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Spinning Particles in the Euclidean Taub-Nut Space

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Abstract. The geodesic motion of pseudo-classical spinning particles in the Euclidean Taub-NUT space is analysed. The equations of motion when $Q = 0$ is analysed in the case of motion on a cone and on a plane. The general solutions corresponding to the trajectories lying on a cone, is given.

1 Introduction

Spinning particles can be described by pseudo-classical mechanics models involving anti-commuting c-numbers for the spin-degree of freedom[1-6]. The general relations between space-time symmetries and the motion of spinning point particles has been analysed in detail in Refs.[7-9].

Generalizations of Riemannian geometry based on anticommuting variables have been found to be of wide mathematical interest; for example, supersymmetric point-particle mechanics has found applications in the area of index theorems, whilst BRST methods are widely used in the study of topological invariants.

Therefore the study of the motion of the spinning particles in curved space-time is well motivated.

In the present paper we investigate the motion of pseudo-classical spinning point par-

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ticle in the Euclidean Taub-NUT space.

The Kaluza-Klein monopole of Gross and Perry[11] and of Sorkin[12] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. Slow Bogomolny Prasad-Sommerfeld monopols move along geodesics in the Euclidean Taub-NUT space [13]. The problem of geodesics motion in this metric has its own interest.

For all these reasons, the extension of the Euclidean Taub-NUT space with additional fermionic dimensions, parametrised by vectorial Grassmann co-ordinates is very interesting to investigate. The plan of this paper is as follows. In Sec.2 we summarise the relevant equations for the motion of spinning points in curved space. For more details see[7-10]. In Sec.3 we analyze the equations of motion in the special case of motion on a cone and in a plane for which we choose $\theta = \frac{\pi}{2}$. This case represents an extension of the scalar particle motions in the usual Taub-NUT space in which the orbits are conic sections. In Sec.4 we summarise the results and present our conclusions.

2 Motion in Spinning Space

An action for the geodesics of spinning space is:

$$S = \int_a^b d\tau \left(\frac{1}{2}\, g_{\mu\nu}(x)\, \dot{x}^\mu\, \dot{x}^\nu \,+\, \frac{i}{2}\, g_{\mu\nu}(x)\, \psi^\mu\, \frac{D\psi^\nu}{D\tau} \right). \tag{1}$$

Here and in the following the overdot denotes an ordinary proper-time derivative $d/d\tau$, whilst the covariant derivative of ψ^μ is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda\, \Gamma^\mu_{\lambda\nu}\, \psi^\nu. \tag{2}$$

The trajectories, which make the action stationary under arbitrary variations δx^μ and $\delta\psi^\mu$ vanishing at the end points, are given by:

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu}\, \dot{x}^\lambda\, \dot{x}^\nu = \frac{1}{2i}\, \psi^\kappa\, \psi^\lambda\, R_{\kappa\lambda}{}^{\mu}{}_{\nu}\, \dot{x}^\nu \tag{3}$$

$$\frac{D\psi^\mu}{D\tau} = 0. \tag{4}$$

The anti-symmetric tensor

$$S^{\mu\nu} = -i\, \psi^\mu \psi^\nu \tag{5}$$

can formally be regarded as the spin-polarization tensor of the particle[1-9]. The equations of motion can be expressed in terms of this tensor and in particular eq.(4) asserts that the spin is covariantly constant

$$\frac{DS^{\mu\nu}}{D\tau} = O \tag{6}$$

The concept of Killing vector can be generalized to the case of spinning manifolds. For this purpose it is necessary to consider variations of x^μ and $\psi\mu$ which leave the action S invariant modulo boundary terms. Let use assume the following forms of these variations:

$$\begin{aligned}\delta x^\mu &= \mathcal{R}^\mu(x,\dot{x},\psi) = R^{(1)\mu}(x,\psi) + \sum_{n=1}^{\infty}\frac{1}{n!}\dot{x}^{\nu_1}\cdots\dot{x}^{\nu_n}R^{(n+1)\ \mu}_{\nu_1\cdots\nu_n}(x,\psi)\\ \delta\psi^\mu &= \mathcal{S}^\mu(x,\dot{x},\psi) = S^{(o)\mu}(x,\psi) + \sum_{n=1}^{\infty}\frac{1}{n!}\dot{x}^{\nu_1}\cdots\dot{x}^{\nu_n}S^{(n)\ \mu}_{\nu_1\cdots\nu_n}(x,\psi)\end{aligned} \tag{7}$$

