Zeitschrift: Helvetica Physica Acta

Band: 67 (1994)

Heft: 2

Artikel: On the size of the state-space for systems of quantum particles with spin

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DOI: https://doi.org/10.5169/seals-116643

Helv Phys Acta
Vol. 67 (1994)

0018-0238/94/020127-17$1.50+0.20/0

On the Size of the State-Space for Systems of Quantum Particles with Spin*

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(18.XI.1993, revised 18.II.1994)

Abstract. The states of spin $1/2$ are parametrized by the points of the sphere S^2, i.e. by the directions in space. For the usual description of the states of higher order spins however, additional (rotation invariant) parameters are needed. We study some consequences of the hypothesis that these parameters are redundant and that the states of any spin are parametrized by the directions in space. This corresponds to the restriction of the statespace to the subset of coherent spin states.

1 Introduction

The standard definition of a particle with spin is originally due to E.P. Wigner [1]. He associates a relativistic particle of spin s to a representation of the Poincaré group that is induced from an irreducible projective representation of order s of the rotation group $SO(3)$. This kinematical definition of a particle with spin can be translated to the Galilean framework [2].

The Galilean particle of spin s ($s = 1/2, 1, \cdots$) is associated with the Hilbert space $L^2\left(R^3, C^{2s+1}; d^3x\right)$, i.e. the set of measurable maps $\psi : R^3 \to C^{2s+1}$ such that $\int \bar{\psi}\psi d^3x <$

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∞. The rotations $R \in SO(3)$ are then represented by the unitary transformations

$$(U(R)\psi)(x) = D(R)\psi(R^{-1}x)$$

where D denotes the irreducible projective representation of $SO(3)$ on C^{2s+1}. The "observables" of the system; the momentum, position and spin are represented by the operators

$$\begin{aligned}(\hat{p}_i\psi)(x) &= \frac{1}{i}\partial_{x^i}\psi(x)\\ (\hat{q}^i\psi)(x) &= x^i\psi(x)\\ (\hat{S}_i\psi)(x) &= S_i\psi(x)\end{aligned}$$

where S_1, S_2, S_3 is a basis for the generators of D.

The state space of a quantum particle is not the Hilbert space $L^2(R^3, C^{2s+1}; d^3x)$, but the complex projective manifold modelled thereon, i.e.

$$\begin{aligned}B(R^3, C^{2s+1}) = \{\psi \in L^2(R^3, C^{2s+1}; d^3x) | \psi \sim \psi e^{i\alpha} \text{ for}\\ \alpha \in R \text{ mod} 2\pi \text{ and } \int |\psi|^2 d^3x = 1\} \ .\end{aligned}$$

To get a better insight into the reason for this choice for state manifold we must look more closely at the structure of the Schrödinger equation. The Hilbert space $L^2(R^3, C^{2s+1}; d^3x)$ has a canonical symplectic structure symbolized by $\int \sum_j i d\psi_j \wedge d\bar{\psi}_j d^3x$ and inherited from the canonical symplectic structure $\sum_j i dz_i \wedge d\bar{z}_j$ on C^{2s+1}. Choosing the function

$$\mathcal{H} \ : \ L^2(R^3, C^{2s+1}; d^3x) \to R \ ; \ (\psi, \bar{\psi}) \mapsto \mathcal{H}(\psi, \bar{\psi}) = \int \bar{\psi} H \psi d^3x$$

as Hamiltonian, where H is the energy operator, it follows that the Schrödinger equations

$$\partial_t\psi_t = \frac{1}{i}H\psi_t \text{ and } \partial_t\bar{\psi}_t = -\frac{1}{i}(H\bar{\psi}_t)$$

are the Hamilton equations with respect to the symplectic form. Also, the Poisson bracket defined by the symplectic form has the property

$$\{\mathcal{A}_1, \mathcal{A}_2\} = \frac{1}{i}\int \bar{\psi}[A_1, A_2]\psi d^3x$$

for any "functions" of the form $\mathcal{A}(\psi, \bar{\psi}) = \int \bar{\psi} A \psi d^3x$.

All Hamiltonians considered in quantum theory are of the above quadratic form. Thus they do not depend on the global phase, which is a cyclic variable. The conjugate variable, the norm, is therefore a constant of motion. Accordingly, one can ignore the global phase and fix the value of the norm once and for all. None of these variables have any dynamical significance, therefore they do not form part of the characterisation of the state.

It follows from the above discussion that $B(R^3, C^{2s+1})$ is a symplectic submanifold of the original Hilbert space. Moreover the restriction of the Schrödinger equations to this submanifold are Hamilton equations.

