 = \frac{1}{2} \sum_{j=1}^{p} \sqrt{jp - j(j-1)} \, (e_{j,j-1} + e_{j-1,j}), \tag{2.2a}$$

$$S_2 = \frac{i}{2} \sum_{j=1}^{p} \sqrt{jp - j(j-1)} \, (e_{j,j-1} - e_{j-1,j}), \tag{2.2b}$$

while p is the usual momentum and W is the superpotential. In order to be complete, let us mention that the notations $e_{j,k}$ refer to matrices (of dimension $p + 1$) containing zeros everywhere except units at the intersection of the j^{th} row and the k^{th} column. Moreover, we ask for the matrix η to satisfy

$$\{\eta, S_a\} = 0, \quad a = 1, 2 \tag{2.3a}$$

and

$$\eta^2 = 1. \tag{2.3b}$$

In particular, with the realization (2.2), η is given by

$$\eta = \mathrm{diag}\,(1, -1, 1, \cdots, (-1)^p). \tag{2.4}$$

It plays a prominent role in connection with the Hamiltonian. Indeed, by using the relations (1.3b), we can convince ourselves that

$$H = \frac{1}{2} p^2 + \frac{1}{2} W^2 + \frac{1}{2} \eta W' \tag{2.5}$$

where the prime refers to the derivative with respect to x. Of course, with this expression (2.5) together with (2.4), it is easy to determine the spectrum of H when the oscillatorlike interaction $W = \omega x$ is under study. We obtain

$$E_n = \omega(n+1) \qquad \text{for } \eta = 1 \tag{2.6a}$$

and

$$E_n = \omega n \qquad \text{for } \eta = -1, \quad n = 0, 1, 2, \cdots, \tag{2.6b}$$

leading to specific degeneracies according to the different values of p.

A last comment associated with this $N = 2$-case concerns some particular values of p. For $p = 1$, i.e. in the supersymmetric context[8], the matrices S_1, S_2 and η coincide with the Pauli ones, an expected result. For $p = 2$ and its subcase[1] $p = 0$, we can point out 3 by 3 and 1 by 1 matrices, respectively, leading to the well known decomposition[11]

$$10 = 1^2 + 3^2 \tag{2.7}$$

of the irreducible representations (irreps) of the Kemmer algebra $K(2)$.

2.B. The case $N = 3$

We have once again to ensure the relations (1.3) but now with the indices a, b and c running from 1 to 3. In order to identify the structure subtended by these relations, we consider, by analogy with (1.4),

$$L_a = -\frac{i}{2}\varepsilon_{abc}\frac{[Q_b, Q_c]}{E} \tag{2.8a}$$

and

$$A_a = \frac{Q_a}{\sqrt{E}}. \tag{2.8b}$$

It is easy, using the relations (1.3) and simple identities, to see that the Lie algebra generated by the operators (2.8) is $so(4)$. In a parallel way to (2.1), we can realize the parasupercharges through

$$Q_a = \frac{1}{\sqrt{2}} A_a\,(p + i\eta\, W(x)), \tag{2.9}$$

where the A_a's are the numerical matrices corresponding to (2.8b) and where η has to anticommute with each of these A_a's . Moreover, the relation (2.3b) is once again required. Such conditions automatically lead to the same Hamiltonian as the one given in (2.5).

In addition to these results, we can also remember[12] that the dimension of the irreps $D(\lambda_1, \lambda_2)$ of so (4) is

$$d = (\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 + 1), \quad \lambda_1 \geq \lambda_2. \tag{2.10}$$

We can then search for a connection between λ_1 and λ_2 and the order p of paraquantization. The $p = 1$-context, as already recalled, is associated with the ordinary supersymmetric quantum mechanics. In other words, for this value of p , the relations (1.3b) can be simplified and reduced to

$$Q_a^2 = \frac{1}{4} H. \tag{2.11}$$

It is then straightforward to see, within this specific context, that $\vec{A}^2$ and $\vec{L}^2$ are just $\frac{3}{4}$ as it is also the case for $\pm \vec{L} \cdot \vec{A}$. This means that $p = 1$ corresponds to $\lambda_1 = \lambda_2 = \frac{1}{2}$ and the matrices A_a essentially coincide with the Pauli ones. Such a result implies, by opposition to the previous $N = 2$-case, that η cannot exist inside this representation. In fact, we have to couple two of these representations and consider 4 by 4 matrices associated with $D(\frac{1}{2}, \frac{1}{2}) \oplus D(\frac{1}{2}, \frac{1}{2})$. The matrix η is then given by, f.i.