and the Lagrangian transforms into a total derivative

$$\delta S = \int_a^b d\tau\frac{d}{d\tau}\left(\delta x^\mu p_\mu - \frac{i}{2}\,\delta\psi^\mu g_{\mu\nu}\psi^\nu - \mathcal{J}(x,\dot{x},\psi)\right) \tag{8}$$

where p_μ is the canonical momentum conjugate to x^μ

$$p_\mu = g_{\mu,\nu}\dot{x}^\nu + \frac{i}{2}\Gamma_{\mu\nu;\lambda}\psi^\lambda\psi^\nu = \Pi_\nu + \frac{i}{2}\Gamma_{\mu\nu;\lambda}\psi^\lambda\psi^\nu \tag{9}$$

Π_μ being the covariant momentum. From Noether's theorem, if the equations of motion are satisfied, the quantity $\mathcal{J}$ is a constant of motion. If we expand $\mathcal{J}$ in a power series in the covariant momentum

$$\mathcal{J}(x,\dot{x},\psi) = \mathcal{J}^{(0)}(x,\psi) + \sum_{n=1}^{\infty}\frac{1}{n!}\Pi^{\mu_1}\cdots\Pi^{\mu_n}\mathcal{J}^{(n)}_{\mu_1\cdots\mu_n}(x,\psi) \tag{10}$$

then $\mathcal{J}$ is a constant of motion if its components satisfy a generalization of the Killing equation[6,7]:

$$\mathcal{J}^{(n)}_{(\mu_1\cdots\mu_n;\mu_{n+1})} + \frac{\partial\mathcal{J}^{(n)}_{(\mu_1\cdots\mu_n}}{\partial\psi^\sigma}\Gamma_{\mu_{n+1})\lambda}{}^{\sigma}\,,\psi^\lambda = \frac{i}{2}\,\psi^\sigma\,\psi^\lambda\,R_{\sigma\lambda\nu(\mu_{n+1}}\mathcal{J}^{(n+1)\nu}_{\mu_1\cdots\mu_n)}. \tag{11}$$

In general the symmetries of a spinning-particle model can be divided into two classes. First, there are conserved quantities which exist in any theory and these are called *generic* constants of motion. The second kind of conserved quantities, called *non-generic*, depend on the explicit form of the metric $g_{\mu\nu}(x)$. It was shown that for a spinning particle model defined by the action (1) there are four generic symmetries [6,7].

1. Propertime translations and the corresponding constant of motion is the Hamiltonian:
$$H = \frac{1}{2}\,g^{\mu\nu}\Pi_\mu\Pi_\nu \tag{12}$$
2. Supersymmetry generated by the supercharge
$$Q = \Pi_\mu\,\psi^\mu \tag{13}$$
3. Chiral symmetry generated by the chiral charge
$$\Gamma_* = \frac{1}{4!}\sqrt{-g}\epsilon_{\mu\nu\lambda\sigma}\,\psi^\mu\,\psi^\nu\,\psi^\lambda\,\psi^\sigma \tag{14}$$
4. Dual supersymmetry, generated by the dual supercharge
$$Q^* = \frac{1}{3!}\,\sqrt{-g}\epsilon_{\mu\nu\lambda\sigma}\,\Pi^\mu\,\psi^\nu\,\psi^\lambda\,\psi^\sigma. \tag{15}$$

3 Euclidean Taub-NUT Spinning Space

In this section we shall apply the results of the previous section to the case of a spinning particle moving on a manifold described by:

$$\begin{aligned} ds_5^2 &= -dt^2 + ds_4^2 \\ &= -dt^2 + V^{-1}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2] \\ &\quad + V(r)[dx^5 + \vec{A}(\vec{r})\, d\vec{r}]^2 \end{aligned} \tag{16}$$

where $\vec{r}$ denotes a three-vector $\vec{r} = (r, \theta, \varphi)$ and the gauge field $\vec{A}$ is that a monopole

$$\begin{aligned} A_r &= A_\theta = 0, \quad A_\varphi = 4m(1 - \cos\theta) \\ \vec{B} &= rot\vec{A} = \frac{4m\vec{r}}{r^3} \end{aligned} \tag{17}$$

where the function V(r) is

$$V(r) = \left(1 + \frac{4m}{r}\right)^{-1} \tag{18}$$

If we make the coordinate transformation

$$4m(\chi + \varphi) = -x^5 \tag{19}$$

with $0 \le \chi \prec 4\pi$ we obtain the following form for ds_4 :

$$ds_4^2 = V^{-1}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2] + 16m^2 V(r)[d\chi + \cos\theta d\varphi]^2 \tag{20}$$

