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Local Perturbations and Limiting Gibbs States of Quantum Lattice Mean–Field Systems

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Abstract: In the frame of operator algebraic quantum statistical mechanics, the limiting Gibbs states for quantum lattice mean–field systems under the influence of weak perturbations are analyzed. For a certain model class it is proved that all homogeneous states which minimize the functional of the free energy density, can be calculated as the thermodynamic limit of perturbed local Gibbs states. For uniformly bounded nets of (not necessarily homogeneous) local perturbations with a well defined asymptotical behaviour in the thermodynamic limit (approximately symmetric, resp. quasi–symmetric nets) the existence of a unique limiting Gibbs state is proved for the considered model class. An inhomogeneous BCS–model and the Josephson junction of coupled superconductors are examples for the applicability of the results. Finally, the relation of the considered local perturbations to extended–valued lower–bounded operators affiliated with a von Neumann algebra as relative Hamiltonians of two normal states is discussed.

1 Introduction

We analyze the set of equilibrium states for a class of quantum lattice mean–field models. The lattice is assumed to consist of finite quantum systems on each site of the lattice with the matrix algebra $\mathcal{B}$ of observables. A model can be characterized in terms of all local Hamiltonians H_Λ, which are assumed to be in the algebra $\mathcal{A}_\Lambda := \otimes_{i\in\Lambda}\mathcal{B}$ of observables of a finite region Λ of the lattice. The treatment of the model requires the analysis of the non–equilibrium dynamics, equilibrium states or thermodynamical functionals of the infinitely extended macroscopic system by calculating the thermodynamic limit. Often it is necessary to choose a certain state which represents the preparation of the system. In an equilibrium situation this can be a certain KMS–state [1, 2]. But in general there exist a lot of KMS–states, especially if there is a phase transition. Therefore one needs additional criteria for a certain choice. Here, the limiting Gibbs states, i.e. the limits of the Gibbs states of the local subsystems, play an important role. They take into account all microscopical aspects of the model in the thermodynamic limit and represent its quantum statistical features as an optimal approximation of all finite size properties, e.g. its symmetries.

For a certain class of mean–field models we will discuss exemplary some methods to determine limiting Gibbs states, resp. their properties. Let us consider the so–called polynomial mean–field models with local Hamiltonian densities $H_\Lambda/|\Lambda|$, being the same polynomial in mean–field operators (i.e. averages of an element in $\mathcal{B}$ over the region Λ) for each region Λ. The existence of the thermodynamical density functionals for internal energy, entropy and free energy were proved, and a minimum principle of the free energy density for limiting Gibbs states was established [3, 4, 5]. This induces a selfconsistency–condition for pure phase states (factor states) in the support of the central measure of the limiting Gibbs state. If a system is prepared in a state, its central decomposition determines the classical distribution of pure phases that may be present in an experimental situation. Concerning the dynamical aspects in the thermodynamic limit, there appear difficulties, because the algebra of the lattice system is not invariant under a limiting dynamics. Therefore, an enlargement of the algebra is required [6, 7]. This has been worked out in refs. [8, 9, 10, 11]. Finally, the KMS–condition for equilibrium states of the limiting dynamics has been proved and the extremal KMS–states have been identified as all solutions of the selfconsistency equations mentioned above [12].

Besides the question whether there exists a unique limiting Gibbs state at all, it turns out that such states show a very sensitive dependence on what we call here a local perturbation: Adding to the local Hamiltonians H_Λ a "small" perturbation h_Λ for each finite region Λ, the limiting Gibbs state may change drastically ("small" means that the density $\|h_\Lambda\|/|\Lambda|$ tends to zero for large regions Λ). Such perturbations are of real physical significance, e.g. as interaction between weakly coupled superconductors in a Josephson junction, inhomogeneities of the interaction on a lattice or boundary effects after symmetrization of short range interactions. Since the limiting Gibbs states of the unperturbed and the perturbed mean–field models both minimize the same functional of the free energy density [5], the sole analysis of this variational principle generally is a too rough a method to find such states. Thus it can only be applied in some special situations; for a counterexample see ref. [13]. This is our motivation to look in more detail on the quantum statistical properties of such models arising in the thermodynamic limit.

There are two ways to attack the arising problems: at first one can refine the method of calculating quasi–averages, originally developed to identify symmetry breaking in the case of phase transitions [14]. In our setting for two given states, resp. distributions of pure phase states, one analyzes local perturbations which allow to go over from one state to the other in the thermodynamic limit. First steps in this direction have been performed in refs. [5, 15] where pure phase states, resp. extremal states with a given symmetry are constructed as limiting Gibbs states. This aspect touches the fundamental question of the richness of classical structures that are accessible in terms of a microscopical defined model. In sharp contrast to this procedure there is the necessity to calculate the limiting Gibbs state if a perturbation is given explicitly by physical reasons. The only available result for our model class can be found in ref. [16], where the given local perturbations lead to states with strictly positive Radon–Nikodym derivative of the central measures of the unperturbed and the perturbed state. Here we will use the two points of view simultaneously to generalize the results of the above stated references and find criteria such that *each* homogeneous state, satisfying the variational principle of the free energy density is a limiting Gibbs state of a locally perturbed model. Especially, this means that all KMS–states with minimal free energy density can be calculated as limiting Gibbs states of these perturbed models.

In Sec. 2 we introduce the quasi–local algebra $\mathcal{A}$ of a quantum lattice system and the relevant part of its state space $\mathcal{S}(\mathcal{A})$ for mean–field systems, the so–called homogeneous or permutation invariant states $\mathcal{S}^P(\mathcal{A})$. Then we define the model class, specify what we call a "local perturbation", and refer the most important results on limiting Gibbs states. As far as possible, the terminology of (approximately–)symmetric nets is used [5, 11], comp. Definitions 2.1, 2.3.

In Sec. 3 the main results on local perturbations of mean–field systems are worked out. The permutation invariant limiting Gibbs states are discussed in terms of their central measures. We construct local perturbations in such a way that a given permutation invariant state with minimal free energy density is the limiting Gibbs state of the locally perturbed model (Theorem 3.1). The local perturbation can be specified to show a well defined asymptotic behaviour for large local regions (Proposition 3.3). Moreover this includes a constructive result that allows to infer the corresponding limiting Gibbs state from the local perturbations. Finally a sufficient condition for the stability of the limiting Gibbs state under local perturbations is given (Proposition 3.2).

The constructive results in Sec. 3 may also be formulated for perturbations which are not locally permutation invariant. In order to do this, in Sec. 4 the notion of quasi–symmetric nets [11] is introduced, and the limiting Gibbs state of an inhomogeneous mean–field system locally perturbed with a quasi–symmetric net is determined.

In Sec. 5 some applications of the foregoing results are presented. We calculate the limiting Gibbs state of an inhomogeneous BCS–model with a nontrivial momentum dependence of its kinetic energies and coupling constants. Furthermore, the influence of a scaling function for the interaction in a Josephson junction of weakly coupled superconductors is now discussed in the thermodynamic limit. We close with some remarks on the connection between local perturbations and relative Hamiltonians [17]. In our context it is necessary to use the so–called extended–valued lower–bounded operators affiliated with a von Neumann algebra [18]. They are affine, weakly lower semicontinuous (possibly infinite) functionals on the normal states of a von Neumann algebra, which generalize the concept of a relative Hamiltonian. In the case of permutation invariant states it is demonstrated that their finite part can be approximated in the sense of strong resolvent convergence by the local perturbations.

2 Algebraic Framework, the Model Class and Preliminary Results

As *quasi–local algebra* of the quantum lattice system we choose the C^*–algebra

$$\mathcal{A} := \bigotimes_{i\in\mathbb{N}} \mathcal{B}_i \ , \text{ with } \mathcal{B}_i \cong \mathcal{B} \cong M_n(\mathbb{C}) \, , \forall i \in \mathbb{N} \, ,$$

with some fixed $n \in \mathbb{N}$. The quasi–local structure of $\mathcal{A}$ is defined as follows: Let $\mathcal{L} := \{\Lambda \subset \mathbb{N} \mid |\Lambda| < \infty\}$ be the family of all finite subsets in $\mathbb{N}$, and $|\Lambda|$ denotes the cardinality of such a *local region* $\Lambda \subset \mathbb{N}$. Consider each algebra $\mathcal{A}_\Lambda := \otimes_{i\in\Lambda}\mathcal{B}_i$ as C^*–subalgebra of $\mathcal{A}$, embedded into $\mathcal{A}$ by $\mathcal{A}_\Lambda \ni A_\Lambda \longrightarrow A_\Lambda \otimes \mathbb{1}_{\mathbb{N}\setminus\Lambda} \in \mathcal{A}$.

The *states* $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ are the positive and normalized linear functionals on $\mathcal{A}$. Since we are interested in the so–called mean–field models, it is appropriate to consider only the set of *permutation invariant (homogeneous) states* $\mathcal{S}^P(\mathcal{A})$, defined by

$$\mathcal{S}^P(\mathcal{A}) = \{\omega \in \mathcal{S}(\mathcal{A}) \mid \omega \circ \Theta_\sigma = \omega \, , \forall \sigma \in \mathcal{P}\} \ .$$

Here we use the following notation: $\mathcal{P}(\Lambda)$ is the set of all permutations of Λ, i.e. the set of all bijections σ on $\mathbb{N}$ with $\sigma(i) = i$, $\forall i \notin \Lambda$, $\mathcal{P} := \bigcup_{\Lambda\in\mathcal{L}} \mathcal{P}(\Lambda)$ and for $\sigma \in \mathcal{P}$ we denote by Θ_σ the automorphism, satisfying $\Theta_\sigma(\otimes_{i\in\mathbb{N}} x_i) := \otimes_{i\in\mathbb{N}} x_{\sigma(i)}$ for elements $\otimes_{i\in\mathbb{N}} x_i \in \mathcal{A}$. We see that the usage of the special lattice $\mathbb{N}$ is adequate for our purposes since the permutation invariant states ignore any further lattice structure.

The set of permutation invariant states has the following well known properties [19]: $\mathcal{S}^P(\mathcal{A})$ is a *Bauer–simplex* with respect to the weak topology on $\mathcal{S}(\mathcal{A})$ with extremal boundary

$$\partial_e \mathcal{S}^P(\mathcal{A}) = \{\bigotimes \varphi \mid \varphi \in \mathcal{S}(\mathcal{B})\} \ , \tag{2.1}$$

where $\otimes\varphi$ is the product state on $\mathcal{A}$, defined by linear continuation of $\langle \otimes\varphi \, ; \otimes_{i\in\mathbb{N}} x_i \rangle = \Pi_{i\in\mathbb{N}} \langle \varphi \, ; x_i \rangle$, $\forall \otimes_{i\in\mathbb{N}} x_i \in \mathcal{A}$. The extremal decomposition of $\omega \in \mathcal{S}^P(\mathcal{A})$ into elements of $\partial_e \mathcal{S}^P(\mathcal{A})$ coincides with the *central decomposition* of ω in $\mathcal{S}^P(\mathcal{A})$, i.e. a decomposition into factor states (pure phases). The *central measure* is denoted by μ_ω; due to the parameterization of $\partial_e \mathcal{S}^P(\mathcal{A})$ by states in $\mathcal{S}(\mathcal{B})$ (2.1), we regard μ_ω as a measure on $\mathcal{S}(\mathcal{B})$ and write $\omega = \int_{\mathcal{S}(\mathcal{B})} \otimes\varphi \, d\mu_\omega(\varphi)$.

