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# Asymptotic Normalization Properties and Mass Independent Renormalization Group Functions

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*Abstract* Mass independent renormalization group functions in massive theories are related to normalization properties of the Green functions in the asymptotic region, where mass effects are neglected. In this special form the renormalization group invariance is restricted to the asymptotic region and consequently mass effects cannot be recovered by a integration of the renormalization group equation. It is shown, that mass independence results in the limit of a large normalization point, whenever a Callan-Symanzik equation exists and contains the same differential operators as the renormalization group equation.

## 1. Introduction

In the development of dimensional regularization schemes have been constructed, e.g. the  $\overline{MS}$ -scheme, which have mass independent  $\beta$ -functions and anomalous dimensions in the renormalization group equation as well as in the Callan-Symanzik equation to all orders of perturbation theory [1].

In this paper we will show that mass independent coefficient functions in the renormalization group equation indicate that one has chosen thereby implicitly the normalization for the independent couplings of the theory in the asymptotic region, where mass effects are negligible. The intrinsic normalization properties of all mass independent schemes are therefore asymptotic ones, and equivalent to the massless theory. The condition for mass independence in the asymptotic region is the existence of a Callan-Symanzik equation

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of the same form as the renormalization group, i.e. they have to contain both the same differential operators with respect to fields and couplings.

This insight is important for the use of the respective equations. Mass independent  $\beta$ -functions in a massive theory express the fact, that one has restricted the renormalization group transformations to the asymptotic region. As a consequence the momentum dependence of the Green functions, which one derives from an integration of the renormalization group equation, is just also the asymptotic one i.e., more specifically, that of the massless theory. Stated differently: Mass dependence cannot be recovered by asymptotic renormalization group transformations. In order to get the full massive invariants of the renormalization group, one has to integrate the renormalization group equation with the general mass dependent  $\beta$ -functions.

In section 1 we give the general renormalization group invariant of order 1-loop in the massive  $\phi^4$ -theory, by solving the renormalization group equation with well-defined boundary conditions. Section 2 contains the proof, that asymptotic normalization properties and mass independent  $\beta$ -functions and anomalous dimensions are equivalent. In the last section we discuss the limit of large momenta or normalization point, and derive an approximation for small deviations from the asymptotic limit.

## 2. Renormalization Group Solution in the Massive $\phi^4$ -Theory

Starting point for the considerations is the massive  $\phi^4$ -theory with the classical action

$$\Gamma_{cl} = \int \frac{1}{2} \partial\phi\partial\phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (2.1)$$

In perturbation theory the Green functions are defined by the Gell-Mann Low formula, a suitable subtraction scheme and normalization conditions according to

$$\begin{aligned} \left. \partial_{p^2} \Gamma_2(p^2) \right|_{p^2=\kappa^2} &= 1 & \left. \Gamma_2(p^2) \right|_{p^2=m^2} &= 0 \\ \left. \Gamma_4(p_1, p_2, p_3, p_4) \right|_{\substack{p_i^2=\kappa^2 \\ p_i p_j = -\frac{\kappa^2}{3}}} &= -\lambda \end{aligned} \quad (2.2)$$

The normalization point for the coupling and the wave function is taken to be in the Euclidean region ( $\kappa^2 < 0$ ).

In perturbation theory the dependence of the Green Functions on the massive parameters is expressed by two partial differential equations, which can be rigorously derived to all orders of perturbation theory [2]: the Callan-Symanzik (CS) equation

$$(m\partial_m + \kappa\partial_\kappa + \beta_\lambda\partial_\lambda - \gamma\mathcal{N})\Gamma(\phi) = \alpha \int [-m^2\phi^2]_2 \cdot \Gamma(\phi) \quad (2.3)$$

and the renormalization group (RG) equation

$$(\kappa \partial_\kappa + \tilde{\beta}_\lambda \partial_\lambda - \tilde{\gamma} \mathcal{N}) \Gamma(\phi) = 0 \quad \text{with} \quad \mathcal{N} = \int \phi \frac{\delta}{\delta \phi}. \quad (2.4)$$

The CS-equation describes the breaking of the dilatational invariance, if one scales the momenta by a common factor  $p_i \rightarrow \epsilon p_i$ . The RG equation on the other hand expresses in differential form invariance of the Green functions under the renormalization group.