The Killing vectors are:

$$D^{\alpha)} = R^{(\alpha)\mu}\,\partial_\mu, \quad \alpha = 1, \cdots, 4 \tag{21}$$

where

$$\begin{aligned} D^{(1)} &= \frac{\partial}{\partial\psi} \\ D^{(2)} &= \sin\varphi\,\frac{\partial}{\partial\theta} + \cos\varphi\,\cot\theta\,\frac{\partial}{\partial\varphi} - \frac{\cos\varphi}{\sin\theta}\,\frac{\partial}{\partial\psi} \\ D^{(3)} &= -\cos\varphi\,\frac{\partial}{\partial\theta} + \sin\varphi\,\cot\theta\,\frac{\partial}{\partial\varphi} - \sin\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\psi} \\ D^{(4)} &= -\frac{\partial}{\partial\varphi} \end{aligned} \tag{22}$$

The isometry group is $SU(2) \times U(1)$ and this can be contrasted with the Schwarzschild space-time where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic mass[12]. In the purely bosonic case we have two constants of motion corresponding to the invariance given above. They are the "relative electric charge" and the angular momentum (13,14)

$$q = 16m^2\,V(r)\,(\dot\psi + \cos\theta\,\dot\varphi), \tag{23}$$

$$\vec{j} = \vec{r} \times \vec{p} + q\vec{r}\frac{1}{r}, \tag{24}$$

The first generalized Killing equation has the form

$$B^{(\alpha)}{}_{,\mu} + \frac{\partial B^{(\alpha)}}{\partial \psi^\sigma} \Gamma^\sigma_{\mu\lambda} \psi^\lambda = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\lambda\mu} R^{(\alpha)\lambda} \tag{25}$$

In Ref[15] this equation was solved without imposing the condition $Q = O$.

The conservation of the supercharge Q is actually crucial for the consistency of the physical interpretation of the theory [7-9].

In our case the supercharge Q has the following form

$$Q = \frac{(4m+r)}{r}\dot{r}\psi^r + (4m+r)r\dot{\theta}\,\psi^\theta + \left[(4m+r)\,r\sin^2\theta\,\dot{\varphi} + q\cos\theta\right]\,\psi^\varphi + q\psi^\psi \tag{26}$$

The supersymmetry constraints $Q = 0$ enables us to solve for ψ^ψ in terms of ψ^φ,ψ^θ,ψ^r

$$\psi^\psi = -\frac{1}{rq}(4m+r)\dot{r}\psi^r - \frac{1}{q}\,(4m+r)r\,\dot{\theta}\psi^\theta - \left[\frac{1}{q}r(4m+r)\,\sin^2\theta\,\dot{\varphi} + \cos\theta\right]\psi^\varphi \tag{27}$$

As a result the chiral charge

$$\Gamma_* = 4mr(4m+r)\,\sin\theta\,\psi^\theta\,\psi^\varphi\,\psi^\psi \tag{28}$$

becomes zero and the dual supercharge has the following form

$$Q^* = -2\frac{1}{q}E\,\psi^r\psi^\theta\,\psi^\varphi\,4mr(4m+r)\sin\theta \tag{29}$$

From $Q = 0$ and from (5) we can deduce the following relations for the independent spin components.

$$\begin{aligned}
S^{r\psi} &= -\frac{1}{q}\,(4m+r)r\dot{\theta}\,S^{r\theta} - \left[\cos\theta + \frac{1}{q}r(4m+r)\sin^2\theta\,\dot{\varphi}\right]\,S^{r\varphi} \\
S^{\theta\psi} &= \frac{1}{qr}(4m+r)\dot{r}\,S^{r\theta} - \left[\frac{1}{q}(4m+r)r\sin^2\theta\,\dot{\varphi} + \cos\theta\right]\,S^{\theta\varphi} \\
S^{\varphi\psi} &= \frac{1}{rq}(4m+r)\dot{r}\,S^{r\varphi} + \frac{1}{q}(4m+r)r\dot{\theta}\,S^{\theta\varphi}
\end{aligned} \tag{30}$$