Now let us have a look at the algebra of observables $\mathcal{A}$ and the definition of the model class. The restriction to permutation invariant states as made above is closely related to the definition of mean–field models on $\mathcal{A}$ by means of local Hamiltonians. We will use here the notion of (approximately) symmetric nets, introduced in ref. [5] (comp. Definition 4.1, where further extensions of this definition are given):

Definition 2.1 ((Approximately) Symmetric Nets) *Let $\Omega, \Lambda \in \mathcal{L}$ with $\Omega \subseteq \Lambda$ and $j^\emptyset_{\Lambda\Omega} := \frac{(|\Lambda|-|\Omega|)!}{|\Lambda|!} \sum \Theta_\sigma$, where the summation runs over all injective maps $\sigma : \Omega \mapsto \Lambda$ (especially $j^\emptyset_{\Lambda\Lambda}$ is the symmetrization operator in $\mathcal{A}_\Lambda$).*

Let $x := (x_\Lambda)_{\Lambda\in\mathcal{L}}$ be a family of local operators with $x_\Lambda \in \mathcal{A}_\Lambda$ and $x_\Lambda = j^{\emptyset}_{\Lambda\Lambda} x_\Lambda$ for all $\Lambda \in \mathcal{L}$. x_Λ and $x_{\Lambda'}$ satisfy $\Theta_\sigma x_\Lambda = x_{\Lambda'}$ for Λ, $\Lambda' \in \mathcal{L}$ with $|\Lambda| = |\Lambda'|$ and $\forall\sigma \in \mathcal{P}$ with $\sigma\Lambda = \Lambda'$. x is called

(i) *a* symmetric net *($x \in \mathcal{Y}$), if there exists a $k \in \mathbb{N}$, such that for all $\Lambda \in \mathcal{L}$ with $|\Lambda| \geq k$, there is an $\Omega \in \mathcal{L}$, $\Omega \subseteq \Lambda$ with $|\Omega| = k$ and $x_\Lambda = j^{\emptyset}_{\Lambda\Omega} x_\Omega$.*

(ii) *an* approximately symmetric net *($x \in \widetilde{\mathcal{Y}}$), if for all $\varepsilon > 0$, there exists a $y \in \mathcal{Y}$ and a $n \in \mathbb{N}$, such that for all $\Lambda \in \mathcal{L}$ with $|\Lambda| \geq n$ it holds: $\|x_\Lambda - y_\Lambda\| \leq \varepsilon$.*

In ref. [5] the algebraical structure of $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ is elaborated. Especially, they are vector spaces and $\|x\| := \lim_{\Lambda\in\mathcal{L}} \|x_\Lambda\|$ ($\lim_{\Lambda\in\mathcal{L}}$ denotes the limit of a net with index set $\mathcal{L}$) defines a seminorm on $\widetilde{\mathcal{Y}}$. The function

$$j : \widetilde{\mathcal{Y}} \longrightarrow \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C}),\ x \longrightarrow [j(x)](\varphi) := \lim_{\Lambda\in\mathcal{L}} \langle \otimes\varphi\, ;\, x_\Lambda \rangle\,,\ \forall\varphi \in \mathcal{S}(\mathcal{B}) \tag{2.2}$$

maps $\widetilde{\mathcal{Y}}$ isometrically onto $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C})$, the continuous functions on $\mathcal{S}(\mathcal{B})$. The seminorm defines an equivalence relation on $\widetilde{\mathcal{Y}}$ and the corresponding quotient space is isomorphic to $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C})$.

The set of symmetric nets can also be characterized in terms of this map j. An element $x \in \mathcal{Y}$ equals a *polynomial Q in the mean-field operators* $m_\Lambda(e) := 1/|\Lambda| \sum_{i\in\Lambda} e \otimes \mathbb{1}_{\Lambda\setminus\{i\}} \in \mathcal{A}_\Lambda$, $e \in \mathcal{B}$ up to a term vanishing in norm for large Λ. If we choose the arguments of the polynomial as mean-field operators corresponding to elements of a fixed basis $e_1, \ldots, e_{n^2}$ of $\mathcal{B}$, the coefficients of Q are uniquely determined and $q = (Q(m_\Lambda(e_1), \ldots, m_\Lambda(e_{n^2-1})))_{\Lambda\in\mathcal{L}}$ defines an approximately symmetric net with $j(q) = j(x)$, being a polynomial in $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C})$.

Proposition 2.2 *Let $h = (h_\Lambda)_{\Lambda\in\mathcal{L}}$ be an approximately symmetric net. Denote by Π_P the partially universal representation of $\mathcal{A}$ corresponding to the folium[1] $\mathcal{F}_P$ of permutation invariant states in the Hilbert space $\mathcal{H}_P := \oplus_{\omega\in\mathcal{F}_P} \mathcal{H}_\omega$ (($\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega$) denoting the GNS-representation to a state ω). The limit* s–$\lim_{\Lambda\in\mathcal{L}} \Pi_P(h_\Lambda)$ *exists in $\mathcal{B}(\mathcal{H}_P)$. Moreover there is a subalgebra of $\mathcal{M}_P := \Pi_P(\mathcal{A})''$ which is isomorphic to $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C}) \bigotimes \Pi_P(\mathcal{A})$. Its center $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C}) \bigotimes \mathbb{1}$ is isomorphic to a subalgebra of the center $\mathcal{Z}_P := \mathcal{M}_P \cap \mathcal{M}'_P$ in $\mathcal{M}_P$. Then we have*

$$\text{s-}\lim_{\Lambda\in\mathcal{L}} \Pi_P(h_\Lambda) \cong j(h) \bigotimes \mathbb{1} \in \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C}) \bigotimes \Pi_P(\mathcal{A})\,,$$

i.e. s–$\lim_{\Lambda\in\mathcal{L}} \Pi_P(h_\Lambda) \in \mathcal{Z}_P$.

PROOF: Concerning the isomorphisms between the various subalgebras of $\mathcal{M}_P$ we refer to refs. [9] and [10]. The strong convergence s–$\lim_{\Lambda\in\mathcal{L}} \Pi_P(m_\Lambda(x)) =: m_P(x) \in \mathcal{Z}_P$ for $x \in \mathcal{B}$ is well known and thus the strong convergence of all polynomials in $m_\Lambda(x)$ follows (they are

[1]A *folium* $\mathcal{F}$ of a C^*-algebra $\mathcal{A}$ is a norm-closed, convex subset of the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$, such that $\omega \in \mathcal{F}$ implies $\omega_B \in \mathcal{F}$, where $\langle \omega_B\, ;\, .\rangle = \langle \omega\, ;\, B^*.\, B\rangle \,/\, \langle \omega\, ;\, B^*B\rangle$, for each $B \in \mathcal{A}$ with $\langle \varphi\, ;\, B^*B\rangle \neq 0$ (i.e. $\mathcal{F}$ is closed under perturbations from $\mathcal{A}$) [20].

uniformly norm bounded). It is easy to see that a symmetric net differs from a polynomial in mean–field operators only up to a part with norm limit zero. Thus we have the strong convergence of symmetric nets with the same limit as the one of the associated polynomials in the mean–field operators.

According to isomorphy we regard $j(h) \in C(\mathcal{S}(\mathcal{B}), \mathbb{C})$ as an element of $\mathcal{Z}_P$ and calculate for arbitrary $\xi \in \mathcal{H}_P$:

$$\|(\Pi_P(h_\Lambda) - j(h))\,\xi\| \leq \|(\Pi_P(h_\Lambda - g_\Lambda))\,\xi\| + \|(\Pi_P(g_\Lambda) - j(h))\,\xi\| \,,$$

with a symmetric net $g = (g_\Lambda)_{\Lambda\in\mathcal{L}}$, satisfying $\|h_\Lambda - g_\Lambda\| \leq \varepsilon$ for large Λ. j is an isometry and thus we have $\|j(g) - j(h)\| \leq \varepsilon$. Consequently:

$$\begin{aligned}
&\|(\Pi_P(h_\Lambda) - j(h))\,\xi\| \\
&\leq \|\Pi_P(h_\Lambda - g_\Lambda)\xi\| + \|(\Pi_P(g_\Lambda) - j(g))\,\xi\| + \|(j(g) - j(h))\,\xi\| \\
&\leq \|h_\Lambda - g_\Lambda\|\,\|\xi\| + \|(\Pi_P(g_\Lambda) - j(g))\,\xi\| + \|j(g) - j(h)\|\,\|\xi\| \\
&\leq 2\varepsilon\,\|\xi\| + \|(\Pi_P(g_\Lambda) - j(g))\,\xi\| \,.
\end{aligned}$$

Using the strong convergence of a symmetric net, Proposition 2.2 is proved. ❑

Definition 2.3 (The Model Class) *The* model class *is specified by families* $(H_\Lambda + h_\Lambda)_{\Lambda\in\mathcal{L}}$ *of locally permutation invariant selfadjoint operators* $H_\Lambda, h_\Lambda \in \mathcal{A}_\Lambda$, $\forall\Lambda \in \mathcal{L}$ *with*

(i) $\left(\frac{H_\Lambda}{|\Lambda|}\right)_{\Lambda\in\mathcal{L}}$ *is a polynomial in mean–field operators.*

(ii) $\left(\frac{h_\Lambda}{|\Lambda|}\right)_{\Lambda\in\mathcal{L}}$ *is a net with* $\lim_{\Lambda\in\mathcal{L}} \|h_\Lambda\| / |\Lambda| = 0$.

The operators H_Λ, *resp.* $H_\Lambda + h_\Lambda$ *are considered as the Hamiltonians, defining the dynamics of the unperturbed, resp. the perturbed mean–field model in the local region* Λ *of the lattice.*

The part $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ in the Hamiltonians is a non–extensive (but in general unbounded in the limit of large regions Λ) perturbation of the mean–field model, locally defined by $(H_\Lambda)_{\Lambda\in\mathcal{L}}$, and we will study the limiting Gibbs state under the influence of these local perturbations $(h_\Lambda)_{\Lambda\in\mathcal{L}}$. A *limiting Gibbs states* of a mean–field model $(H_\Lambda + h_\Lambda)_{\Lambda\in\mathcal{L}}$ (according to Definition 2.3) at inverse temperature $\beta > 0$ is a w^*–accumulation point of the net $(\omega^{\beta H_\Lambda + h_\Lambda})_{\Lambda\in\mathcal{L}} \subset \mathcal{S}(\mathcal{A})$, with the *local Gibbs states* $\langle \omega^{\beta H_\Lambda + h_\Lambda}\,;\, A\rangle := \tau(\exp(-\beta(H_\Lambda + h_\Lambda))A) / \tau(\exp(-\beta(H_\Lambda + h_\Lambda)))$, $\forall A \in \mathcal{A}$. τ is the trace state on $\mathcal{A}$. A necessary condition for a permutation invariant state to be a limiting Gibbs states is given by a variational principle for the free energy density functional on permutation invariant states [5] (concerning the thermodynamical density functionals, comp. [4]):

Proposition 2.4 (Limiting Gibbs States) *Each limiting Gibbs state* ω^β *at* $\beta > 0$ *of a model according to Definition 2.3 is an element of* $S^P(\mathcal{A})$ *and a minimizer of the free* *energy density functional, defined on* $\mathcal{S}^P(\mathcal{A})$ *by*

$$\mathcal{S}^P(\mathcal{A}) \ni \omega \longrightarrow f(\beta,\omega) := \int_{\mathcal{S}(\mathcal{B})} \left([j(\left(\frac{H_\Lambda + h_\Lambda}{|\Lambda|}\right)_{\Lambda\in\mathcal{L}})](\varphi) - \frac{1}{\beta}s(\varphi)\right) d\mu_\omega(\varphi)$$
$$= \int_{\mathcal{S}(\mathcal{B})} \left([j(\left(\frac{H_\Lambda}{|\Lambda|}\right)_{\Lambda\in\mathcal{L}})](\varphi) - \frac{1}{\beta}s(\varphi)\right) d\mu_\omega(\varphi),$$

with the entropy of a state in $\mathcal{S}(\mathcal{B})$*, given by* $\mathcal{S}(\mathcal{B}) \ni \varphi \to s(\varphi) := -\mathrm{tr}(\varrho_\varphi \ln(\varrho_\varphi))$*. Here* ϱ_φ *is the density matrix which represents the state* φ *as element in* $M_n(\mathbb{C})$*.*

In some situations the variational principle allows to calculate the limiting Gibbs state of a given mean–field model. But if there occurs a phase transition a lot of permutation invariant states minimizing the free energy density are possible and one needs further information to find the *unique* limiting Gibbs state (if it exists at all). In general only the detailed analysis of the symmetries of the model allows to determine this state. Especially if there is an internal symmetry group and the set of all pure phases minimizing the free energy density consists of exactly one orbit to the internal symmetry, the limiting state is uniquely determined; for an example see e.g. [13]. In order to develop a general theory of limiting Gibbs states for quantum lattice mean–field models according to Definition 2.3, we need an additional assumption on the limiting Gibbs state of the unperturbed mean–field model with local Hamiltonians $(H_\Lambda)_{\Lambda\in\mathcal{L}}$:

General Assumption 2.5 *Use the notation of Definition 2.3. Throughout this paper assume that the unperturbed mean–field model with local Hamiltonians* $(H_\Lambda)_{\Lambda\in\mathcal{L}}$ *possesses an unique limiting Gibbs state* ω^β *at inverse temperature* $\beta > 0$ *with central measure* μ_β *and support* $\mathcal{K}_\beta = \mathrm{supp}(\mu_\beta)$*.*

Finally we state a fundamental result on local perturbations and limiting Gibbs states [16].