At the Euclidean symmetric momentum  $p_i^2 = p^2, p_i p_j = -\frac{p^2}{3}, p^2 < 0$  the 4-point vertex in 1-loop order is given by

$$\begin{aligned} \Gamma_4\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) &= -\lambda - \frac{1}{16\pi^2} \frac{3}{2} \lambda^2 \left( \sqrt{1 - \frac{3m^2}{p^2}} (\ln(\sqrt{1 - \frac{3m^2}{p^2}} + 1) - \ln(\sqrt{1 - \frac{3m^2}{p^2}} - 1)) \right. \\ &\quad \left. - \sqrt{1 - \frac{3m^2}{\kappa^2}} (\ln(\sqrt{1 - \frac{3m^2}{\kappa^2}} + 1) - \ln(\sqrt{1 - \frac{3m^2}{\kappa^2}} - 1)) \right) + O(\lambda^3) \\ &\equiv -\lambda + \lambda^2 (Q^{(1)}\left(\frac{m^2}{p^2}\right) - Q^{(1)}\left(\frac{m^2}{\kappa^2}\right)) + O(\lambda^3), \end{aligned} \quad (2.5)$$

It satisfies the normalization condition (2.2) by construction to all orders. With the normalization condition (2.2) the 2-point function is zero in 1-loop order and consequently the anomalous dimensions of the CS- and the RG-equation are vanishing.

$$\Gamma_2(p^2, m^2, \kappa^2) = p^2 - m^2 + O(\lambda^2) \Rightarrow \gamma^{(1)} = \tilde{\gamma}^{(1)} = 0. \quad (2.6)$$

From (2.5) and (2.6) the RG-function  $\tilde{\beta}_\lambda^{(1)}$  is calculated to be

$$\tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = \frac{1}{16\pi^2} \frac{3}{2} \lambda^2 \left( \frac{3m^2}{\kappa^2} \frac{1}{\sqrt{1 - \frac{3m^2}{\kappa^2}}} \ln \frac{\sqrt{1 - \frac{3m^2}{\kappa^2}} + 1}{\sqrt{1 - \frac{3m^2}{\kappa^2}} - 1} + 2 \right) \quad (2.7)$$

Only in the limit of an asymptotic normalization point  $\frac{m^2}{\kappa^2} \rightarrow 0$  the RG-function  $\tilde{\beta}_\lambda(\frac{m^2}{\kappa^2})$  becomes  $\kappa$ -independent and coincides with the CS-function  $\beta_\lambda^{(1)}$ :

$$\beta_\lambda^{(1)} = \lim_{\kappa^2 \rightarrow -\infty} \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = \frac{3}{16\pi^2} \lambda^2, \quad (2.8)$$

This is a feature of the massless limit we will discuss later on in detail. In the limit  $\kappa^2 \rightarrow 0$  the RG-function vanishes:

$$\lim_{\kappa^2 \rightarrow 0} \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = 0 \quad (2.9)$$

According to (2.6) the RG-equation is a homogeneous differential equation in order 1-loop, which can be solved analytically with standard methods [3]. Introducing the variables

$$t = \ln \frac{p^2}{\kappa^2} \quad \text{and} \quad u = \frac{m^2}{p^2} \quad (2.10)$$

the RG-equation reads

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\tilde{\beta}_\lambda^{(1)}(ue^t)\right)\Gamma = 0 \quad (2.11)$$

The characteristic equations are integrated immediately:

$$\begin{aligned} \frac{d\lambda}{dt} &= -\frac{1}{2}\tilde{\beta}_\lambda^{(1)}(ue^t) \implies -\frac{1}{\lambda(t, u_o, \lambda_o)} + \frac{1}{\lambda_o} = Q^{(1)}(u_o e^t) - Q^{(1)}(u_o) \\ \frac{du}{dt} &= 0 \implies u(t, u_o, \lambda_o) = u_o \\ \frac{d\Gamma_4}{dt} &= 0 \implies \Gamma_4(t, u_o, \lambda_o) = -\lambda_o, \end{aligned} \quad (2.12)$$

with the function  $Q^{(1)}(y)$  defined in (2.5). In the integration it is unavoidable to fix the integration constants  $\lambda_o$  and  $u_o = \frac{m^2}{p^2}$  of the characteristic equations by boundary conditions. According to our construction the normalization conditions (2.2) imposed on the Green functions are satisfied to all orders and act here as well-defined boundary conditions by specifying the 2-dimensional boundary manifold, where the RG-solution has to start in. The equations (2.12) can be solved with respect to  $\Gamma_4$ ,  $\lambda$  and  $u$  in order to calculate the solution  $\bar{\Gamma}_4(t, u, \lambda)$  of the RG-equation:

$$\begin{aligned} \bar{\Gamma}_4(t, u, \lambda) &= -\frac{\lambda}{1 + \lambda(Q^{(1)}(u) - Q^{(1)}(ue^t))} \\ &= -\sum_{i=0}^{\infty} \lambda^{i+1} (Q^{(1)}(ue^t) - Q^{(1)}(u))^i \end{aligned} \quad (2.13)$$

As one easily verifies  $\bar{\Gamma}_4(t, u, \lambda)$  is a RG invariant for an arbitrary momentum  $p^2$  and normalization point  $\kappa^2$ , i.e. it satisfies the general multiplication law of the renormalization group:

$$\bar{\Gamma}_4(t + t_1, u, \lambda) = \bar{\Gamma}_4(t, u, \bar{\Gamma}_4(t_1, ue^t, \lambda)) \quad (2.14)$$

It continues the massive 1-loop Green function in a unique, RG-invariant way to all orders of perturbation theory.