After long calculations we obtain the expressions for the Killing scalars $B^{(\alpha)}$ which satisfied the equation (25) when $Q = O$:

$$B^1 = -\frac{1}{q(4m+r)}32m^3r\dot{\theta}S^{r\theta} - \frac{1}{q(4m+r)}32m^3\sin^2\theta\dot{\varphi}S^{r\varphi} - \frac{1}{(4m+r)}8m^2r\sin\theta S^{\theta\varphi}$$

$$
\begin{aligned}
B^{(3)} &= -\frac{\partial}{\partial\varphi}B^2 \\
B^{(2} &= [-2m\sin\theta\cos\theta\cos\varphi - \frac{4mr(8m^2+8mr+r^2)\dot{\varphi}\sin^3\theta\cos\varphi}{(4m+r)q} \\
&- \frac{4mr(2m+r)\dot{r}\sin\theta\sin\varphi}{q}]S^{r\varphi} \\
&+ [\frac{8m^2r\sin^2\theta\cos\varphi}{4m+r} - \frac{4mr^2(2m+r)\sin^2\theta\cos\theta\cos\varphi\dot{\varphi}}{q} \\
&- \frac{4mr^2(2m+r)\dot{\theta}\sin\theta\sin\varphi}{q}]S^{\theta\varphi} \\
&+ [-2m\sin\varphi - \frac{4mr(8m^2+8mr+r^2)\dot{\theta}\sin\theta\cos\varphi}{q(4m+r)} + \frac{4m(2m+r)\dot{r}\cos\theta\cos\varphi}{q}]S^{r\theta} \\
B^{(4)} &= \left[32m^3r\cos\theta\frac{\dot{\theta}}{(4m+r)q} + \frac{8m^2\sin\theta\dot{r}}{q}\right]S^{r\theta} \\
&+ \left[(-2m+r)\sin^2\theta + \frac{32m^3r\sin^2\theta\cos\theta\dot{\varphi}}{(4m+r)q}\right]S^{r\varphi} \\
&+ \left[-\frac{8m^2r^2\sin^3\theta\dot{\varphi}}{q} - \frac{8m^2r+8mr^2+r^3}{4m+r}\sin\theta\cos\theta\right]S^{\theta\varphi}
\end{aligned}
\tag{31}
$$

Taking into account the contribution of the Killing scalars one finds four conserved quantities $\mathcal{J}^\alpha$

$$
\begin{aligned}
\mathcal{J}^{(1)} &= B^{(1)} + q \\
\mathcal{J}^{(2)} &= B^{(2)} + (4m+r)\sin\varphi\,\dot{\theta} + r(4m+r)\sin\theta\cos\theta\cos\varphi\,\dot{\varphi} - q\sin\theta\cos\varphi \\
\mathcal{J}^{(3)} &= B^{(3)} - (4m+r)r\cos\varphi\,\dot{\theta} + r(4m+r)\sin\theta\cos\theta\sin\varphi\,\dot{\varphi} - q\sin\theta\,\sin\varphi \\
\mathcal{J}^{(4)} &= B^{(4)} - r(4m+r)\sin^2\theta\,\dot{\varphi} - q\cos\theta
\end{aligned}
\tag{32}
$$

We remark that the "relative electric charge" q is no longer conserved contrasting with the purely bosonic case.