Theorem 2.6 *Let* $(H_\Lambda)_{\Lambda\in\mathcal{L}}$ *be the local Hamiltonians of an unperturbed mean–field model according to 2.3 and 2.5. If* h_Λ *is a polynomial in the mean–field operators* $m_\Lambda(e_1),\ldots,m_\Lambda(e_k)$ *to a fixed basis* $\{e_1,\ldots,e_k\}$ *of* $\mathcal{B}$ *and in the elements of* $\mathcal{A}_\Omega$ *for some* $\Omega \in \mathcal{L}$*, we have*

$$\mathrm{w}^*\text{-}\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h_\Lambda} = (\omega^\beta)^{h^\beta}.$$

$h^\beta \in \mathcal{M}_\beta = \Pi_\beta(\mathcal{A})''$ *is the limit of* h_Λ *in the strong topology induced by the Hilbert space* $\mathcal{H}_\beta$ *of the GNS–representation* $(\mathcal{H}_\beta, \Pi_\beta, \Omega_\beta)$ *of* ω^β*.* $(\omega^\beta)^{h^\beta}$ *is the perturbation of* ω^β *with* h^β *in terms of perturbation theory for KMS–states on the von Neumann algebra* $\mathcal{M}_\beta$*, i.e.* $\left\langle (\omega^\beta)^{h^\beta} ; A \right\rangle$ *is given for all* $A \in \mathcal{A}$ *by* $\left\langle (\omega^\beta)^{h^\beta} ; A \right\rangle = \left\langle \omega^\beta ; A\Gamma^{h^\beta}_{i\beta} \right\rangle / \left\langle \omega^\beta ; \Gamma^{h^\beta}_{i\beta} \right\rangle$*, with*

$\Gamma_{i\beta}^{h^\beta} := \sum_{n=0}^\infty (-1)^n \int_0^\beta ds_n \ldots \int_0^{s_2} ds_1 \tau_\beta^{is_1}(h^\beta) \ldots \tau_\beta^{is_n}(h^\beta)$, *where* τ_β^t *is the limiting KMS–dynamics [21]. If* $h^\beta \in \mathcal{Z}_\beta = \mathcal{M}_\beta \cap \mathcal{M}_\beta'$*, the perturbed limiting Gibbs state is given by*

$$\left\langle (\omega^\beta)^{h^\beta} ; A \right\rangle = \frac{\left\langle \omega^\beta ; Ae^{-\beta h^\beta} \right\rangle}{\left\langle \omega^\beta ; e^{-\beta h^\beta} \right\rangle} = \frac{\int_{S(\mathcal{B})} \langle \otimes\varphi ; A \rangle \, e^{-\beta[j(h)](\varphi)} \, d\mu_\beta(\varphi)}{\int_{S(\mathcal{B})} e^{-\beta[j(h)](\varphi)} \, d\mu_\beta(\varphi)} \, .$$

PROOF: This Theorem was proved in [16, Theorem 2.3] for $H_\Lambda = |\Lambda| Q_\Lambda$, with a quadratic polynomial Q_Λ in mean–field operators. Therefore, we have to consider here the generalization to arbitrary polynomials. Since all techniques used in ref. [16] remain unchanged, we only sketch the main steps of the proof.

The limiting dynamics for the unperturbed model, i.e. the strong limit

$$\underset{\Lambda \in \mathcal{L}}{\text{s–lim}}\, \Pi_\beta(\tau_\Lambda^t(A)) = \underset{\Lambda \in \mathcal{L}}{\text{s–lim}}\, \Pi_\beta(e^{itH_\Lambda} A e^{-itH_\Lambda})) = \tau_\beta^t(\Pi_\beta(A)) \in \mathcal{M}_\beta \, ,$$

exists for all $t \in \mathbb{R}$, $A \in \mathcal{A}$ and τ_β is a σ–weak–contiuous W^*–automorphism group of $\mathcal{M}_\beta$. This is established directly by following the argumentation in [8], [21], where the case of a quadratic polynomial Q_Λ in the mean–field operators is treated, in combination with [9]. Another way to determine the limiting dynamics is to use the techniques as performed in ref. [11] by working in the context of mean–field dynamical semigroups and application of Theorem 4.2 below. Since $\omega^{\beta H_\Lambda}$ is a β–KMS–state for the dynamics τ_Λ^β one can calculate the β–KMS–condition for ω^β as state on $\mathcal{M}_\beta$ and the dynamics τ^β.

The essential part of the proof is now to show for all $A \in \mathcal{A}_0 = \bigcup_{\Lambda \in \mathcal{L}} \mathcal{A}_\Lambda$

$$\lim_{\Lambda \in \mathcal{L}} \left\langle \omega^{\beta H_\Lambda + h_\Lambda} ; A \right\rangle = \lim_{\Lambda \in \mathcal{L}} \left\langle \omega^{\beta H_\Lambda} ; A \Gamma_{i\beta\Lambda}^{h_\Lambda} \right\rangle = \left\langle \omega^\beta ; A \Gamma_{i\beta}^{h^\beta} \right\rangle = \left\langle (\omega^\beta)^{h^\beta} ; A \right\rangle , \tag{2.3}$$

with

$$\Gamma_{i\beta\Lambda}^{h_\Lambda} := \sum_{n=0}^\infty (-1)^n \int_0^\beta ds_n \ldots \int_0^{s_2} ds_1 \tau_\Lambda^{is_1}(h_\Lambda) \ldots \tau_\Lambda^{is_n}(h_\Lambda) \, . \tag{2.4}$$

This follows from:

(i) For all $n \in \mathbb{N}$, $z \in \mathcal{D}_\beta^n = \{ z \in \mathbb{C}^n \mid \beta > \mathrm{Im}(z_n) > \cdots > \mathrm{Im}(z_1) > 0 \}$, and $A \in \mathcal{A}$ we have

$$\begin{aligned} &\lim_{\Lambda \in \mathcal{L}} \left\langle \omega^{\beta H_\Lambda} ; A \tau_\Lambda^{z_1}(h_\Lambda) \cdots \tau_\Lambda^{z_n}(h_\Lambda) \right\rangle \\ &= \lim_{\Lambda \in \mathcal{L}} \left\langle \omega^\beta ; A \tau_\Lambda^{z_1}(h_\Lambda) \cdots \tau_\Lambda^{z_n}(h_\Lambda) \right\rangle = \left\langle \omega^\beta ; A \tau_\beta^{z_1}(h^\beta) \cdots \tau_\beta^{z_n}(h^\beta) \right\rangle \\ &= \left\langle \Omega_\beta \,|\, \Pi_\beta(A) \tau_\beta^{z_1}(h^\beta) \cdots \tau_\beta^{z_n}(h^\beta) \Omega_\beta \right\rangle . \end{aligned}$$

These limits are obtained from the expansion

$$\tau_\Lambda^z(h_\Lambda) = \sum_{n=0}^\infty \frac{(-iz)^n}{n!} [H_\Lambda, [\cdots [H_\Lambda, h_\Lambda] \cdots]]$$

for $z \in \mathcal{D}_C^n = \{z \in \mathbb{C}^n \mid |Z_i| < 1/C \text{ for } i = 1, \ldots, n\}$. $C > 0$ is a constant appearing in the following estimation of the n–fold commutators

$$\|[H_\Lambda, [\cdots [H_\Lambda, h_\Lambda] \cdots]]\| \leq M n! C^n \,, \tag{2.5}$$

for all $\Lambda \in \mathcal{L}$ and some constants $M, C > 0$, see e.g. [9, Lemma 3.2], resp. [22, Chapter 6.2]. For $z \in \mathcal{D}_\beta^n$ the result is estalished with the help of the identity Theorem of holomorphic functions.

(ii) The limits in Eq. (2.3) resp. the summation and integration in Eq. (2.4) can be interchanged due to

$$\Big| \int_0^\beta ds_n \cdots \int_0^{s_2} ds_1 \, \langle \omega^{\beta H_\Lambda} \, ; \, A\tau_\Lambda^{is_1}(h_\Lambda) \cdots \tau_\Lambda^{is_n}(h_\Lambda) \rangle \Big| \leq \|A\| \, \frac{(\beta M)^n}{n!}$$

and the uniform convergence of $\langle \omega^{\beta H_\Lambda} \, ; \, A\tau_\Lambda^{is_1}(h_\Lambda) \cdots \tau_\Lambda^{is_n}(h_\Lambda) \rangle$. □

3 Local Perturbations and Limiting Gibbs States

After having established the general frame of our model class, we formulate the main result. All possible permutation invariant limiting Gibbs states $\omega \in \mathcal{S}^P(\mathcal{A})$ will be discussed in terms of their central measures μ_ω. As a reference, the limiting Gibbs state ω^β of the unperturbed model (comp. Assumption 2.5), resp. its central measure μ_β with support $\mathcal{K}_\beta$, are used. The analysis is done by *constructing local perturbations* according to Definition 2.3 such that a given permutation invariant state ω becomes a limiting Gibbs state (Theorem 3.1). This is an extreme generalization of the method of calculating the so–called *quasi–averages*, because it allows to find pure phase states as well as classical statistical mixtures of them as limiting Gibbs states. The proof is divided into several steps. In certain cases one finds that it is possible to obtain constructive results, i.e. to conclude from the local structure of the model to the limiting Gibbs state. This variation in the point of view allows to determine in Proposition 3.2 the stability of such states under local perturbations. In Proposition 3.3 (i) the limiting Gibbs state is calculated explicitly for approximately symmetric nets as local perturbations (comp. also Sec. 4 for a generalization to the case where the perturbations are no longer permutation invariant). These results are used for the rest of the section to construct more and more general perturbations with uniquely determined limiting Gibbs state until Theorem 3.1 is proved.

Theorem 3.1 *Given a model with local Hamiltonians $(H_\Lambda)_{\Lambda \in \mathcal{L}}$ and unique limiting Gibbs state ω_β it holds:*
For every permutation invariant state ω with central measure μ_ω and support $\mathrm{supp}(\mu_\omega) \subseteq \mathcal{K}_\beta$*, there exists a family of locally permutation invariant selfadjoint operators $(h_\Lambda)_{\Lambda \in \mathcal{L}}$ according to Definition 2.3 such that ω is the limiting Gibbs state of the locally perturbed mean–field model $(H_\Lambda + h_\Lambda)_{\Lambda \in \mathcal{L}}$:*

$$\mathrm{w}^*\text{–}\lim_{\Lambda \in \mathcal{L}} \omega^{\beta H_\Lambda + h_\Lambda} = \omega.$$

Before proving Theorem 3.1 we start with a result on the stability of limiting Gibbs states under perturbations. The difficulty lies in the fact that the considered limiting Gibbs states are w^*–limits of the local Gibbs states, whereas the known continuity properties of perturbed states consider either a fixed unperturbed state ω or a norm convergent net ω_α with limit ω, e.g. [1, Theorem 5.4.4], [23, Theorem 12.3], [24, Theorem 1.9]. Here we need a modification of a result in Proposition 3.5 of ref. [24]. In this reference the case of a w^*–convergent net of normal states on a von Neumann algebra is treated which is perturbed by a norm convergent net of selfadjoint operators. Unfortunately the limiting Gibbs states can only be formulated via w^*–convergence on the C^*–algebra $\mathcal{A}$. Thus we have to reformulate the stated convergence properties in a weakened form for the case of C^*–algebras.