Considering the purely asymptotic situation  $-p^2 \gg m^2$  and  $-\kappa^2 \gg m^2$ , we get the limit of the massless theory. The respective invariant charge is often called the “running coupling”  $\bar{\lambda}(t)$  [4]:

$$\bar{\lambda}(t) \equiv -\bar{\Gamma}_{4,\infty}(t, 0, \lambda) = \frac{\lambda}{1 - \lambda \frac{1}{16\pi^2} \frac{3}{2}t} \quad (2.15)$$

This expression one would have obtained, too, if one had calculated the RG-solution with mass independent  $\beta$ -function  $\tilde{\beta}_\lambda^{(1)}(0) = \beta_\lambda^{(1)}$  (cf. (2.8)) and according to the derivation we have given here it is obvious that (2.15) is an asymptotic result, in the sense that the coupling is thought to be fixed in the asymptotic region *and* the momenta are considered in the asymptotic region, too, where all mass effects are neglected. Moreover it can be shown

quite generally, that with mass independent functions  $\tilde{\beta}_\lambda$  one only gets the asymptotic invariant charge of the massless theory. In order to proceed we therefore consider once again the solution of the RG-equation especially (2.12). As we have pointed out there, it is absolutely mandatory to fix the integration constants by well-defined boundary conditions. For this reason one has to know a point  $\kappa^2 = p^2 f(\frac{m^2}{p^2})$  to all orders, where the invariant charge has normalization properties, i.e. coincides with the coupling. For well defined normalization conditions as (2.2) such boundary conditions are known from the beginning ( $\kappa^2 = p^2$ ). In a scheme without specific normalization conditions one has to find a point  $\kappa^2$  where the invariant charge has normalization properties in order to be able to solve the characteristic equations. Examples for such schemes are the MS- and  $\overline{\text{MS}}$ -scheme [5] or the BPHZL-scheme with  $s - 1$  at  $s = 0$  [6]. To all these three schemes it is common that the  $\beta$ -functions and anomalous dimensions of the CS-equation and RG-equation are the same and mass independent. In the next section we will prove quite generally that in such schemes one has normalization properties in the asymptotic region.

$$\begin{aligned} \lim_{p^2 \rightarrow -\infty} \partial_{p^2} \Gamma_2(p^2) \Big|_{p^2=\kappa^2} &= 1 (+\rho_{2,1}\lambda + \rho_{2,2}\lambda^2 + \dots) \\ \lim_{p^2 \rightarrow -\infty} \Gamma_4(p_1, p_2, p_3, p_4) \Big|_{\substack{p_i^2=\kappa^2 \\ p_i p_j = -\frac{\kappa^2}{3}}} &= -\lambda (+\rho_{4,1}\lambda^2 + \rho_{4,2}\lambda^3 + \dots) \end{aligned} \quad (2.16)$$

The expressions in the brackets indicate, that we have to allow also reparametrizations of the coupling with mass independent coefficients  $\rho_{i,j}$ . As an example we calculate the 1-loop 4-point function (2.5) in the schemes mentioned above and one can easily verify that they have indeed normalization properties in the asymptotic region as described by (2.16):

$$\begin{aligned} \text{MS} : \Gamma_4\left(\frac{p^2}{\mu^2}, \frac{m^2}{p^2}, \lambda\right) &= -\lambda + \lambda^2 \left( Q^{(1)}\left(\frac{m^2}{p^2}\right) + \frac{1}{16\pi^2} \frac{3}{2} \ln\left(\frac{\mu^2}{m^2}\right) + \ln 4\pi - \gamma_E \right) \\ \overline{\text{MS}} : \Gamma_4\left(\frac{p^2}{\bar{\mu}^2}, \frac{m^2}{p^2}, \lambda\right) &= -\lambda + \lambda^2 \left( Q^{(1)}\left(\frac{m^2}{p^2}\right) + \frac{1}{16\pi^2} \frac{3}{2} \ln\left(\frac{\bar{\mu}^2}{m^2}\right) \right) \\ \text{BPHZL} : \Gamma_4\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) &= -\lambda + \lambda^2 \left( Q^{(1)}\left(\frac{m^2}{p^2}\right) + \frac{1}{16\pi^2} \frac{3}{2} \ln\left(-\frac{4\kappa^2}{3m^2}\right) \right), \end{aligned} \quad (2.17)$$