At the same time the total angular momentum is the difference of the Poincaré contribution and the spin angular momentum:

$$
\vec{\mathcal{J}} = \vec{B} - \vec{j} \tag{33}
$$

From eqs (32) we derive two very interesting relations

$$
\mathcal{J}^2\sin\varphi - \mathcal{J}^3\cos\varphi = -2mS^{r\theta} - \frac{1}{q}4m(2m+r)\dot{r}S^{r\varphi} - r(4m+r)\dot{\theta} \tag{34}
$$

and

$$
\mathcal{J}^1 + \frac{\vec{\mathcal{J}}.\vec{r}}{r} = \left[-\frac{1}{q}4mr(4m+r)\,\sin^2\theta\,\dot{\theta} + \frac{1}{q}4m(4m+r)\sin\theta\cos\theta\,\dot{r}\right]S^{r\theta}
$$

$$
\begin{aligned}
&+ \left[-(4m+r)\sin^2\theta\, cos\theta - \frac{1}{q}4mr(4m+r)\sin^4\theta\,\dot{\varphi}\right] S^{r\varphi} \\
&+ \left[-r(4m+r)\sin\theta\cos^2\theta - \frac{1}{q}4mr^2(4m+r)\sin^3\theta\cos\theta\,\dot{\varphi}\right] S^{\theta\varphi}
\end{aligned}
\tag{35}
$$

Eqs (35) express the fact that the total angular momentum in the radial direction receives a contribution only from spin components.

The expressions for the $\dot{\theta}$ and $\dot{\varphi}$ can be obtained using the relations from (31,32):

$$
\begin{aligned}
\dot{\theta} &= \frac{1}{(4m+r)r}\left[-\mathcal{J}^2\sin\varphi + \mathcal{J}^3\cos\varphi - B^3\cos\varphi + B^2\sin\varphi\right] \\
\dot{\varphi} &= \frac{1}{r(4m+r)\sin^2\theta}\left[-\mathcal{J}^4 - \mathcal{J}^1\cos\theta + B^1 + B^4\right]
\end{aligned}
\tag{36}
$$

The relative electric charge q is given by

$$
q = \mathcal{J}^1 + \frac{1}{(4m+r)q}32m^2r\dot{\theta}S^{r\theta} + \frac{1}{(4m+r)q}32m3r\,\sin^3\theta\,\dot{\varphi}S^{r\varphi} + \frac{1}{(4m+r)}8m2r\sin\theta S^{\theta\varphi}
\tag{37}
$$

From(37) and (36) we can deduce the expression for the $\dot{\psi}$ The expression for the energy is

$$
E = \frac{1}{2r}(4m+r)\,\dot{r}^2 + \frac{1}{2}(4m+r)r\,\dot{\theta}^2 + \frac{1}{2}(4m+r)r\sin^2\theta\,\dot{\varphi}^2 + 8m^2\frac{r}{(4m+r)}(\cos\theta\,\dot{\varphi}+\dot{\psi})^2 \tag{38}
$$

Finally $\dot{r}$ can be determined from (38).

Keeping in mind that ψ^μ is covariantly constant, the rate of change of spin is:

$$
\begin{aligned}
\dot{\psi}^r &= r^2\frac{1}{4m+r}\dot{\theta}\psi^\theta + \frac{r^2}{4m+r}\sin^2\theta\dot{\varphi}\psi^\varphi \\
\dot{\psi}^\theta &= \left[8m^2\frac{1}{r(4m+r)q}\dot{\varphi}\dot{r}\sin\theta - \frac{(2m+r)}{r(4m+r)}\dot{\theta}\right]\psi^r \\
&+ \left[\frac{8m^2r}{(4m+r)}\sin\theta\,\dot{\theta}\,\dot{\varphi} - \frac{(r+2m)}{r(4m+r)}\dot{r}\right]\psi^\theta \\
&+ \left[\frac{8m^2+8mr+r^2}{(4m+r)^2}\sin\theta\cos\theta\dot{\varphi} - \frac{8m^2\sin\theta}{(4m+r)^2}\dot{\psi} + \frac{8m^2\sin^3\theta\, r}{(4m+r)q}\dot{\varphi}^2\right]\psi^\varphi \\
\dot{\psi}^\varphi &= -\left[(r+2m)\frac{1}{r(4m+r)}\dot{\varphi} + \frac{8m^2}{\sin\theta(4m+r)rq}\dot{r}\dot{\theta}\right]\psi^r \\
&+ \left[-\cos\theta\frac{1}{\sin\theta}\dot{\varphi} + \frac{q}{2r(4m+r)\sin\theta} - \frac{8m^2r}{\sin\theta(4m+r)q}\dot{\theta}^2\right]\psi^\theta
\end{aligned}
$$

$$+ \left[-(2m+r)\frac{1}{r(4m+r)}\dot{r} - \frac{\cos\theta}{\sin\theta}\dot{\theta} - \frac{8m^2 r\sin\theta}{(4m+r)q}\dot{\theta}\dot{\varphi}\right]\psi^{\varphi} \tag{39}$$

The complicated equations for the velocities given by (39) can be integrated to solve the equations of motion for the coordinates and spin.