Since we use the relative entropy of states on a C^*–algebra we specify the following notations (for an overview on relative entropy, see [23]):
For states ω_1, ω_2 on a C^*–algebra $\mathcal{A}$, $S(\omega_1|\omega_2)$ is the relative entropy between the unique normal extensions of ω_1, ω_2 in the universal enveloping von Neumann algebra $\mathcal{A}^{**}$ [25] (we use the notation and choice of sign as in [1]).
For a state ω on a C^*–algebra $\mathcal{A}$ and $h = h^* \in \mathcal{A}$, we denote by ω^h the state obtained from ω by perturbation with a selfadjoint element $h \in \mathcal{A}$. This state ω^h is uniquely determined by the condition that it maximizes the functional $\mathcal{S}(\mathcal{A}) \ni \sigma \to S(\sigma|\omega) - \langle \sigma\,;\, h\rangle$ and the maximum is denoted by $c(\omega, h) := S(\omega^h|\omega) - \langle \omega^h\,;\, h\rangle$, cf. ref. [18].

Proposition 3.2 *Let $\mathcal{A}$ be a C^*-algebra with identity, $(\omega_\alpha)_{\alpha\in\mathcal{I}} \subset \mathcal{S}(\mathcal{A})$ be a net with* $\text{w}^*\text{–}\lim_{\alpha\in\mathcal{I}} \omega_\alpha = \omega$, *and $(k_\alpha)_{\alpha\in\mathcal{I}} \subset \mathcal{A}_{sa}$ with $\lim_{\alpha\in\mathcal{I}} \|k_\alpha\| = 0$. Then it follows that* $\text{w}^*\text{–}\lim_{\alpha\in\mathcal{I}} \omega_\alpha^{k_\alpha} = \omega$.

PROOF: The following estimation of the relative entropy is valid:

$$\begin{aligned} 0 \;\geq\; -\frac{1}{2}\left\|\omega_\alpha - \omega_\alpha^{h_\alpha}\right\|^2 \;&\geq\; S(\omega_\alpha^{h_\alpha} \mid \omega_\alpha) \;=\; c(\omega_\alpha, h_\alpha) + \left\langle \omega_\alpha^{h_\alpha}\,;\, h_\alpha\right\rangle \\ &\geq\; -\left\langle \omega_\alpha\,;\, h_\alpha\right\rangle + \left\langle \omega_\alpha^{h_\alpha}\,;\, h_\alpha\right\rangle \;\geq\; -2\left\|h_\alpha\right\| . \end{aligned}$$

For the second inequality, see e.g. [23, Theorem 5.23]. The estimation of $c(\omega_\alpha, h_\alpha)$ is a generalization of the Peierls–Bogoliubov inequality, same reference, Chapter 12. The w^*–convergence of $\omega_\alpha^{h_\alpha}$ then follows immediately. ❑

Proposition 3.2 gives a criterion for the robustness of a limiting Gibbs state if the microscopically defined model is perturbed locally. We have to remark that the above condition is not at all necessary. A nontrivial situation is given for example if the limiting Gibbs state is an extremal state, which is invariant with respect to a given group of internal symmetries. Then all perturbations according to the Assumption 2.3 which are invariant under these symmetry transformations do not affect the limiting Gibbs state of the model [15]! Now we use the above stability condition for the further analysis of the interplay between local perturbations and limiting Gibbs states. This includes constructive as well as pure existence results (quasi–averages):

Proposition 3.3 (i) *Let* $h := (h_\Lambda)_{\Lambda\in\mathcal{L}}$ *be an approximately symmetric net of selfadjoint operators. Then the limiting Gibbs state of the perturbed system is given by:* w*–$\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h_\Lambda} = (\omega^\beta)^{h^\beta}$, *with* $h^\beta = \text{s--}\lim_{\Lambda\in\mathcal{L}} \Pi_\beta(h_\Lambda) \in \mathcal{M}_\beta \cap \mathcal{M}'_\beta$ *and* $(\omega^\beta)^{h^\beta}$ *being the state arising from* ω^β *by perturbation with* h^β*.* $\mathcal{M}_\beta$ *is the von Neumann algebra* $\Pi_\beta(\mathcal{A})''$ *and* $(\mathcal{H}_\beta, \Pi_\beta, \Omega_\beta)$ *is the GNS–representation of* ω^β*.*

(ii) *If* $\omega = \int_{\mathcal{S}(\mathcal{B})} \otimes\varphi\, \varrho(\varphi)\, d\mu_\beta(\varphi)$ *with a positive continuous function* ϱ*, satisfying* $\|\varrho\|_1 := \int_{\mathcal{S}(\mathcal{B})} |\varrho(\varphi)|\, d\mu_\beta(\varphi) = 1$*, then the net of local perturbations in Theorem 3.1 can be chosen in such a way that for* $\varrho_\Lambda : \mathcal{S}(\mathcal{B}) \mapsto \mathbb{R}, \varphi \to \varrho_\Lambda(\varphi) := e^{-\beta\langle\otimes\varphi\,;\,h_\Lambda\rangle}$ *it holds:*

$$\|\cdot\| - \lim_{\Lambda\in\mathcal{L}} \varrho_\Lambda = \varrho\,.$$

(For $f \in \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C})$ *it is* $\|f\| := \sup\{|f(\varphi)| \mid \varphi \in \mathcal{S}(\mathcal{B})\}$*.)*

(iii) *If* $\omega = \int_{\mathcal{S}(\mathcal{B})} \otimes\varphi\, \varrho(\varphi)\, d\mu_\beta(\varphi)$ *with a positive lower semicontinuous integrable function, satisfying* $\|\varrho\|_1 = 1$*, then the net of local perturbations in Theorem 3.1 can be chosen such that:*

$$\lim_{\Lambda\in\mathcal{L}} \varrho_\Lambda(\varphi) = \varrho(\varphi), \quad \forall\varphi \in \mathcal{S}(\mathcal{B})\,.$$

PROOF: (i) The proof runs as follows: Construct with the help of Theorem 2.6 an approximately symmetric net h' with $j(h) = j(h')$ such that w*–$\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h'_\Lambda} = (\omega^\beta)^{h^\beta}$. Then use Proposition 3.2 because h and h' only differ in terms vanishing in norm for large Λ. The details of the proof can be found in the one of Theorem 4.3 below, where a more general situation is treated, including non permutation invariant perturbations.

(ii) There exists a sequence of strictly positive functions $\varrho_n \in \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{R})$ with $\|\cdot\|$–$\lim_{n\to\infty} \varrho_n = \varrho$ and consequently

$$\|\cdot\|\text{--}\lim_{n\to\infty} \frac{\int_{\mathcal{S}(\mathcal{B})} \bigotimes\varphi\, \varrho_n(\varphi)\, d\mu_\beta(\varphi)}{\int_{\mathcal{S}(\mathcal{B})} \varrho_n(\varphi)\, d\mu_\beta(\varphi)} = \int_{\mathcal{S}(\mathcal{B})} \bigotimes\varphi\, \varrho(\varphi)\, d\mu_\beta(\varphi) = \omega\,. \tag{3.1}$$

Without loss of generality, the ϱ_n can be chosen such that

$$\|-\ln(\varrho_n)\| \le \ln(n)\,. \tag{3.2}$$

for n large enough.

For each $n \in \mathbb{N}$ there exists a $h_n \in \widetilde{\mathcal{Y}}$ with $e^{-\beta j(h_n)} = \varrho_n$. The approximately symmetric nets h_n satisfy $\lim_{\Lambda\in\mathcal{L}} \|h_{n,\Lambda}\| = \|-1/\beta \ln(\varrho_n)\|$ and thus there is an increasing sequence $(N_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with

$$\|h_{n,\Lambda}\| \le \frac{1}{\beta}\|-\ln(\varrho_n)\| + \frac{1}{n}\,, \tag{3.3}$$

$$\left\|e^{-\beta\langle\otimes\cdot\,;\,h_{n,\Lambda}\rangle} - \varrho_n\right\| \le \frac{1}{n}\,, \tag{3.4}$$

for all $\Lambda \in \mathcal{L}$, $|\Lambda| \geq N_n$.

According to (i), the limiting Gibbs states of the models locally defined by $(H_\Lambda + h_{n,\Lambda})_{\Lambda\in\mathcal{L}}$ exist and are given by $\omega_n^\beta := \int_{\mathcal{S}(\mathcal{B})} \otimes\varphi\, \varrho_n(\varphi)\, d\mu_\beta(\varphi) / \int_{\mathcal{S}(\mathcal{B})} \varrho_n(\varphi)\, d\mu_\beta(\varphi)$.

Using the metric $d(\cdot,\cdot)$ of the w^*–topology in $\mathcal{S}(\mathcal{A})$ ([26, Chapter 3.4], $\mathcal{A}$ is separable), there exists a family of regions $\Lambda_n \in \mathcal{L}$, $n \in \mathbb{N}$ with $|\Lambda_n| \geq N_n$, $\Lambda_n \subset \Lambda_{n+1}$, $\bigcup_{n\in\mathbb{N}} \Lambda_n = \mathbb{N}$, and $d\left(\omega_n^\beta, \omega^{\beta H_\Lambda + h_{n,\Lambda}}\right) \leq 1/2n$ for all $\Lambda_n \subseteq \Lambda \in \mathcal{L}$. Now set

$$h_\Lambda := h_{n,\Lambda}\,, \quad \forall \Lambda \in \mathcal{L} \text{ with } |\Lambda_n| \leq |\Lambda| < |\Lambda_{n+1}|\,. \tag{3.5}$$

The convergence of $\omega^{\beta H_\Lambda + h_\Lambda}$ follows immediatelly.

It remains to prove $\lim_{\Lambda\in\mathcal{L}} \|h_\Lambda\| / |\Lambda| = 0$ and the norm convergence of the ϱ_Λ. Take a $\Lambda \in \mathcal{L}$ with $N_n \leq |\Lambda_n| \leq |\Lambda| < |\Lambda_{n+1}|$:

$$\begin{aligned}
\frac{\|h_\Lambda\|}{|\Lambda|} &= \frac{\|h_{n,\Lambda}\|}{|\Lambda|} \leq \frac{1}{|\Lambda|}\left(\frac{1}{\beta}\|-\ln(\varrho_n)\| + \frac{1}{n}\right), \text{ with (3.3)},\\
&\leq \frac{1}{n}\left(\frac{1}{\beta}\|-\ln(\varrho_n)\| + \frac{1}{n}\right) \leq \frac{1}{n}\left(\frac{1}{\beta}\ln(n) + \frac{1}{n}\right), \text{ with (3.2)},\\
&\longrightarrow 0 \text{ for large } \Lambda\,.
\end{aligned}$$

To show the convergence of ϱ_Λ, take an arbitrary $\varphi \in \mathcal{S}(\mathcal{B})$:

$$\begin{aligned}
\left|e^{-\beta\langle\otimes\varphi\,;\,h_\Lambda\rangle} - \varrho(\varphi)\right| &\leq \left|e^{-\beta\langle\otimes\varphi\,;\,h_{n,\Lambda}\rangle} - \varrho_n(\varphi)\right| + |\varrho_n(\varphi) - \varrho(\varphi)|\\
&\leq \frac{1}{n} + \|\varrho_n - \varrho\| \text{ with (3.4)},\\
&\longrightarrow 0 \text{ for large } \Lambda,
\end{aligned}$$