where we denoted the normalization point, i.e. the unit mass, according to the general conventions  $\mu$  and  $\bar{\mu}$  respectively. For the qualitative understanding of the consequences of this innocent looking fact it is useful to observe that with the special normalization properties (2.16) the boundary manifold of the RG-equation is 1-dimensional ( $\frac{m^2}{p^2} = 0$ ) [3] and one cannot reach the non-asymptotic region by a RG-transformation due to the singular character of the boundary conditions, but one will stay in the 1-dimensional asymptotic region. The solutions of the system of characteristic equations is no more single valued anymore as required in order to get the full mass parameter dependent solution. The introduction of an anomalous mass dimension term will not change the situation.

### 3. Equivalence of Asymptotic Normalization Properties and Mass Independent $\beta$ -Functions

In this section we will prove the equivalence of asymptotic normalization properties and mass independent coefficient functions in the CS- and RG-equation to all orders of perturbation theory. The proof is quite general and not restricted to the  $\phi^4$ -theory, although we take only one coupling for reasons of transparency. For this proof we will not specify the mass normalization condition and introduce instead a  $\beta$ -function for the mass in the RG equation:

$$(\kappa\partial_\kappa + \tilde{\beta}_\lambda\partial_\lambda + \gamma_m m\partial_m - \tilde{\gamma}\mathcal{N})\Gamma(\phi) = 0, \quad (3.1)$$

the CS-equation is not changed:

$$(m\partial_m + \kappa\partial_\kappa + \beta_\lambda\partial_\lambda - \gamma\mathcal{N})\Gamma(\phi) = \alpha \int [-m^2\phi^2]_2 \cdot \Gamma(\phi) \quad (3.2)$$

In the first part we impose normalization conditions in the asymptotic region and show, that the  $\beta$ -functions and anomalous dimensions of the CS- and RG-equation are the same and mass independent. In the second part, we start from the assumption that we have calculated the Green functions in a scheme with mass independent coefficient functions in the RG- and CS-equation, and prove that the Green functions of the 4-point vertex and the residuum of the 2-point function have normalization properties in the asymptotic region. This means especially, that the differentiation with respect to the mass of the theory is soft.

To be specific we assume now to have calculated the Green functions with the following asymptotic normalization condition:

$$\begin{aligned} \lim_{p^2 \rightarrow \infty} \partial_{p^2} \Gamma_2(p^2) \Big|_{p^2 = \kappa^2} &= 1 \\ \lim_{p^2 \rightarrow \infty} \Gamma_4(p_1, p_2, p_3, p_4) \Big|_{\substack{p_i^2 = \kappa^2 \\ p_i p_j = -\frac{\kappa^2}{3}}} &= -\lambda \end{aligned} \quad (3.3)$$

where  $\lim_{p^2 \rightarrow \infty}$  means just  $-p^2 \gg m^2$ . We subtract the CS-equation and the RG-equation,

$$(m\partial_m + (\beta_\lambda - \tilde{\beta}_\lambda)\partial_\lambda - (\gamma - \tilde{\gamma})\mathcal{N} - \gamma_m m\partial_m)\Gamma = \Delta_m \cdot \Gamma(\phi) \quad (3.4)$$

and test (3.4) at the normalization point choosing first  $p^2$  in the asymptotic region and setting then  $p^2 = \kappa^2$ . Thereby the right-hand-side, the soft mass insertion,  $\Delta_m = \alpha \int [-m^2\phi^2]_2$ , is vanishing for the 4-point function  $\Gamma_4$  and for  $\partial_{p^2}\Gamma_2$  and we remain with

$$\left. \begin{aligned} (\beta_\lambda - \tilde{\beta}_\lambda) - 4\lambda(\gamma - \tilde{\gamma}) &= 0 \\ -2(\gamma - \tilde{\gamma}) &= 0 \end{aligned} \right\} \implies \begin{aligned} \beta_\lambda &= \tilde{\beta}_\lambda \\ \gamma &= \tilde{\gamma} \end{aligned} \quad (3.5)$$