4 Special Solutions

In this section we solve the equations given above in the case of motion on a cone and for the case of motion in a plane.

Let us choose the z axis along $\vec{J}$ so that the motion of the particle may be conveniently described in terms of polar coordinates

$$\vec{r} = r\vec{e}(\theta,\varphi) \tag{40}$$

with

$$\vec{e} = (\sin\theta\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta) \tag{41}$$

In this case from (32) we have

$$S^{r\theta} - 2r(2m+r)\frac{1}{(4m+r)}\sin\theta S^{\varphi\psi} = 0 \tag{42}$$

and this relation implies $\Gamma_* = 0$

The equations of motion for the spin components when $Q \neq 0$ are

$$\begin{aligned}
\frac{d}{dt}\left[(4m+r)S^{r\varphi}\right] &= r\frac{1}{4m+r}qS^{\varphi\psi} \\
\frac{d}{dt}\left[\cos\theta S^{r\varphi} + S^{r\psi}\right] &= 0 \\
\frac{d}{dt}\left[r(4m+r)S^{\theta\varphi}\right] &= -2\frac{\sin\theta}{\cos\theta}\frac{r}{4m+r}q\,S^{\varphi\psi} \\
\frac{d}{dt}\left[r\cos\theta\,S^{\theta\varphi} + r\,S^{\theta\psi}\right] &= -\frac{\sin\theta}{4m}\frac{r}{4m+r}q\,S^{\varphi\psi}
\end{aligned} \tag{43}$$

The equations given above may be integrated but the solutions are very complicated.

The general solutions may be expressed in the following form

$$S^{r\varphi} = \frac{1}{4m+r}C_4\exp^{-(\int\Phi dt)} + \frac{1}{4m+r}\exp^{-\int\Phi dt}\int\Psi\,\exp^{\int\Phi dt}dt$$

$$
\begin{aligned}
S^{\theta\varphi} &= 2\left[C_1 - (4m+r)S^{r\varphi}\right] \, sin\theta \frac{1}{\cos\theta r(4m+r)} \\
S^{r\psi} &= C_3 - \cos\theta \, S^{r\varphi} \\
S^{\theta\psi} &= \frac{\sin\theta}{4mr}\left[C_2 - \frac{8m}{4m+r}C_1 + (4m+r)S^{r\varphi}\right] \\
S^{\varphi\psi} &= -S^{r\varphi}\frac{1}{4mr}\left[\frac{1}{4m} + \frac{r}{\cos^2\theta} + \frac{\sin^2\theta}{(4m+r)\cos^2\theta}\right]\frac{q}{r\dot{r}} \\
&+ \frac{q}{\dot{r}r}\left[\frac{C_2}{8mr} + \frac{1}{2r(4m+r)^2\cos\theta}\left((4m+r)^2 + r^2\right)C_3\right] \\
&+ C_1 q\frac{1}{r(4m+r)\dot{r}\cos^2\theta}(\sin^2\theta + \frac{r\sin^2\theta}{4m+r}) \\
S^{r\theta} &= -\frac{2r(2m+r)}{4m+r}\sin\theta \, S^{\varphi\psi}
\end{aligned} \tag{44}
$$

where C_1,C_2,C_3,C_4 are Grassmann constants and

$$
\begin{aligned}
\Phi &= \frac{q^2}{(4m+r)\,\dot{r}}\left[\frac{1}{4m} + \frac{1}{r\cos^2\theta} + \frac{\sin^2\theta}{cos^2\theta}\frac{1}{4m+r}\right] \\
\Psi &= q^2\frac{1}{(4m+r)\,\dot{r}}\left[\frac{C_2}{8mr} + \frac{1}{2r(4m+r)^2\cos\theta}\left((4m+r)^2 + r^2\right)\right]C_3 \\
&+ q^3 C_1\frac{1}{r(4m+r)^2\,\dot{r}}(1 + \frac{r}{4m+r})\sin^2\theta\frac{1}{\cos^2\theta}
\end{aligned} \tag{45}
$$

We are very interested to obtain an exact solution and in the case $Q = 0$.