(iii) Any positive lower semicontinuous function ϱ is the pointwise limit of a monotone increasing sequence of continuous functions $(\varrho_n)_{n\in\mathbb{N}}$ which may be assumed to be positive [27, §4]. Using the monotone convergence theorem, we have $\lim_{n\to\infty} \|\varrho - \varrho_n\|_1 = 0$ (consider ϱ, ϱ_n, $n \in \mathbb{N}$ as elements in $L^1(\mathcal{S}(\mathcal{B}), \mu_\beta)$) and $\lim_{n\to\infty} \|\varrho_n\|_1 = \|\varrho\|_1 = 1$. So we can replace ϱ_n by $\varrho_n / \|\varrho_n\|_1$, still being positive continuous functions with pointwise limit ϱ. Then we have for $\omega_n^\beta := \int_{\mathcal{S}(\mathcal{B})} \otimes\varphi\, \varrho_n(\varphi)\, d\mu_\beta(\varphi)$:

$$\|\cdot\|\text{-}\lim_{n\to\infty} \omega_n^\beta = \omega\,.$$

Just as above take a family $h_n := (h_{n,\Lambda})_{\Lambda\in\mathcal{L}}$ of local perturbations according to (ii) to approximate the states ω_n^β as limiting Gibbs states. Then construct the family of operators $(h_\Lambda)_{\Lambda\in\mathcal{L}}$, we are looking for, from h_n, $n \in \mathbb{N}$ as in Eqs. (3.3) – (3.5) with the help of the metric $d(\cdot,\cdot)$ in $\mathcal{S}(\mathcal{A})$. □

In the situations described in the Propositions 3.2 and 3.3 we have a maximum of information about the asymptotic behaviour of the constructed families of local perturbations. In the general case of Theorem 3.1 this information is almost completely lost. Before starting the proof of Theorem 3.1, we give the following

Lemma 3.4 *Let $\omega \in \mathcal{S}^P(\mathcal{A})$ and ν a regular probability measure on $\mathcal{S}(\mathcal{A})$ with* $\mathrm{supp}(\nu) \subseteq \mathrm{supp}(\mu_\omega) =: \mathcal{K}_\omega$. *Then there exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ of positive and normalized elements in $L^1(\mathcal{K}_\omega, \mu_\omega)$, such that*

$$\text{w}^*\text{–}\lim_{n\to\infty} \int_{\mathcal{K}_\omega} \bigotimes \varphi\, \varrho_n(\varphi)\, d\mu_\omega(\varphi) = \int_{\mathcal{K}_\omega} \bigotimes \varphi\, d\nu(\varphi)\,.$$

PROOF: Look at the C^*–algebra of continuous functions $\mathcal{C}(\mathcal{K}_\omega, \mathbb{C})$ with the state space $M^1_+(\mathcal{K}_\omega)$, the set of all positive normalized regular Borel measures on $\mathcal{K}_\omega$. Thus each positive and normalized $\varrho \in L^1(\mathcal{K}_\omega, \mu_\omega)$ defines a state ω^ϱ on this algebra by $\langle \omega^\varrho ; f\rangle := \int_{\mathcal{K}_\omega} f(\varphi)\varrho(\varphi)\, d\mu_\omega(\varphi)$. The set $\mathcal{N} := \{\omega^\varrho \mid \varrho \in L^1(\mathcal{K}_\omega, \mu_\omega)$ positive and normalized$\}$ is a full set of states, cf. [1, Definition 3.2.9], because we have $\langle \omega^\varrho ; f\rangle \geq 0$, $\forall \omega^\varrho \in \mathcal{N} \Longrightarrow f \geq 0$. This follows from [28, page 231] by decomposing an arbitrary continuous f into its positive and negative part.

Using [1, Proposition 3.2.10] it follows that $\mathcal{N}$ is w^*–dense in the state space of $\mathcal{C}(\mathcal{K}_\omega, \mathbb{C})$. Lemma 3.4 then follows from the separability and the continuity of $\mathcal{S}(\mathcal{B}) \ni \varphi \to \langle \otimes\varphi ; A\rangle$ for all $A \in \mathcal{A}$. ❑

PROOF OF THEOREM 3.1: Let μ_ω be absolutely continuous with respect to μ_β, i.e. the Radon–Nikodym derivative exists and is an element in $L^1(\mathcal{S}(\mathcal{B}), \mu_\beta)$. Using Proposition 3.3 (ii) we can construct a family of local perturbations $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ such that ω is the limiting Gibbs state of the locally perturbed system and $\|h_\Lambda\|$ has the asymptotic behaviour according to Definition 2.3. The proof runs exactly in the same way as in Proposition 3.3, using the fact that the continuous functions on $\mathcal{S}(\mathcal{B})$ are $\|\cdot\|_1$–dense in $L^1(\mathcal{S}(\mathcal{B}), \mu_\beta)$. We omit this step. Finally Theorem 3.1 follows with Lemma 3.4 and the above constructions, which serve to find the local perturbations. ❑

Corollary 3.5 *If the support of the central measure of the limiting Gibbs state of the unperturbed system $(H_\Lambda)_{\Lambda\in\mathcal{L}}$ is equal to the set of all product states with minimal free energy density, then for each permutation invariant state ω, minimizing the free energy density, there exists a family of local perturbations according to Definition 2.3 such that ω is the limiting Gibbs state of the locally perturbed system.*

We see that a large set of limiting Gibbs states can be calculated by means of a sub–extensive family of local perturbations not influencing the thermodynamical density functionals. But, except for the situation of local perturbations arising from approximately symmetric nets, there is no concrete information on the perturbations available, besides the asymptotic behaviour of some expectation values. Especially in the case where a limiting Gibbs state should be perturbed in such a way that the central measure becomes singular relative to the unperturbed one, there is hardly any information on the local perturbations accessible. Therefore, we will give another scheme to construct local perturbations leading to limiting Gibbs states with a singular central measure. The interesting point is that this can be done in terms of a scaling function, i.e. only the strength of a local perturbation is varied, but each local operator itself remains the same. We will replace an approximately symmetric net $h = (h_\Lambda)_{\Lambda\in\mathcal{L}}$ by another family of local perturbations $h' := (n_\Lambda h_\Lambda)_{\Lambda\in\mathcal{L}}$, where $n_\Lambda \in \mathbb{R}^+$ and n_Λ tends to infinity for large regions $\Lambda \in \mathcal{L}$ (but h' still fulfills the conditions of Definition 2.3).

Proposition 3.6 *Let $h := (h_\Lambda)_{\Lambda\in\mathcal{L}}$ be an approximately symmetric net such that either $j(h)|_{\mathcal{K}_\beta}$ attains a unique minimum m in $\varphi_0 \in \mathcal{K}_\beta$ or the set $N := \{\varphi \in \mathcal{K}_\beta \mid [j(h)](\varphi) = m\}$ fulfills $\mu_\beta(N) > 0$. Then, there exists a net $\Lambda \ni \mathcal{L} \to n_\Lambda \in \mathbb{R}^+$ such that for all $\varepsilon > 0$:*

$$\lim_{\Lambda\in\mathcal{L}} \frac{n_\Lambda}{|\Lambda|^\varepsilon} = 0 \tag{3.6}$$

and $\otimes\varphi_0$, resp. $\int_N \otimes\varphi d\mu_\beta(\varphi)/\mu_\beta(N)$ is the limiting Gibbs state of the system $(H_\Lambda + n_\Lambda h_\Lambda)_{\Lambda\in\mathcal{L}}$.

Note that this condition allows a nearly arbitrary slow growing of the perturbation h_Λ but nevertheless $\|n_\Lambda h_\Lambda\|$ in general tends to infinity while $\lim_{\Lambda\in\mathcal{L}} \frac{\|n_\Lambda h_\Lambda\|}{|\Lambda|} = \lim_{\Lambda\in\mathcal{L}} \frac{n_\Lambda}{|\Lambda|} \lim_{\Lambda\in\mathcal{L}} \|h_\Lambda\| = 0$.

PROOF: The main part of the proof consists of the construction of a convenient sequence of permutation invariant states with limit $\otimes\varphi_0$ resp. $\int_N \otimes\varphi d\mu_\beta(\varphi)/\mu_\beta(N)$. Then we can repeat the tricks in the construction of families of local perturbations as before. Without loss of generality we assume that $\exp(-\beta m) = 1$ and set $\varrho := \exp(-\beta j(h))$. With

$$(\omega^\beta)^{nh^\beta} = \frac{\int_{S(\mathcal{B})} \bigotimes \varphi\, \varrho^n(\varphi)\, d\mu_\beta(\varphi)}{\int_{S(\mathcal{B})} \varrho^n(\varphi)\, d\mu_\beta(\varphi)}, \quad \forall n \in \mathbb{N},$$

it is w*-$\lim_{n\to\infty}(\omega^\beta)^{nh^\beta} = \otimes\varphi_0$ resp. w*-$\lim_{n\to\infty}(\omega^\beta)^{nh^\beta} = \int_N \otimes\varphi d\mu_\beta(\varphi)/\mu_\beta(N)$. To show this, analyze the convergence of $\varrho^n(\varphi)/\int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi)$ for $\varphi \in \mathcal{K}_\beta$. Since $\varrho(\varphi) \le 1$ for all $\varphi \in \mathcal{K}_\beta$ it follows

$$\begin{aligned}
\int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi) &\ge \int_{\mathcal{K}_\beta} \varrho^{n+1}(\varphi)d\mu_\beta(\varphi)\,, \\
\Big(\int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi)\Big)^{\frac{1}{n}} =: \|\varrho\|_n &\le \|\varrho\|_{n+1}\,, \quad \text{[28, Page 106]}, \\
\lim_{n\to\infty} \|\varrho\|_n &= \|\varrho\|_\infty\,, \quad \text{[28, Page 105]}.
\end{aligned}$$

From these estimates we obtain

$$\lim_{n\to\infty} \int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi) = \mu_\beta(N)$$

and

$$\frac{\varrho^n(\varphi)}{\int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi)} \xrightarrow{n\to\infty} \begin{cases} \infty & \varphi \in N \\ 0 & \varphi \in \mathcal{K}_\beta \backslash N \end{cases} \quad \text{if } \mu_\beta(N) = 0\,,$$

resp.

$$\frac{\varrho^n(\varphi)}{\int_{\mathcal{K}_\beta} \varrho^n(\varphi) d\mu_\beta(\varphi)} \xrightarrow{n\to\infty} \begin{cases} \frac{1}{\mu_\beta(N)} & \varphi \in N \\ 0 & \varphi \in \mathcal{K}_\beta \backslash N \end{cases} \quad \text{if } \mu_\beta(N) > 0 .$$

Since $\mathcal{S}(\mathcal{B}) \ni \varphi \to \langle \otimes\varphi \,;\, A\rangle$ is continuous for all $A \in \mathcal{A}$, we have to look at the convergence of $\int_{\mathcal{K}_\beta} f(\varphi)\varrho^n(\varphi)d\mu_\beta(\varphi)/\int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi)$ for continuous $f \in \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C})$. If $\mu_\beta(N) > 0$, the convergence is an immediate consequence of [28, Proposition 3.1.5] and we have w*-$\lim_{n\to\infty}(\omega^\beta)^{nh^\beta} = \int_N \otimes\varphi d\mu_\beta(\varphi)/\mu_\beta(N)$.

In the case of $\mu_\beta(N) = 0$ and $N = \{\varphi_0\}$, after some elementary estimates we find that $\lim_{n\to\infty} \frac{\int_{\mathcal{K}_\beta} f(\varphi)\varrho^n(\varphi)d\mu_\beta(\varphi)}{\int_{\mathcal{K}_\beta} \varrho^n(\varphi)d\mu_\beta(\varphi)} = f(\varphi_0)$, i.e. w*-$\lim_{n\to\infty}(\omega^\beta)^{nh^\beta} = \otimes\varphi_0$.

Concerning the construction of the net $(n_\Lambda)_{\Lambda\in\mathcal{L}}$ we can repeat all estimates as previously done in the proofs of Proposition 3.3 to find the local perturbations. All estimates, occurring there are of the form $d(\omega^{\beta H_\Lambda + nh_\Lambda}, \omega) < \varepsilon$ for some state ω and all Λ with $|\Lambda|$ greater than some $N_n \in \mathbb{N}$. Thus they remain valid, even if Λ become arbitrarily large. In this way the net $\Lambda \to n_\Lambda$ may be constructed such that it shows the asymptotics according to Eq. (3.6). □

We will finish this section with a remark on lattice systems consisting of a finite number of coupled quantum systems. Such a system may be described by using a quasi-local algebra $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, each of the $\mathcal{A}_i$ being the infinite tensor product of finite quantum systems $\mathcal{B}_i$. Since an equivalent formulation of Theorem 2.6 is still valid for mean-field models of such systems, all foregoing results can be obtained as well in this case. Then the local perturbations may be considered as Hamiltonians of the interaction between these systems. An example of such a model is the Josephson-junction of two BCS-superconductors below a critical temperature, see Sect. 5.