$$\eta = \mathrm{diag}(1, 1, -1, -1). \tag{2.12}$$

The $p = 2$-context is characterized by analogous features. We can indeed be convinced of relations like $\vec{L}^3 = \vec{L}$, $\vec{A}^3 = \vec{A}$ and $(\vec{L} \cdot \vec{A})^3 = 4(\vec{L} \cdot \vec{A})$ in this case, simply by using the relations (1.3).

As a consequence of these results, we can point out the following irreps of so (4)

$$D(0,0),\ D(1,0),\ D(1,1), D'(1,1), \tag{2.13}$$

in order to describe the $p = 2$- and $N = 3$-case, these irreps being in perfect agreement with the decomposition[11]

$$35 = 1^2 + 3^2 + 3^2 + 4^2 \tag{2.14}$$

of the Kemmer algebra $K(3)$. However, it has to be noticed that the *twin* representations[11] $D(1,1)$ and $D'(1,1)$ have to be coupled in order to ensure the existence of the matrix η.

To conclude the discussion on $N = 3$-parasupercharges, we would like to insist on the fact that these parasupercharges essentially behave like the operators A_a's $(a = 1, 2, 3)$ of $so(4)$. This is nothing but the behavior of the components of the Runge-Lenz vector[9] appearing when the Hydrogen atom is under study. The latter can thus be seen as particularly interesting *physical* parasupercharges.

2.C. The case $N = 4$

Very similar results to the ones of the $N = 2$-context hold when four parasupercharges are considered. Indeed, we can define

$$Z_j = \frac{i}{2} \varepsilon_{jkm} \frac{[Q_k, Q_m]}{E}, \quad Y_j = i \frac{[Q_j, Q_4]}{E}, \quad j, k, m = 1, 2, 3 \tag{2.15}$$

and

$$X_j = \frac{Q_a}{\sqrt{E}}, \quad a = 1, \cdots, 4, \tag{2.16}$$

and understand through the relations (1.3) that these operators generate an $so(5)$-algebra. The corresponding parasupercharges can be easily realized via

$$Q_a = \frac{1}{\sqrt{2}} X_a \left(p + i\eta\, W(x)\right) \tag{2.17}$$

where the X_a's $(a = 1, \cdots, 4)$ are numerical matrices associated with $so(5)$ (in fact, the complement of $so(4)$ into this $so(5)$). The matrix η has to anticommute with each of the X_a's and has to satisfy the relation (2.3b). In what concerns the associated Hamiltonian, it takes one more time the form (2.5).

The dimension of the irreps $D(\lambda_1, \lambda_2)$ of so (5) is given by[12]

$$d = \frac{1}{6}(2\lambda_1 + 3)(2\lambda_2 + 1)(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 + 2), \quad \lambda_1 \geq \lambda_2. \tag{2.18}$$

Exploiting once again the $p = 1$-features (in particular the relation (2.11)), we can be convinced that supersymmetric quantum mechanics generated by four supercharges deals with the $D(\frac{1}{2}, \frac{1}{2})$ irrep of so (5) characterized by 4 by 4 matrices. This is, of course, in perfect agreement with what is known on the (minimal) Poincaré superalgebra[10] whose physical interest is prominent in connection with the Wess-Zumino model[13], for example. We have also to add that, in this case and by opposition to the $N = 3$ one, there is no problem to put in evidence a convenient matrix η. It can take the form

$$\eta = \mathrm{diag}(1, -1, -1, 1). \tag{2.19}$$

The $p = 2$-context can also be considered and gives rise to the irreps

$$D(0,0),\ D(1,0),\ D(1,1) \tag{2.20}$$

of so (5) and to the expected decomposition[11]

$$126 = 1^2 + 5^2 + 10^2 \tag{2.21}$$

of the Kemmer algebra $K(4)$. The corresponding matrix η can evidently be determined in each case.