(3.4) simplifies to

$$(m\partial_m - \gamma_m m\partial_m)\Gamma = \Delta_m \cdot \Gamma, \quad (3.6)$$

expressing the fact, that the differentiation with respect to the mass is soft. To show that the RG-coefficients are mass independent one has to use the consistency equation between the CS- and RG-equation, which has the general form

$$\begin{aligned} & (1 - \gamma_m)((\kappa\partial_\kappa\beta_\lambda)\partial_\lambda - (\kappa\partial_\kappa\gamma)\mathcal{N})\Gamma \\ &= (\tilde{\beta}_\lambda\partial_\lambda\beta_\lambda - \beta_\lambda\partial_\lambda\tilde{\beta}_\lambda)\partial_\lambda\Gamma - (\tilde{\beta}_\lambda\partial_\lambda\gamma - \beta_\lambda\partial_\lambda\tilde{\gamma})\mathcal{N}\Gamma \\ & - (\beta_\lambda\partial_\lambda\gamma_m)m\partial_m\Gamma + (\kappa\partial_\kappa + \tilde{\beta}_\lambda\partial_\lambda + \gamma_m m\partial_m - \tilde{\gamma}\mathcal{N})(\Delta_m \cdot \Gamma). \end{aligned} \quad (3.7)$$

We insert into the consistency equation the result of (3.5), test with respect to the Green functions  $\Gamma_4$  and  $\partial_{p^2}\Gamma_2$  and for asymptotic momenta ( $p^2 \neq \kappa^2!$ ), where all soft terms,  $\Delta_m$  and according to (3.6)  $(\beta_\lambda\partial_\lambda\gamma_m)m\partial_m\Gamma$ , are vanishing. In this case (3.7) simplifies to

$$\begin{aligned} & (1 - \gamma_m)((\kappa\partial_\kappa\beta_\lambda)\partial_\lambda - 4(\kappa\partial_\kappa\gamma))\Gamma_4 \stackrel{-p^2 \gg m^2}{=} 0 \\ & (1 - \gamma_m)((\kappa\partial_\kappa\beta_\lambda)\partial_\lambda - 2(\kappa\partial_\kappa\gamma))\partial_{p^2}\Gamma_2 \stackrel{-p^2 \gg m^2}{=} 0 \end{aligned} \quad (3.8)$$

which leads order by order to the result

$$\kappa\partial_\kappa\beta_\lambda = 0 \quad \text{and} \quad \kappa\partial_\kappa\gamma = 0. \quad (3.9)$$

(3.5) and (3.9) finish the first part of the proof.

To prove the opposite direction we start from the assumption to have a scheme where the  $\beta$ -functions and anomalous dimensions are mass-independent to all orders. Concerning equivalence of the CS-functions and RG-functions we have only to require that the  $\beta$ -functions agree in their lowest non-vanishing order:

$$\begin{aligned} 1) \quad & \kappa\partial_\kappa\beta_\lambda = \kappa\partial_\kappa\tilde{\beta}_\lambda = 0 \quad \kappa\partial_\kappa\gamma = \kappa\partial_\kappa\tilde{\gamma} = 0 \\ 2) \quad & \beta_\lambda^{(1)} = \tilde{\beta}_\lambda^{(1)} \end{aligned} \quad (3.10)$$

Again in the RG-equation we allow the appearance of a  $\beta$ -function for the mass term, as it is common to the generally used mass independent schemes. We wish to show that asymptotic normalization properties are a consequence. The consistency equation (3.7) with the assumption (3.10(1)) inserted reads now:

$$\begin{aligned} 0 &= (\tilde{\beta}_\lambda\partial_\lambda\beta_\lambda - \beta_\lambda\partial_\lambda\tilde{\beta}_\lambda)\partial_\lambda\Gamma - (\tilde{\beta}_\lambda\partial_\lambda\gamma - \beta_\lambda\partial_\lambda\tilde{\gamma})\mathcal{N}\Gamma \\ & - (\beta_\lambda\partial_\lambda\gamma_m)m\partial_m\Gamma + (\kappa\partial_\kappa + \tilde{\beta}_\lambda\partial_\lambda + \gamma_m m\partial_m - \tilde{\gamma}\mathcal{N})(\Delta_m \cdot \Gamma). \end{aligned} \quad (3.11)$$

We apply it in 2-loop order on  $\partial_{p^2}\Gamma_2$ , take the limit to an asymptotic momentum, where the soft insertion vanishes, and get:

$$\gamma^{(1)} = \tilde{\gamma}^{(1)} \quad (3.12)$$



Subtracting now CS- and RG-equation, i.e. (3.2) and (3.1), and using thereby (3.10) and (3.12) we get in 1-loop order that the the differentiation with respect to the mass is soft:

$$\left((m\partial_m - \gamma_m m\partial_m)\Gamma\right)^{(1)} = \left(\Delta_m \cdot \Gamma\right)^{(1)}, \quad (3.13)$$