4 Local Perturbations with Quasi-Symmetric Nets

In the previous sections we discussed local perturbations $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ with each h_Λ being invariant with respect to all permutations of Λ. There, of course, each limiting Gibbs state of a mean-field model with local Hamiltonians $(H_\Lambda + h_\Lambda)_{\Lambda\in\mathcal{L}}$ is a permutation invariant state on the quasi-local algebra $\mathcal{A}$. We will generalize now some of the convergence properties to the case of not necessarily permutation invariant perturbations. We will not treat in detail the question of the calculation of arbitrary quasi-averages, but will restrict our attention to the case, where the connection between perturbation and limiting Gibbs state can be made explicit, i.e. we consider a generalization of Proposition 3.3 (i). The extension of approximately symmetric nets as local perturbations from the permutation invariant to the general case is performed by the introduction of the so-called *quasi-symmetric nets* [11]. This terminology is also based on the definition of a well defined asymptotic behaviour for large regions. We show, that for each quasi-symmetric net $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ the analogue to Proposition 3.3 (i) is valid, i.e. the limiting Gibbs state of the model, locally perturbed with a quasi–symmetric net exists and is the perturbation of the original limiting Gibbs state with the strong limit of the local perturbations. Then this state is no longer permutation invariant.

To define the nets of local perturbations over the set $\mathcal{L}$ of finite regions Λ we need more structure, than given by the ordering on $\mathcal{L}$ in terms of set–inclusion. Therefore a *tagging* of a finite region Λ is introduced to mark regions where the permutation invariance is not satisfied. From now on we regard pairs $(\Lambda, \Lambda^T) \in \mathcal{L} \times \mathcal{L}$ with $\Lambda^T \subseteq \Lambda$. Λ^T is called the tagging of $\Lambda \in \mathcal{L}$. Now choose a tagging on the lattice, i.e. specify for all $\Lambda \in \mathcal{L}$ a $\Lambda^T \subseteq \Lambda$ in such a way that $|\Lambda^T|/|\Lambda|$ tends to zero in the limit of large Λ , $\Lambda_1^T \subseteq \Lambda_2^T$ for all $\Lambda_1 \subset \Lambda_2$, and $\cup_{\Lambda \in \mathcal{L}} \Lambda^T = \mathbb{N}$. This tagging of the lattice is assumed to be fixed for the rest of the section.

Now consider $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \in \mathcal{L}$ and define the operator $j^{\Lambda_1}_{\Lambda_3\Lambda_2} : \mathcal{A}_{\Lambda_2} \mapsto \mathcal{A}_{\Lambda_3}$ which symmetrizes elements of $\mathcal{A}_{\Lambda_2}$ in $\mathcal{A}_{\Lambda_3}$, but excludes the region Λ_1 from the symmetrization:

$$j^{\Lambda_1}_{\Lambda_3\Lambda_2} := \frac{|\Lambda_3 \backslash \Lambda_2|!}{|\Lambda_3 \backslash \Lambda_1|!} \sum \Theta_\sigma \,. \tag{4.1}$$

The sum runs over all injective mappings $\sigma : \Lambda_2 \mapsto \Lambda_3$, leaving Λ_1 pointwise invariant. If Λ_1 is chosen to be equal to Λ_2, the resulting operator is the canonical embedding of $\mathcal{A}_{\Lambda_2}$ into $\mathcal{A}_{\Lambda_3}$: $\mathcal{A}_{\Lambda_2} \ni A_{\Lambda_2} \longrightarrow j^{\Lambda_2}_{\Lambda_3\Lambda_2}(A_{\Lambda_2}) = A_{\Lambda_2} \otimes \mathbb{1}_{\Lambda_3\backslash\Lambda_2}$. If $\Lambda_1 = \emptyset$, an element of $A_{\Lambda_2} \in \mathcal{A}_{\Lambda_2}$ becomes completely symmetrized, i.e. $j^{\emptyset}_{\Lambda_3\Lambda_2}(A_{\Lambda_2})$ is invariant with respect to all permutations in Λ_3 (we see that this notation agrees with the one introduced in Definition 2.1). Finally, for a tagged set $(\Lambda_1, \Lambda_1^T) \in \mathcal{L} \times \mathcal{L}$ and $\Lambda_2 \in \mathcal{L}$ with $\Lambda_1 \subseteq \Lambda_2$ we write

$$j_{\Lambda_2\Lambda_1} := j^{\Lambda_1^T}_{\Lambda_2\Lambda_1} : \mathcal{A}_{\Lambda_1} \mapsto \mathcal{A}_{\Lambda_2} \,. \tag{4.2}$$

The limit properties of the operators $j_{\Lambda_2\Lambda_1}$ for large tagged regions are the main tool for working with quasi–symmetric nets. Let us collect the basic definitions and properties [29, 11]:

Definition and Proposition 4.1 *Let $\mathcal{L} \ni \Lambda \rightarrow h_\Lambda$ be a net with $h_\Lambda \in \mathcal{A}_\Lambda$ for all $\Lambda \in \mathcal{L}$.*

(i) *$h = (h_\Lambda)_{\Lambda \in \mathcal{L}}$ is called Ω–symmetric ($h \in \widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$) with $\Omega \in \mathcal{L}$, if* $\lim_{\Lambda_1 \in \mathcal{L}} \limsup_{\Lambda_2 \in \mathcal{L}} \left\| h_{\Lambda_2} - j^{\Omega}_{\Lambda_2\Lambda_1}(h_{\Lambda_1}) \right\| = 0.$

(ii) *$h = (h_\Lambda)_{\Lambda \in \mathcal{L}}$ is called quasi–symmetric ($h \in \widetilde{\mathcal{Y}}(\mathcal{A})$) if* $\lim_{\Lambda_1 \in \mathcal{L}} \limsup_{\Lambda_2 \in \mathcal{L}} \| h_{\Lambda_2} - j_{\Lambda_2\Lambda_1}(h_{\Lambda_1}) \| = 0.$

The following criteria are valid [11, Lemma 2.3]:
$h = (h_\Lambda)_{\Lambda \in \mathcal{L}}$ is a quasi–symmetric net iff for all $\varepsilon > 0$ there exists a tagged set Ω (depending on ε) and a $g_\Omega \in \mathcal{A}_\Omega$, such that

$$\limsup_{\Lambda \in \mathcal{L}} \| h_\Lambda - j_{\Lambda\Omega}(g_\Omega) \| \leq \varepsilon \,. \tag{4.3}$$

h is Ω_0-symmetric, iff for all $\varepsilon > 0$ there exists a $\Omega \in \mathcal{L}$ (depending on ε) and a $g_\Omega \in \mathcal{A}_\Omega$, such that

$$\limsup_{\Lambda \in \mathcal{L}} \left\| h_\Lambda - j^{\Omega_0}_{\Lambda\Omega}(g_\Omega) \right\| \leq \varepsilon . \tag{4.4}$$

Note that $\widetilde{\mathcal{Y}}(\mathcal{A}_\Omega) \subset \widetilde{\mathcal{Y}}(\mathcal{A})$, due to Proposition 2.4 in [11] and that a permutation invariant Ω-symmetric net for $\Omega = \emptyset$ is an approximately symmetric net as introduced in Section 2.

In analogy to the relations for approximately symmetric nets $\|h\| := \lim_{\Lambda \in \mathcal{L}} \|h_\Lambda\|$ exists for all quasi-symmetric nets h and defines a C^*-seminorm on $\widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$, resp. $\widetilde{\mathcal{Y}}(\mathcal{A})$. If we equip the nets with the pointwise algebraic operations and factorize by the subspace of zero seminorm nets, we get a C^*-algebra which is isomorphic to $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A}) \cong \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbb{C}) \otimes \mathcal{A}$ (the continuous functions on $\mathcal{S}(\mathcal{B})$ with values in $\mathcal{A}$). The isomorphism is a generalization of the map j from Eq. (2.2):

$$j : \widetilde{\mathcal{Y}}(\mathcal{A}) \to \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A}) \,, h \to [j(h)](\varphi) := \lim_{\Lambda \in \mathcal{L}} \mathbb{E}^{\varphi}_{\Lambda\setminus\Lambda^T}(h_\Lambda) \,, \quad \forall \varphi \in \mathcal{S}(\mathcal{B}) \,. \tag{4.5}$$

$\mathbb{E}^{\varphi}_{\Lambda\setminus\Lambda^T} : \mathcal{A}_\Lambda \to \mathcal{A}_{\Lambda^T}$ is defined by means of the expectation values for all $\sigma \in \mathcal{S}(\mathcal{A}_{\Lambda^T})$: $\mathcal{A}_\Lambda \ni A \to \langle\, \sigma \,;\, \mathbb{E}^{\varphi}_{\Lambda\setminus\Lambda^T}(A) \,\rangle := \langle \sigma \otimes (\otimes_{i \in \Lambda\setminus\Lambda^T} \varphi) \,;\, A \rangle$. The map j is surjective and the limit in Eq. (4.5) is uniform in $\varphi \in \mathcal{S}(\mathcal{B})$. j is an isometry in the sense that $\|h\| = \lim_{\Lambda \in \mathcal{L}} \|h_\Lambda\| = \|j(h)\| = \sup\{\|[j(h)](\varphi)\| \mid \varphi \in \mathcal{S}(\mathcal{B})\}$.

The surjective mapping j leads to the strong convergence of a quasi-symmetric net in the partially universal representation to the folium generated by the permutation invariant states. This is exactly the same property as the strong convergence of approximately symmetric nets as stated in Proposition 2.2:

Proposition 4.2 *For all quasi-symmetric nets $(h_\Lambda)_{\Lambda \in \mathcal{L}}$, the limit $\text{s-}\lim_{\Lambda \in \mathcal{L}} \Pi_P(h_\Lambda)$ exists in $\mathcal{M}_P$ and is an element of a subalgebra isomorphic to $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A})$. Conversely every element of $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A})$ considered as element of $\mathcal{M}_P$ is the strong limit of the net $\Pi_P(h_\Lambda)$ for some $(h_\Lambda)_{\Lambda \in \mathcal{L}} \in \widetilde{\mathcal{Y}}(\mathcal{A})$*

SKETCH OF THE PROOF: First we show the convergence of special Ω-symmetric nets h_Λ. Let $\Lambda_0 \in \mathcal{L}$ with $\Lambda_0 \cap \Omega = \emptyset$ and $\Lambda \supseteq \Omega \cup \Lambda_0$. Then define $h_\Lambda := j^{\Omega}_{\Lambda\,\Omega\cup\Lambda_0}(A_\Omega \otimes y_{\Lambda_0}) = A_\Omega \otimes j^{\emptyset}_{\Lambda\setminus\Omega\,\Lambda_0}(y_{\Lambda_0})$. This means, that h_Λ is separated into the tagged part $A_\Omega \in \mathcal{A}_\Omega$ and a symmetric net $j^{\emptyset}_{\Lambda\setminus\Omega\,\Lambda_0}(y_{\Lambda_0})$ on $\otimes_{j \in \mathbb{N}\setminus\Omega^T} \mathcal{B}_i$; thus it is convergent (use Prop. 2.2).