As a final comment, we would like to mention that the 10-dimensional irrep of so (5) has been recognized as a fundamental one in the (minimal) Poincaré para-superalgebra[14] and its interpretation of a generalization of the Wess-Zumino model.

3. THE GENERAL CONTEXT OF AN ARBITRARY NUMBER OF PARASUPERCHARGES

Having in mind what happens in the cases $N = 2, 3$ and 4, we can now generalize the previous results to the case of an arbitrary number of parasupercharges. In particular, the coupling of some representations in order to ensure the existence of the matrix η in the case $N = 3$ suggests to separate the discussion into two subcases according to the parity of N.

3.A. The case $N = 2n$

It is easy to see that commutators of the different N parasupercharges (up to constant factors) generate an so (2n)-algebra. Moreover, when they are supplemented by the parasupercharges themselves, these commutators lead to so (2n + 1). The dimension of the irreps $D(\lambda_1, \cdots, \lambda_n)$ of so (2n + 1) is given by[12]

$$d = \prod_{j=1}^{n} \frac{(2\lambda_j + 2n - 2j + 1)}{(2n - 2j + 1)!} \prod_{k=j+1}^{n} (\lambda_j - \lambda_k + k - j)(\lambda_j + \lambda_k + 2n - j - k + 1),$$
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{3.1}$$

The parasupercharges can be realized in an analogous way to (2.1) (or equivalently (2.9) and (2.17)) and the matrix η does exist for any value of n and p, ensuring the expression (2.5) to be valid for characterizing the Hamiltonian.

The $p = 1$-context is relatively easy to handle because it deals with $\lambda_1 \; (= \frac{p}{2}) = \cdots = \lambda_n = \frac{1}{2}$ and leads to

$$d = \prod_{j=1}^{n} \frac{2n - 2j + 2}{(2n - 2j + 1)!} \frac{(2n - 2j + 1)!}{n - j + 1} = 2^n, \tag{3.2}$$

in perfect agreement with what we know on the irreps of the Clifford algebras $\mathscr{C}\ell_{2n}$[15] associated with supersymmetric considerations.

The $p = 2$-context is more elaborate but we can convince ourselves that it corresponds to $\lambda_1 = \cdots = \lambda_n = 0$ (leading evidently to $d = 1$) <u>and</u> to $\lambda_1 \; (= \frac{p}{2}) = \lambda_2 = \cdots = \lambda_j = 1, \; \lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_n = 0$ for $j = 1, \cdots, n$, giving as it can be verified

$$d = \frac{(2n+1)!}{j!(n+1-j)!} \equiv C^j_{2n+1}. \tag{3.3}$$

In particular, the decomposition into irreps of the Kemmer algebra K (2n) is now clear. It is

$$[\text{order of } K(2n)] = \sum_{j=0}^{n} (C^j_{2n+1})^2, \tag{3.4}$$

a new result, at our knowledge, coming from the connection between parasupersymmetries and orthogonal Lie algebras.

3.B. The case N = 2n - 1

As already suggested by the N = 3- context, this case is more complicated than the even one. Nevertheless, we can easily see that commutators of the $N = 2n - 1$ parasupercharges together with the parasupercharges themselves give rise to the algebra so (2n) whose irreps $D(\lambda_1, \cdots, \lambda_n)$ are characterized by the dimension[12]

$$d = 2^{n-1} \prod_{j=1}^{n} \frac{1}{(2n-2j)!} \prod_{k=j+1}^{n} (\lambda_j - \lambda_k + k - j)(\lambda_j + \lambda_k + 2n - j - k),$$
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{3.5}$$

If the realization of the corresponding parasupercharges and Hamiltonian is clear through (2.1) and (2.5), it is also evident from our analysis of the case $N = 3$ that the existence of η has to be examined carefully. In fact, the *twin* representations must be coupled for each value of p in order to ensure this existence of η and they are transformed one to each other through this η .

For example, the p = 1-context is associated with $\lambda_1 \ (= \frac{p}{2}) = \cdots = \lambda_n = \frac{1}{2}$ and is characterized by

$$d = 2^{n-1} \tag{3.6}$$

but, inside such a representation, it is impossible to find a convenient η . So, we finally have to consider matrices of dimension 2^n, again in agreement with the irreps of the Clifford algebra $\mathscr{Cl}_{2n}$.