Specifically this means:

$$\begin{aligned} m\partial_m \Gamma_4^{(1)}(p_i, \kappa^2, m^2) &= (\Delta_m \cdot \Gamma)_4^{(1)}(p_i, \kappa^2, m^2) \\ m\partial_m \partial_{p^2} \Gamma_2^{(1)}(p^2, m^2, \kappa^2) &= \partial_{p^2} (\Delta_m \cdot \Gamma)_2^{(1)}(p^2, \kappa^2, m^2). \end{aligned} \quad (3.13a)$$

Tested at asymptotic symmetric momentum  $-p^2 \gg m^2$  the soft insertions vanish, showing that the asymptotic expressions only depend on the ratio  $\frac{p^2}{\kappa^2}$ :

$$\begin{aligned} \Gamma_4^{(1)}(p^2, \kappa^2, m^2, \lambda) &\stackrel{-p^2 \gg m^2}{\simeq} \lambda^2 \Gamma_{4,as}^{(1)}\left(\frac{p^2}{\kappa^2}\right) \\ \partial_{p^2} \Gamma_2^{(1)}(p^2, \kappa^2, m^2, \lambda) &\stackrel{-p^2 \gg m^2}{\simeq} \lambda \omega_{2,as}^{(1)}\left(\frac{p^2}{\kappa^2}\right) \end{aligned} \quad (3.14)$$

The consistency equation (3.11) of order 3-loop (3.10) tested in the asymptotic region ( $m\partial_m \Gamma_4^{(1)}$  and  $m\partial_m \partial_{p^2} \Gamma_2^{(1)}$  is soft according to (3.13)) gives the equalities:

$$\begin{aligned} 0 &= 5 \beta_\lambda^{(1)} (\tilde{\beta}_\lambda^{(2)} - \beta_\lambda^{(2)}) \\ 0 &= 2 \beta_\lambda^{(1)} (\tilde{\gamma}^{(2)} - \gamma^{(2)}) \end{aligned} \quad (3.15)$$

and  $\tilde{\beta}_\lambda^{(2)} = \beta_\lambda^{(2)}$  and  $\tilde{\gamma}^{(2)} = \gamma^{(2)}$  follows immediately. Subtracting the CS- and RG-equation in 2-loop order we find the same equation as above in 1-loop order

$$\left((m\partial_m - \gamma_m m\partial_m)\Gamma\right)^{(2)} = \left(\Delta_m \cdot \Gamma\right)^{(2)} \quad (3.16)$$

and are able to argue in the same way again (cf. (3.13{14})):

$$\begin{aligned} \Gamma_4^{(2)}(p^2, \kappa^2, m^2, \lambda) &\stackrel{-p^2 \gg m^2}{\simeq} \lambda^3 \Gamma_{4,as}^{(2)}\left(\frac{p^2}{\kappa^2}\right) \\ \partial_{p^2} \Gamma_2^{(2)}(p^2, \kappa^2, m^2, \lambda) &\stackrel{-p^2 \gg m^2}{\simeq} \lambda^2 \omega_{2,as}^{(2)}\left(\frac{p^2}{\kappa^2}\right) \end{aligned} \quad (3.17)$$

The proof to all orders is just the same, yielding that for asymptotic momenta and mass-independent  $\beta$  functions and anomalous dimensions the variation with respect to the mass is always soft for the Green functions in consideration. The asymptotic functions depend therefore only on the dimensionless ratio  $\frac{p^2}{\kappa^2}$ . Using now the RG-equation for the asymptotic Green functions we get the result that they have the following structure:

$$\begin{aligned} \Gamma_{4,as}^{(n)} &= \lambda^{n+1} \sum_{i=0}^n \rho_{n,i} \ln^i\left(\frac{p^2}{\kappa^2}\right) \\ \omega_{2,as}^{(n)} &= \lambda^n \sum_{i=0}^n \tilde{\rho}_{n,i} \ln^i\left(\frac{p^2}{\kappa^2}\right) \end{aligned} \quad (3.18)$$

where  $|p^2| \geq |p_\infty^2| \gg m^2$ . With  $p_\infty^2$  we denote the smallest momentum from where on one can neglect all mass dependence. Taking  $\kappa^2$  also in the asymptotic region, e.g.  $\kappa^2 = p_\infty^2$  one has the normalization properties we have required in (2.16).