In this case between the spin components are the following relations

$$
\begin{aligned}
p_r S^{r\theta} &= p_\varphi S^{\theta\varphi} + qS^{\theta\psi} \\
p_r S^{r\varphi} &= qS^{\varphi\psi} \\
p_r S^{r\psi} &= -p_\varphi S^{\varphi\psi}
\end{aligned} \tag{46}
$$

where $\vec{p} = \frac{\dot{\vec{r}}}{V}$

The condition $Q = 0$ modifies drastically the form of the exact solutions.

In spite of the complexity of the equations, we are able to present an exact solution which has the physical interpretation

$$
S^{r\theta} = S^{\varphi\psi} = S^{r\varphi} = S^{r\psi} = 0
$$

$$\begin{aligned} S^{\theta\varphi} &= \frac{C_1}{r(4m+r)} \\ S^{\theta\psi} &= \frac{1}{r}C_2 - \cos\theta\frac{C_1}{r(4m+r)} \end{aligned} \tag{47}$$

where C_1,C_2 are Grassmann constants. In this case $\mathcal{J}^1 = 0$ and

$$q = \frac{1}{(4m+r)^2}8m^2\sin\theta C_1 \tag{48}$$

From (32) we can deduce the expressions for the $\dot{\varphi}$ and $\dot{\psi}$. From energy we can deduce $\dot{r}$.

An exact solution was presented in Ref [15] but, unfortunately, when we impose the condition $Q = 0$ all the components of the spin tensor become zero. In this case there is a conserved vector analogous to the Runge-Lenz vector of the Coulomb problem whose existence is rather surprising in view of the complexity of the equations of motions. This conserved vector is:

$$\vec{\mathcal{K}} = \vec{p}\times\vec{j} + \left(\frac{q^2}{4m} - 4mE\right)\frac{\vec{r}}{r}. \tag{49}$$

The trajectories hence lie simultaneously on the cone $\vec{j}\cdot\vec{r}\frac{1}{r} = q$ and also in the plane perpendicular to

$$\vec{n} = q\vec{\mathcal{K}} + \left(4mE - \frac{q^2}{4m}\right)\vec{j}. \tag{50}$$

They are thus conic sections. Unfortunately, for the spinning particles this form turns out to be inadequate. One way to solve this problem is to use the Runge-Lenz vector from[16,17] but from my point of view it is inadequate. We consider that in the spinning case the Runge-Lenz vector are deeply related to the Killing-Yano tensors [10].

Another very interesting case is the motion in a plane.

For scalar particles any solution would actually describe planar motion, because the orbital angular momentum of scalar particles is always conserved. For spinning particle only the total angular momentum is a constant of motion. In conclusion the planar motion of spinning particles is strictly possible only in special cases, in which orbital and spin angular momentum are separately conserved [9].

When $\theta = \frac{\pi}{2}$ the equations of motion become

$$\begin{aligned} \frac{d}{dt}\left[(4m+r)S^{r\theta}\right] &= \frac{-8m^2}{4m+r}\dot{\varphi}\,S^{r\psi} - r(2m+r)\,\dot{\varphi}\,S^{\theta\varphi} \\ \frac{d}{dt}\left[rS^{\varphi\psi}\right] &= -2m + r\frac{1}{4m+r}\dot{\varphi}\,S^{r\psi} - \frac{\dot{\varphi}\,r}{2}S^{\theta\varphi} \\ \frac{d}{dt}\left[r(4m+r)S^{\theta\varphi}\right] &= -2r^2\,\dot{\varphi}\,S^{\varphi\psi} \end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left[S^{r\psi}\right] &= 0\\ \frac{d}{dt}\left[rS^{\theta\psi}\right] &= 0\\ \frac{d}{dt}\left[(4m+r)S^{r\varphi}\right] &= 0\end{aligned} \tag{51}$$