The p = 2-context has also the same features in the sense that, besides the trivial irrep corresponding to $\lambda_1 = \cdots = \lambda_n = 0$ and leading to $d = 1$, we have to consider the cases $\lambda_1 \ (= \frac{p}{2}) = \lambda_2 = \cdots = \lambda_j = 1$, $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_n = 0$ for $j = 1, \cdots, n-1$ and we obtain

$$d = \frac{(2n)!}{j!(2n-j)!} \equiv C^j_{2n}. \tag{3.7}$$

The *twin* representations dealing with $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$ and characterized by

$$d = \frac{1}{2} C^n_{2n} \tag{3.8}$$

have to be coupled as already mentioned. The final result concerning the decomposition into irreps of the Kemmer algebra K (2n - 1) then writes

$$[\text{order of } K(2n-1)] = \sum_{j=0}^{n-1} (C^j_{2n})^2 + 2\left(\frac{1}{2} C^n_{2n}\right)^2. \tag{3.9}$$

4. COMMENTS

We have analyzed the connection between the so-called parasupersymmetric quantum mechanics , generated by an arbitrary number N of parasupercharges and characterized by an arbitrary order p of paraquantization, and orthogonal Lie algebras. With the knowledge of the irreps of such Lie structures, we have specified the Hamiltonian in each case and the dimension of the matrices involved in this model. A consequence of this analysis is that we can discuss the number of (para)supercharges. For instance, in the supersymmetric context corresponding to $p = 1$, we know that the cases $N = 3$ and $N = 4$ both lead to 4 by 4 matrices and to the same Hamiltonian (2.5). In the first case, η is given by

$$\eta = \sigma_3 \otimes \sigma_2 \text{ or } \eta = I \otimes \sigma_1, \tag{4.1}$$

while, in the second case, it coincides with

$$\eta = \sigma_3 \otimes \sigma_3 . \tag{4.2}$$

This means that the two Hamiltonians are unitarily equivalent through

$$U = \frac{1}{\sqrt{2}} I \otimes (\sigma_3 + \sigma_2) \text{ or } U = \frac{1}{\sqrt{2}} (I \otimes \sigma_3 + \sigma_3 \otimes \sigma_1), \tag{4.3}$$

respectively. In other words, it is impossible to find, in the supersymmetric context, a set of three supercharges : this set is automatically completed by a fourth supercharge. This result is in perfect agreement with the Baake et al. paper[16] based on tensor calculus and Clifford algebras.

As a last comment, we would like to insist on the fact that our developments deal with the one-dimensional context. Going to the d-dimensional context means that we have to consider, in the case $N = 2$, for example,

$$Q_a = \frac{1}{\sqrt{2}} \vec{\varphi}_a \cdot (\vec{p} + i \eta \vec{W}(x)), \quad a = 1, 2, \tag{4.4}$$

with

$$\{ \varphi_a^j , \varphi_a^k \} = \frac{1}{2} \delta^{jk}, \quad \{ \eta , \eta \} = \frac{1}{2}, \tag{4.5a}$$

$$\{ \varphi_1^j , \varphi_2^k \} = \Xi^{jk}, \quad \Xi^{jk} = - \Xi^{kj}, \tag{4.5b}$$

$$\{ \eta , \varphi_a^j \} = 0 . \tag{4.5c}$$

We know[17] that, in this case, the set $\vec{\varphi}_a$ $(a = 1, 2)$ generates the superalgebra $su(2^{m-1}|2^{m-1})$ if $d = 2m$ or $d = 2m - 1$ while η is the so-called canonical element[17]. We thus notice that, in order to go to this d-dimensional context, we have to take account of 2^{d-3} (2^{d-2}) irreps $D(\frac{1}{2})$ of $so(3)$ if $d = 2m$ $(d = 2m - 1)$. In particular, the physical situation $d = 3$ is just a question of a doubling of the $d = 1$ results presented in this paper. The one-dimensional context is thus the fundamental one in such developments. The same kind of results hold for $N = 3, 4, \cdots$ but the situation can be complicated by the introduction of many superpotentials. Work is in progress in this direction.

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