Hence we have shown that mass independence of coefficient functions and asymptotic normalization are equivalent. This – we repeat – means in particular that we can never recover the non-asymptotic behavior by a RG-transformation, but remain “trapped” in asymptotics. This result is not restricted to the  $\phi^4$  theory, but is valid also for theories with more than one coupling. Following the arguments of the proof the most important ingredient is the existence of the Callan-Symanzik equation with a truly soft mass-term at the right-hand side and a RG-equation, which contains – up to an anomalous mass dimension – the same differential operators as the CS-equation. Therefore the same results hold true for spontaneously broken theories in a completely unphysical parametrization, i.e. fixing all the couplings as in the symmetric theory. If one chooses in such theories physical normalization conditions specifying the masses of the fields, the CS-equation contains additional differential operators with the consequence, that the  $\beta$ -functions depend on the normalization point also in the asymptotic region [7]. Another important input for recovering mass independent  $\beta$ -functions is the choice of a non-exceptional momentum as normalization point. The symmetric point is non-exceptional and soft terms vanish, when it is driven to infinity. (This is also the reason, why we have never tested for the full 2-point function  $\Gamma_2$ .)

Therefore it is obvious that – especially also in theories with spontaneous breaking – mass-independent RG-coefficients are not appropriate to determine mass effects, because all RG-integrations are carried out in the asymptotic region, where mass effects are not visible anymore.

#### 4. Discussion and Applications

With the general renormalization group invariant (2.13) one is able to study the limit of large  $p^2$  neglecting all terms of order  $\frac{m^2}{p^2}$  and  $\frac{m^2}{p^2} \ln |\frac{m^2}{p^2}|$ :

$$\begin{aligned} \bar{\Gamma}_4(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)^{-p^2 \gg m^2} &\simeq \frac{-\lambda}{1 + \lambda - (\frac{1}{16\pi^2} \frac{3}{2} \ln(-\frac{4p^2}{3m^2}) - Q^{(1)}(\frac{m^2}{\kappa^2}))} \\ &\approx \frac{-\lambda}{1 - \lambda \frac{1}{16\pi^2} \frac{3}{2} \ln(-\frac{p^2}{m^2})} \end{aligned} \quad (4.1)$$

where in the last approximation we have assumed that  $\kappa$  is somewhere in the low energy region and all  $\kappa$ -dependent constants multiplied by  $\lambda$  are negligible in comparison to the logarithmic term. Furthermore we require to be in a perturbative domain, i.e.  $\frac{1}{16\pi^2} \frac{3}{2} \lambda \ln(-\frac{p^2}{m^2}) < 1$ . The asymptotic formula (4.1) is related to a region where in a massive theory with positive 1-loop  $\beta$ -function the concept of improvement works best: For a small coupling  $\lambda \ll 1$  fixed at a low energy point  $\kappa$ , e.g.  $\kappa \simeq 0$ , it is then ensured for large  $-p^2$  that

$$Q^{(1)}(\frac{m^2}{\kappa^2}) - Q^{(1)}(\frac{m^2}{p^2}) > 1. \quad (4.2)$$

As a rough estimate shows, this implies that under these circumstances the 1-loop induced invariant charge takes into account the *leading* terms in any order and one is in a situation of summing up truly leading logarithms in the asymptotic region of the massive theory. For the electromagnetic coupling, where one has a similar situation as in the above one, such concepts have been successfully applied in the standard model (see e.g. [8]).

On the other hand choosing the normalization point at large  $-\kappa^2 \gg m^2$ , i.e. fixing the coupling in the asymptotic region, one finds a corresponding expression:

$$\bar{\Gamma}_4(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)^{-\kappa^2 \gg m^2} \frac{-\lambda}{1 + \lambda(Q^{(1)}(\frac{m^2}{p^2}) + \frac{1}{16\pi^2} \frac{3}{2} \ln(-\frac{4\kappa^2}{3m^2}))} \quad (4.3)$$

Due to the alternating sign of the perturbative power series for finite  $p^2$  improvement is now not naively applicable in the sense of summing up all the leading contributions, but one rather has to estimate the additional higher order contributions quite carefully. (In a UV-asymptotic free theory the situation of (4.1) and (4.3) is reversed, because the restriction (4.2) has the opposite sign.)

Finally we want to use the exact solution of the RG-equation, in order to study small deviations from the logarithmic asymptotic behavior affected by mass terms. This approximation can be derived without taking care of the sign of the  $\beta$ -functions and should be relevant especially in an UV-asymptotically free theory. For these purposes we start from (4.3) with asymptotic normalization conditions and expand  $Q^{(1)}(\frac{m^2}{p^2})$  for large Euclidean  $p^2$ :

$$Q^{(1)}(\frac{m^2}{p^2}) = \frac{1}{2}b_1 \ln|\frac{3m^2}{4p^2}| + b_{1,1} \frac{m^2}{p^2} \ln|\frac{m^2}{p^2}| + b_{1,2} \frac{m^2}{p^2} + O(\frac{m^4}{p^4}) \quad (4.4)$$