An arbitrary Ω-symmetric net can be approximated by nets $(j^{\Lambda_0}_{\Lambda\,\Omega\cup\Lambda_0}(y_{\Omega\cup\Lambda_0}))_{\Lambda \in \mathcal{L}}$ with fixed $y_{\Omega\cup\Lambda_0} \in \mathcal{A}_{\Omega\cup\Lambda_0}$ up to an arbitrarily small norm difference in the limit of large Λ, comp. Eq. (4.4). Using this fact and [11, Theorem 2.5 (ii)], the convergence of an arbitrary Ω-symmetric net is proved. The limit is an element of a subalgebra of $\mathcal{M}_P$, isomorphic to $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A}_\Omega)$. The convergence of a quasi-symmetric net in $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A})$ finally results from Eq. (4.3). □

Now we can generalize Proposition 3.3 (i) to local perturbations, which no longer have to be permutation invariant. The limiting Gibbs state of the locally perturbed mean-field model is expressed in terms of the perturbation of the limiting Gibbs state of the unperturbed model by the strong limit of the local perturbations:

Theorem 4.3 *Let $(H_\Lambda)_{\Lambda\in\mathcal{L}}$ be the local Hamiltonians of an unperturbed mean-field model according to the Assumption 2.5 and let $h = (h_\Lambda)_{\Lambda\in\mathcal{L}}$ be a quasi-symmetric net with $h_\Lambda = h_\Lambda^*$, $\forall\Lambda \in \mathcal{L}$. Then there is an unique limiting Gibbs state of the model with local Hamiltonians $(H_\Lambda + h_\Lambda)_{\Lambda\in\mathcal{L}}$, given by*

$$\text{w*-}\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h_\Lambda} = (\omega^\beta)^{h^\beta},$$

with $h^\beta := \text{s-}\lim_{\Lambda\in\mathcal{L}} \Pi_\beta(h_\Lambda) \in \mathcal{M}_\beta$.

PROOF: The idea of the proof is, to construct some $h' \in \widetilde{\mathcal{Y}}(\mathcal{A})$ with $j(h') = j(h)$ and $\text{w*-}\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h'_\Lambda} = (\omega^\beta)^{h^\beta}$. Then, the w^*-convergence of $\omega^{\beta H_\Lambda + h_\Lambda}$ follows from Proposition 3.2. The construction of h' runs stepwise, by considering perturbations $h \in \mathcal{Y}(\mathcal{A}_\Omega)$, a special subset of $\widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$, then turning towards arbitrary elements in $\widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$ and finally reaching $\widetilde{\mathcal{Y}}(\mathcal{A})$.

(i) Let $\Omega \in \mathcal{L}$. $\mathcal{Y}(\mathcal{A}_\Omega)$ is defined to be the set of all $h \in \widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$ for which a $\Lambda_0 \in \mathcal{L}$, $\Omega \subset \Lambda_0$ exists with:
1.) $h_\Lambda = j^\Omega_{\Lambda\Lambda_0}(g_{\Lambda_0})$ for some $g_{\Lambda_0} \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda \supseteq \Lambda_0$.
2.) If $\Lambda \in \mathcal{L}$ with $|\Lambda_0| \le |\Lambda|$ and $\Lambda_0 \not\subseteq \Lambda$, then h_Λ shall satisfy $h_\Lambda = j^\emptyset_{\Lambda\sigma(\Lambda_0)}\Theta_\sigma(g_{\Lambda_0})$ with an permutation $\sigma \in \mathcal{P}(\Lambda \cup \Lambda_0)$ with $\sigma(\Lambda_0) \subseteq \Lambda$.

The second condition in the definition of $\mathcal{Y}(\mathcal{A}_\Omega)$ is only of technical relevance. The character of nets $h \in \mathcal{Y}(\mathcal{A}_\Omega)$ can be found in the following example: For $\Omega \subset \Lambda_0 \subset \Lambda$, h_Λ is the linear combination of $j^\Omega_{\Lambda\Lambda_0}(A_\Omega \otimes g_{\Lambda_0\setminus\Omega})$ with $A_\Omega \in \mathcal{A}_\Omega$ and $g_{\Lambda_0\setminus\Omega} \in \mathcal{A}_{\Lambda_0\setminus\Omega}$. With $j^\Omega_{\Lambda\Lambda_0}(A_\Omega \otimes g_{\Lambda_0\setminus\Omega}) = A_\Omega \otimes j^\emptyset_{\Lambda\setminus\Omega\,\Lambda_0\setminus\Omega}(g_{\Lambda_0\setminus\Omega})$ we see that these nets are the analogue to symmetric nets in the permutation invariant case.

Obviously for each element $h \in \mathcal{Y}(\mathcal{A}_\Omega)$, there exists a net $(h'_\Lambda)_{\Lambda\in\mathcal{L}}$, each h'_Λ being the same polynomial in mean-field operators $m_\Lambda(x)$ and elements of $\mathcal{A}_\Omega$, with

$$\lim_{\Lambda\in\mathcal{L}} \|h_\Lambda - h'_\Lambda\| = 0.$$

Application of Theorem 2.6 and Proposition 3.2 for $h \in \mathcal{Y}(\mathcal{A}_\Omega)$ gives:

$$\text{w*-}\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h_\Lambda} = (\omega^\beta)^{h^\beta} = \text{w*-}\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h'_\Lambda}. \tag{4.6}$$

(ii) With Eq. (4.4) in Proposition 4.1 it follows that for all $h \in \widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$ and all $\varepsilon > 0$, there is a $g \in \mathcal{Y}(\mathcal{A}_\Omega)$ and $\Lambda_0 \in \mathcal{L}$ such that

$$\|h_\Lambda - g_\Lambda\| \le \varepsilon, \quad \text{for all } \Lambda \supseteq \Lambda_0.$$

Now, let h be an arbitrary element in $\widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$. Then there exists a sequence $(h_n)_{n\in\mathbb{N}} \subset \mathcal{Y}(\mathcal{A}_\Omega)$, such that

$$\|j(h) - j(h_n)\| \le \frac{1}{n}, \tag{4.7}$$

i.e. $\|\cdot\|$–$\lim_{n\to\infty} j(h_n) = j(h)$ and $\|\cdot\|$–$\lim_{n\to\infty} h_n^\beta = h^\beta$ with the strong limits h^β, h_n^β in $\mathcal{M}_\beta$. From [24, Theorem 1.1] it follows that $\|\cdot\|$–$\lim_{n\to\infty}(\omega^\beta)^{h_n^\beta} = (\omega^\beta)^{h^\beta}$. By the help of the techniques as used in the proof of Theorem 3.3, one can construct now a net $h' \in \widetilde{\mathcal{Y}}(\mathcal{A}_\Omega)$ with the required convergence properties, i.e. $j(h) = j(h')$ and w*–$\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h'_\Lambda} =$ w*–$\lim_{\Lambda\in\mathcal{L}} \omega^{\beta H_\Lambda + h_\Lambda} = (\omega^\beta)^{h_\beta}$

(iii) Since all elements $j(h) \in \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathcal{A})$, $h \in \widetilde{\mathcal{Y}}(\mathcal{A})$ are approximated uniformly by $j(h_n)$, each $h_n \in \widetilde{\mathcal{Y}}(\mathcal{A}_{\Lambda_n})$ for some $\Lambda_n \in \mathcal{L}$, the steps in (ii) can be repeated with replacing elements in $\mathcal{Y}(\mathcal{A}_\Omega)$ by elements in $\widetilde{\mathcal{Y}}(\mathcal{A}_{\Lambda_n})$ for variable $\Lambda_n \in \mathcal{L}$ (they can be chosen increasingly). Again a new family of local perturbations $h' = (h'_\Lambda)_{\Lambda\in\mathcal{L}}$ is constructed as above. The model, locally perturbed with $(h'_\Lambda)_{\Lambda\in\mathcal{L}}$ leads to the unique limiting Gibbs state $(\omega^\beta)^{h^\beta}$. After having proved the quasi–symmetry of h' and $j(h) = j(h')$, the w^*–convergence of the local Gibbs states $\omega^{\beta H_\Lambda + h_\Lambda}$ follows with Proposition 3.2 . □

5 Applications

5.1 The Inhomogeneous BCS–Model

Using the above introduced methods, it is possible to treat a certain kind of inhomogeneous BCS–model with local Hamiltonians

$$K_\Lambda := \sum_{k\in\Lambda} \varepsilon_k \left(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}\right) - \frac{1}{|\Lambda|}\sum_{k,k'\in\Lambda} g_{kk'}\, c^*_{k\uparrow}c^*_{-k\downarrow}c_{-k'\downarrow}c_{k'\uparrow}\,, \tag{5.1}$$

where the summation runs over a region in momentum space near the Fermi–surface and $c^*_{k\sigma}$ is the creation operator of the corresponding Bloch wave function. There exists a variety of results on the thermodynamical properties of this model, which determine for suitable chosen parameters ε_k, $g_{kk'}$ a phase transition at some critical inverse temperature β_c [30, 31]. But in contrast to the homogeneous strong coupling situation with $\varepsilon_k = \varepsilon$ and $g_{kk'} = g > 0$ [32, 33] the limiting Gibbs state is not calculated explicitly.

The Hamiltonians in Eq. (5.1) can be transformed into our frame by averaging ε_k, resp. $g_{kk'}$ over the lattice, i.e.

$$g = \lim_{\Lambda\in\mathcal{L}} \frac{1}{|\Lambda|^2}\sum_{k,k'\in\Lambda} g_{kk'} > 0\,, \quad \text{and} \quad \varepsilon = \lim_{\Lambda\in\mathcal{L}} \frac{1}{|\Lambda|}\sum_{k\in\Lambda} \varepsilon_k\,.$$

Making explicit the Jordan–Wigner–representation of the CAR–algebra on the lattice $\mathbb{N}$ with $\mathcal{B} = \mathbb{M}_2(\mathbb{C})$, resp. $\mathcal{B} = \mathbb{M}_4(\mathbb{C})$ if we combine electrons with opposite momenta and spin, we find that the model with the averaged Hamiltonian

$$H_\Lambda := \sum_{k\in\Lambda} \varepsilon\,(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}) - \frac{1}{|\Lambda|}\sum_{k,k'\in\Lambda} g\, c^*_{k\uparrow}c^*_{-k\downarrow}c_{-k'\downarrow}c_{k'\uparrow}$$

satisfies the condition in our Assumption 2.5, i.e. is a mean–field model with a unique limiting Gibbs state ω^β, which is given below the critical temperature by

$$\omega^\beta = \int_{[0,2\pi]} \omega^\beta_\vartheta \frac{d\vartheta}{2\pi},$$

with the permutation invariant product states ω^β_ϑ on $\mathcal{A}$. The density matrix $\varrho^\vartheta_\Lambda$ of its local restriction to $\mathcal{A}_\Lambda$ becomes

$$\varrho^\vartheta_\Lambda = \frac{\exp\{ -\beta(\sum_{k\in\Lambda}\varepsilon\,(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}) - \Delta\sum_{k\in\Lambda}(e^{-i\vartheta}c^*_{k\uparrow}c^*_{-k\downarrow} + e^{i\vartheta}c_{-k\downarrow}c_{k\uparrow}))\}}{\mathrm{tr}(\exp\{ -\beta(\sum_{k\in\Lambda}\varepsilon\,(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}) - \Delta\sum_{k\in\Lambda}(e^{-i\vartheta}c^*_{k\uparrow}c^*_{-k\downarrow} + e^{i\vartheta}c_{-k\downarrow}c_{k\uparrow}))\})}, \quad (5.2)$$

where Δ is the positive solution of the well known gap equation.