At this point we need the expressions for $B^{(\alpha)}$ with $(\alpha = 1, \cdots 4)$ in the case when $Q \neq O$

$$\begin{aligned}B^{(1)} &= \frac{32m^3}{(4m+r)^2}S^{r\psi} - \frac{8m^2r}{4m+r}S^{\theta\varphi}\\ B^{(3)} &= -\frac{\partial}{\partial\varphi}B^{(2)}\\ B^{(2)} &= -2m\sin\varphi S^{r\theta} + \frac{4m(8m^2+8mr+r^2)}{(4m+r)^2}\sin\theta\cos\varphi S^{r\psi} + (\frac{8m^2r}{4m+r}\sin^2\theta\cos\varphi)S^{\theta\varphi}\\ &- \frac{4mr(2m+r)}{4m+r}\sin\theta\sin\varphi S^{\varphi\psi}\\ B^{(4)} &= -\left[(2m+r)\sin^2\theta\right]S^{r\varphi} + \frac{8m^2r}{4m+r}\sin\theta S^{\theta\psi}\end{aligned} \tag{52}$$

In this case the system of equations (51) may be integrated.

When $\dot{\varphi}$=0 the solutions follow without difficulties from (34,35,52) and from condition $\dot{B}^{(4)}$=o[9]:

$$\begin{aligned}S^{r\theta} &= S^{\varphi\psi} = 0\\ S^{r\psi} &= \frac{\mathcal{J}^1}{4m}\\ S^{r\varphi} &= \frac{\mathcal{J}^4}{4m+r}\\ S^{\theta\psi} &= -\frac{\mathcal{J}^4}{4mr}\\ S^{\theta\varphi} &= \frac{C}{r(4m+r)}\end{aligned} \tag{53}$$

where C is a Grassmann constant.

When and $Q = 0$ using (30,34,35) and from $\dot{B}^{(4)} = 0$ the solutions become

$$\begin{gathered}S^{r\theta} = S^{r\psi} = S^{\theta\psi} = S^{r\theta} = S^{\varphi\psi} = 0\\ S^{\theta\varphi} = C\frac{1}{r(4m+r)}\end{gathered} \tag{54}$$

For a distant observer the motion of the particle is described by

$$\dot{r} = \sqrt{2}Er\frac{1}{4m+r} \tag{55}$$

as in the case of spinless particle. This solution describes a particle moving along a radius.

From (32) we can deduce the expression for the "relative electric charge"

$$q = \frac{8m^2C}{(4m+r)^2} \tag{56}$$

When $\dot{\varphi} \neq 0$ a general solution may be given.

Unfortunately, the equations of motion are quite intricate and the general solution is by no means illuminating.

Instead of the general solution we present here a special solution which is very simple.

$$\begin{aligned} S^{r\theta} &= S^{\varphi\psi} = S^{\theta\varphi} = S^{r\psi} = 0 \\ S^{r\varphi} &= \frac{C}{4m+r} \\ S^{\theta\psi} &= -\frac{C}{4mr} \end{aligned} \tag{57}$$

If we impose the condition $Q = 0$ after a simple calculation we obtain that all components of spin tensor become zero. In conclusion the spin and orbital angular momentum are parallel.

5 Concluding Remarks

The spinning particles can be described by pseudo-classical mechanics models involving anti-commuting c-numbers for the spin-degrees of freedom.

The main aim of this paper has been to show, how generalization of the usual equations and Noether's theorem for particles with internal degrees of freedom like spin can be used to obtain information about the solutions of the equations of motion of these particles in Euclidean Taub-NUT space. This space is deeply connected to the study of the monopole scattering and Kaluza-Klein monopole.

In this paper we have restricted ourselves to the contribution of the spin contained in the Killing scalars $B^\alpha(x,\psi)$.

A general solution for the motion on a cone is presented when $Q \neq 0$.

The case when $Q = 0$ was investigated in detail. In spite of the complexity of the equations, we are able to give an exact solution for the motion on a cone.

The planar motion was investigated and an exact solutions in the cases $Q = 0$ and $Q \neq 0$ was presented.

In the case of Taub-NUT metric the condition $Q = 0$ plays a crucial role and changes dramatically the form of the exact solutions.

On the other hand it is desirable to have a deeper understanding of the role of the Runge -Lenz vector for the motion of spinning particles. The existence of this vector can be related to a Killing-Yano 2 form [10]. A very interesting case to investigate is when a scalar field is coupled to the metric tensor field.

The work is in progress [18].

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