( $b_1$  is determined by the CS- $\beta$ -function,  $\beta_\lambda^{(1)} = \lambda^2 b_1$ , i.e. in the  $\phi^4$ -theory  $b_1 = \frac{3}{16\pi^2}$ .) The first approximation, where one takes the normalization point *and* the momenta in the asymptotic region, is given in (2.15)

$$\bar{\Gamma}_{4,\infty}(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)^{-\kappa^2, p^2 \gg m^2} \frac{\lambda}{1 - \lambda \frac{1}{16\pi^2} \frac{3}{2} \ln \frac{p^2}{\kappa^2}} \equiv \bar{\lambda}(\ln \frac{p^2}{\kappa^2}) \quad (4.5)$$

In the next step we want to take into account small deviations from the asymptotic behavior, i.e. we have to consider terms of order  $(Q^{(1)}(\frac{m^2}{p^2}) + \frac{1}{2}b_1 \ln(-\frac{4p^2}{3m^2}))$ , but neglect all terms of order  $(Q^{(1)}(\frac{m^2}{p^2}) + \frac{1}{2}b_1 \ln(-\frac{4p^2}{3m^2}))^2$ . Specifically, this means to take into account the terms of order  $\frac{m^2}{p^2}$  and  $\frac{m^2}{p^2} \ln|\frac{m^2}{p^2}|$  and to neglect the terms of order  $\frac{m^4}{p^4}$  for large  $p^2$ . From (4.3) one finds:

$$\begin{aligned} \bar{\Gamma}_4(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)^{-\kappa^2 \gg m^2} & \frac{-\lambda}{1 + \lambda(Q^{(1)}(\frac{m^2}{p^2}) + \frac{1}{2}b_1 \ln|\frac{4p^2}{3m^2}|) - \lambda \frac{1}{2}b_1 \ln \frac{p^2}{\kappa^2}} \\ & \stackrel{|\frac{m^4}{p^4}| \ll 1}{\approx} -\bar{\lambda}(t) + \bar{\lambda}^2(t)(Q^{(1)}(\frac{m^2}{p^2}) + \frac{1}{2}b_1 \ln|\frac{4p^2}{3m^2}|) \\ & = -\bar{\lambda}(t) + \bar{\lambda}^2(t)(b_{1,1} \frac{m^2}{p^2} \ln|\frac{m^2}{p^2}| + b_{1,2} \frac{m^2}{p^2}) + O(\frac{m^4}{p^4}) \end{aligned} \quad (4.6)$$

where  $t = \ln \frac{p^2}{\kappa^2}$ . In (4.6) we have assumed that the “running coupling”  $\bar{\lambda}(t)$  is sufficiently small in the enlarged region of momenta, we consider now. The result, we have calculated in a well-defined approximation, one would have achieved, too, by solving the RG-equation with a mass-independent  $\beta$ -function and inserting the solution in the massive 1-loop Green function. Starting from the complete massive 1-loop invariant charge the approximation one has performed is obvious, namely one has neglected all terms of order  $\frac{m^4}{p^4}$ . This emphasizes once more that RG-solutions calculated with mass independent  $\beta$ -functions have to be used very carefully outside the asymptotic region.

## REFERENCES

- [1] G. 'T HOOFT, *Nucl.Phys.* **B 61** (1973) 455;  
J.C. COLLINS, *Phys. Rev.* **D 10** (1974) 1213 .
- [2] K. SYMANZIK, *Comm. Math. Phys.* **18** (1970) 227;  
K. SYMANZIK, *Comm. Math. Phys.* **23** (1971) 49;  
C.G. CALLAN, *Phys. Rev.* **D2** (1970) 1541.
- [3] R. COURANT, D. HILBERT, *Methoden der Mathematischen Physik II*, Springer Verl. Berlin 1968;  
E. KAMKE, *Differentialgleichungen II*, Akad. Verlagsgesellsch. Leipzig 1959.
- [4] D. GROSS, Applications of the renormalization group, in: *Methods in field theory*, Les Houches 1975; ed. R.Balian and J.Zinn-Justin, North-Holland 1976.
- [5] J. COLLINS, *Renormalization*, Cambridge University Press 1984.
- [6] J.H. LOWENSTEIN, W. ZIMMERMANN, *Comm. Math. Phys.* **44** (1975) 73; J.H. LOWENSTEIN, *Comm. Math. Phys.* **47** (1976) 53.
- [7] E. KRAUS, *Z. Phys. C* **60** (1993) 741 - 750;  
E. KRAUS, The structure of the invariant charge in massive theories with one coupling; Bern Preprint BUTP-93/26.
- [8] F. JEGERLEHNER, Renormalizing the standard model, in: *Proceedings of the 1990 Theoretical Advanced Study Institute in Elementary Particle Physics*, Boulder, Colorado, ed. M. Cvetič and P. Langacker, 1991, Singapore.