The inhomogeneities will be specified by the requirement of quasi–symmetry of the net

$$\Lambda \longrightarrow h_\Lambda := \sum_{k\in\Lambda}(\varepsilon_k - \varepsilon)\,(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}) - \frac{1}{|\Lambda|}\sum_{k,k'\in\Lambda}(g_{kk'} - g)\, c^*_{k\uparrow}c^*_{-k\downarrow}c_{-k\downarrow}c_{k\uparrow}. \quad (5.3)$$

This includes especially h_Λ with

$$\begin{aligned} h^\beta &:= \operatorname*{s-lim}_{\Lambda\in\mathcal{L}} \Pi_\beta(h_\Lambda) \\ &= \Big(\lim_{\Lambda\in\mathcal{L}}\sum_{k\in\Lambda}(\varepsilon_k - \varepsilon)\Pi_\beta(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow})\Big) \\ &\quad -\frac{\Delta}{g}\Big\{\int^\oplus_{[0,2\pi]} e^{-i\vartheta}\mathbb{1}_\vartheta \frac{d\vartheta}{2\pi}\Big(\lim_{\Lambda\in\mathcal{L}}\sum_{k\in\Lambda}\delta g_k \Pi_\beta(c^*_{k\uparrow}c^*_{-k\downarrow})\Big) + h.c.\Big\}. \end{aligned}$$

In general δg_k are complex numbers, since we have not demanded that $g_{kk'} \in \mathbb{R}$. By use of Theorem 4.3 we calculate the limiting Gibbs state of the inhomogeneous model as perturbation of the homogeneous limiting Gibbs state ω^β with h^β. Evaluation of the perturbational expansion gives for the density matrix of the restriction of $(\omega^\beta)^{h^\beta}$ to $\mathcal{A}_\Lambda$

$$\varrho'^\vartheta_\Lambda = \frac{e^{-\beta(\sum_{k\in\Lambda}\varepsilon_k(c^*_{k\uparrow}c_{k\uparrow}+c^*_{-k\downarrow}c_{-k\downarrow}) - \sum_{k\in\Lambda}\Delta_k(e^{-i\vartheta_k}c^*_{k\uparrow}c^*_{-k\downarrow}+e^{i\vartheta_k}c_{-k\downarrow}c_{k\uparrow}))}}{\mathrm{tr}(e^{-\beta(\sum_{k\in\Lambda}\varepsilon_k(c^*_{k\uparrow}c_{k\uparrow}+c^*_{-k\downarrow}c_{-k\downarrow}) - \sum_{k\in\Lambda}\Delta_k(e^{-i\vartheta_k}c^*_{k\uparrow}c^*_{-k\downarrow}+e^{i\vartheta_k}c_{-k\downarrow}c_{k\uparrow}))})}, \quad (5.4)$$

with $\Delta_k = \left|1 + \frac{\delta g_k}{g}\right|\Delta$ and $\vartheta_k = \vartheta + \arg(\delta g_k)$.

This way of determining a limiting Gibbs state is remarkable due to the following point: The representation Π_β induced by the homogeneous model fixes the collective features, such as symmetry breaking resp. condensed particles in the macroscopical system, whereas the microscopical details of the model are incorporated via perturbation theory *in this* representation. Such a state is the key to discuss macroscopical (classical) effects against the background of nontrivial microscopical features of the model. The corresponding KMS–automorphsim group is accessible and via unitary implementation the spectrum and eigenstates of the corresponding generator can be discussed as quasiparticle excitations. The detailed analysis of these questions is in preparation [34].

5.2 Josephson–Junctions

The calculation of the limiting Gibbs state of weakly coupled superconductors allows to determine a representation and an interaction of two superconductors, where the Josephson relations for tunneling current and the phase difference become operator valued expressions in the center of the corresponding von Neumann algebra [35]. If the tunneling interaction of the finite system is determined by the Hamiltonian

$$h_\Lambda := \frac{g_s}{|\Lambda_1|^2|\Lambda_2|^2} \sum_{k_1\in\Lambda_1, k_2\in\Lambda_2} (c^*_{k_1\uparrow}c^*_{-k_1\downarrow}c_{-k_2\downarrow}c_{k_2\uparrow} + c_{-k_1\downarrow}c_{k_1\uparrow}c^*_{k_2\uparrow}c^*_{-k_2\downarrow})$$

($\Lambda = (\Lambda_1, \Lambda_2)$, with finite regions Λ_1, Λ_2 in the momentum spaces of the two superconductors) and the single superconductors are described as in the above example, we find for the limiting Gibbs state of the coupled system:

$$\omega^\beta = \int_{[0,2\pi]} \omega_{\delta\vartheta} e^{-\xi - 2\beta g_s \frac{\Delta_1}{g_1}\frac{\Delta_2}{g_2}\cos(\delta\vartheta)} \frac{d\delta\vartheta}{2\pi},$$

with the normalization $e^{-\xi}$ and

$$\omega_{\delta\vartheta} = \int_{[0,2\pi]} \omega^1_\vartheta \otimes \omega^2_{\vartheta+\delta\vartheta} \frac{d\vartheta}{2\pi} \in \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2).$$

ω^1_ϑ, ω^2_ϑ are states as in Eqs. (5.2), resp. (5.4) for the two sides of the junction. The derivation of the Josephson relations principally requires the variability of the phase difference $\delta\vartheta$ between the two superconductors, which is guaranteed in this representation. But if we consider the special internal symmetries of such a model (i.e. the invariance under simultaneous gauge transformations in the two subsystems, comp. [15]) it is possible to find the analogue result as in Proposition 3.6 to approximate the state ω_π, i.e. $\delta\vartheta = \pi$ as limiting Gibbs state, i.e. there exists a net $\Lambda \to n_\Lambda$ with $\lim_{\Lambda\to\mathbb{N}\times\mathbb{N}} \frac{n_\Lambda}{|\Lambda_1|^\varepsilon|\Lambda_2|^\varepsilon} = 0$ for all $\varepsilon > 0$ and ω_π is the limiting Gibbs state of the model with local tunneling interaction $n_\Lambda h_\Lambda$. Obviously in the corresponding representation the free variability of the phase difference is lost and the Josephson current is destroyed. This demonstrates the very sensitive reaction of the effect on the microscopical properties of the finite systems and suggests that for another ansatz of the interaction a completely different treatment and physical interpretation is necessary. Such effects have been discussed up to now only in the case of an extensive tunneling interaction [36].

5.3 A Remark on Relative Hamiltonians

Within the frame of the Tomita–Takesaki theory and perturbation theory it can be shown that under certain conditions for two given normal states ω_1, ω_2 on a von Neumann algebra $\mathcal{M}$ there exists a selfadjoint element $h \in \mathcal{M}$, such that $\omega_1^h = \omega_2$, where ω_1^h is the state arising from ω_1 by perturbation with h in terms of the perturbation series [17]. The operator h is called a *relative Hamiltonian.* The main criterion for the (normal, faithful) states ω_1, ω_2 is that there exist $0 < l_2 \leq l_1$ such that $l_1\omega_1 \geq \omega_2 \geq l_2\omega_1$. Having in mind a situation as in Section 3 with permutation invariant states, this requires $\text{supp}(\mu_{\omega_1}) = \text{supp}(\mu_{\omega_2})$, which in general is not fulfilled.

To find relative Hamiltonians for a more general class of states, there exists a generalization of this concept, using perturbations with so–called *extended–valued lower–bounded operators affiliated with* $\mathcal{M}$ [18], see also [23]. These are affine, weakly lower semicontinuous functionals on the normal states $\mathcal{M}_*^{+1}$ of $\mathcal{M}$ which can formally be written as $h_s + \infty P$, where P is a projection in $\mathcal{M}$ and h_s is a lower bounded, selfadjoint operator affiliated with $\mathcal{M}$. The perturbation of a state ω with such an extended–valued lower–bounded operator h affiliated with $\mathcal{M}$ is defined in terms of a variational principle [18, Theorem 3.1]: If $\omega \in \mathcal{M}_*^{+1}$, h is an extended–valued lower–bounded operator affiliated with $\mathcal{M}$, and there exists a state $\sigma \in \mathcal{M}_*^{+1}$ with $S(\sigma|\omega) - h(\sigma) > -\infty$, then there exists a unique state $\omega^h \in \mathcal{M}_*^{+1}$ which maximizes the functional $\sigma \to S(\sigma|\omega) - h(\sigma)$ (we assume $\beta = 1$ and S denotes as in Sec. 3 the relative entropy). This state coincides (up to the choice of sign) with the one defined in terms of perturbation theory, if h is defined by $h(\sigma) := \langle \sigma\, ;\, h' \rangle$ with some selfadjoint $h' \in \mathcal{M}$. Moreover in [18] it is shown that for two normal states ω_1, ω_2 with a $c \in \mathbb{R}$ and $c\,\omega_1 \geq \omega_2$, there exists an extended–valued lower–bounded operator h affiliated with $\mathcal{M}$, such that $\omega_1^h = \omega_2$ and $S(\sigma|\omega_1) - h(\sigma) = S(\sigma|\omega_2)$ for all $\sigma \in \mathcal{M}_*^{+1}$.

It should be remarked that the use of extended–valued lower–bounded operators affiliated with a von Neumann algebra in the theory of quantum lattice mean–field systems is also possible on the microscopical level, i.e. one can work with such affine functionals as local Hamiltonians, see [37]. Again, this leads to a variational principle for limiting Gibbs states like the one, given in Proposition 2.4 (the contiuous function $j((\frac{H_\Lambda}{|\Lambda|})_{\Lambda\in\mathcal{L}})$ for the internal energy density is then replaced by a lower semicontiuous one). Although this leads to a rather large class of models with a rich reservoir of possible phase structures in the thermodynamic equilibrium, it still remains an unsolved problem to determine the limiting Gibbs state. Therefore we will not follow these ideas, but will have a look on the concept of such kinds of relative Hamiltonians as perturbations on the macroscopical level, i.e. as operators affiliated with $\mathcal{M}_\beta$ in the representation of some unperturbed limiting Gibbs state ω^β:

We have seen in Proposition 3.3 (i) that the system, locally perturbed with an approximately symmetric net $h \in \widetilde{\mathcal{Y}}$, possesses the limiting Gibbs state $(\omega^\beta)^{h^\beta}$, with $h^\beta = \text{s–lim}_{\Lambda\in\mathcal{L}} \Pi_\beta(h_\Lambda)$. Moreover, the strong limit of the local perturbations is exactly the relative Hamiltonian in $\mathcal{M}_\beta$ (up to an additive constant for normalization).

For $\omega = \int_{\mathcal{S}(\mathcal{B})} \otimes\varphi\, \varrho(\varphi)\, d\mu_\beta(\varphi) \in \mathcal{S}^P(\mathcal{A})$ with a positive normalized $\varrho \in C(\mathcal{S}(\mathcal{B}), \mathbb{R})$, in general there exists no relative Hamiltonian $h \in \mathcal{M}_\beta$ such that $(\omega^\beta)^h = \omega$, but it is an easy calculation to find an extended–valued lower–bounded operator h, affiliated with $\mathcal{M}_\beta$ and $(\omega^\beta)^h = \omega$. In Proposition 3.3 (ii) there was constructed a net of local perturbations $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ such that ω is the limiting Gibbs state of the locally perturbed system. We have seen that the norm of the h_Λ tends to infinity for large regions Λ. This net $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ can be chosen in a way such that it approximates the extended–valued lower–bounded operator h with $(\omega^\beta)^h = \omega$. It can be shown that there exists a net $(h_\Lambda)_{\Lambda\in\mathcal{L}}$ of local perturbations with the same properties as the one in Proposition 3.3 (ii), but additionally for all $\varepsilon > 0$:

$$\text{s.res.--}\lim_{\Lambda\in\mathcal{L}} \Pi_\beta(h_\Lambda)P_\varepsilon = h_s P_\varepsilon = -\frac{1}{\beta}\int_{\mathcal{S}(\mathcal{B})}^{\oplus} \ln(\varrho(\varphi))\chi_{M_\varepsilon}(\varphi)\mathbb{1}_\varphi\, d\mu_\beta(\varphi)\,, \tag{5.5}$$

with $M_\varepsilon := \{\varphi \in \mathcal{K}_\beta \mid \varrho(\varphi) \geq \varepsilon\}$, $P_\varepsilon := \int_{\mathcal{S}(\mathcal{B})}^{\oplus} \chi_{M_\varepsilon}(\varphi)\mathbb{1}_\varphi\, d\mu_\beta(\varphi)$, and the characteristic function χ_{M_ε}.

We will omit the proof, because we stated this result only to demonstrate the close connection between the local perturbations of a limiting Gibbs state ω^β and perturbations performed with extended–valued lower–bounded operators, affiliated with the center $\mathcal{Z}_\beta$ of $\Pi_\beta(\mathcal{A})''$.

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