This paper is devoted to a study of "further" restrictions of the *a priori* linear state spaces for particles with spin $s > 1/2$. In standard quantum mechanics the state of a particle of spin s is assumed to be described by $2s+1$ complex functions, that is by $4s+2$ real functions. The particle degrees of freedom are described by the density and the "global phase". The other parameters which correspond to the points on the $4s$ dimensional sphere describe the degrees of freedom of the spin s. For spin 1/2 we thus get a two-dimensional sphere, and the spin degrees of freedom can be identified with directions in space. For higher order spins we get more parameters, all of which can be chosen rotationally invariant. The question is then what could be the geometrical and physical significance of these: how could they be measured? So far nobody seems to have an answer to these questions: could it be that they have no physical significance at all, but are introduced simply because of *a priori* imposed conditions of "linearity'? In the following we study some of the theoretical consequences of the hypothesis that the additional parameters (to the direction in space) are redundant and should be fixed. It turns out that there exists a canonical choice of value for these parameters that reduces the original state space to the submanifold consisting of "coherent" spin states.

There are two kinds of consequences of this choice, the formal and the "physical". The main formal consequence is that the submanifold of the coherent spin states (for any spin s) is a rotationally invariant symplectic submanifold of the state manifold $B(R^3, C^{2s+1})$ of standard quantum mechanics. Thus the main structure of the theory as it is explained above is preserved by the reduction. This is also to some extent the case for the dynamics. The (quadratic) Hamiltonian function defined on the big space by a hermitean operator can be restricted to a Hamiltonian function on the reduced space. When the Schrödinger operator is linear in the spin operators, the dynamics on the big space leaves invariant the reduced state space, and induces a dynamics on this space which is "the same" as the dynamics defined by the reduced Hamiltonian. In general however, the Schrödinger evolution on the big space does not move a coherent spin state into a coherent spin state.

As concerns the "physical" consequences for the description of physical systems, our hypothesis is that it is realising the "physically realisable states". It is not possible to prove such a hypothesis, only to look for counterexamples. In this paper we show that the application of quantum mechanics to atomic physics is not in contradiction with this hypothesis. In fact, its application to the system of two particles shows that the energies of the stationary states will be the same on the restricted as on the big space. However, the parametric degeneracy of the stationary solutions is reduced for rotationally invariant models, and one must take into account non-stationary solutions to reproduce the "energy spectrum" for systems in external fields. The construction presented shows that also in the case of the restricted theory it will be possible to interpret the internal angular momentum as spin.

2 Mathematical preliminaries

Let $E \to X;\ (x;y) \mapsto (x)$ be a fibered manifold, and denote by $\tilde{\gamma}\ :\ X \to E;\ (x \to (x);\gamma(x))$ a section of E. Certain subsets of sections can be considered as function spaces by defining appropriate topologies. The construction is functorial, that is the topological function space $\mathcal{F}(E)$ constructed on the sections of a fibered manifold E inherits a part of its structure from E; if E is a vectorbundle on X, then $\mathcal{F}(E)$ will be a topological vector space. Moreover, morphisms

$$\Phi\ :\ E \to F\ ;\ (x;y) \mapsto (\varphi(x);\Phi(x;y))$$

induce morphisms $\mathcal{F}(\Phi)\ :\ \mathcal{F}(E) \to \mathcal{F}(E)$ [3].

Examples of such constructions are the Hilbert spaces $L^2(R^3, C^{2s+1}; d^3x)$ for $s = 0, 1/2, 1, \cdots$. The fibered manifolds are

$$R^3 \times C^{2s+1} \to R^3\ ;\ (x;z) \mapsto (x)$$

where C^{2s+1} is endowed with the norm $|z|^2 = \sum_j \bar{z}_j z_j$. The functor L^2 associates the square-integrable Lebesque measurable sections of $R^3 \times C^{2s+1} \to R^3$ with vectors in the Hilbert space $L^2(R^3, C^{2s+1}; d^3x)$. This space inherits its complex linear structure and norm from C^{2s+1}. The morphisms

$$\Phi\ :\ R^3 \times C^{2s+1} \to R^3 \times C^{2s+1}\ ;\ (x;z_i) \mapsto (\varphi(x);\Phi(x)_i^j z_j)$$

induce morphisms

$$L^2(\Phi)\ :\ L^2(R^3, C^{2s+1}; d^3x) \to L^2(R^3, C^{2s+1}; d^3x')\ .$$

In particular, if Φ is an isomorphism, then $L^2(\Phi)$ defines an isometry. Thus, if

$$\Phi\ :\ G \times R^3 \times C^{2s+1} \to R^3 \times C^{2s+1}\ ;\ (g;x;z_i) \mapsto ((\varphi_g(x);\Phi_g(x)_i^j z_j)$$

defines an unitary action of G on $R^3 \times C^{2s+1}$, then

$$\begin{aligned} &L^2(\Phi) : G \times L^2(R^3, C^{2s+1}; d^3x) \to L^2(R^3, C^{2s+1}; d^3x')\ ; \\ &(g;\psi) \mapsto \Phi_{gi}^{\ j}\psi_j \end{aligned}$$

defines an isometric action of G on $L^2(R^3, C^{2s+1}; d^3x)$. By applying the canonical identification, we get the standard form

$$\begin{aligned} U\ :\ G \times L^2(R^3, C^{2s+1}; d^3x) &\to L^2(R^3, C^{2s+1}; d^3x)\ ; \\ (g;\psi_i(x)) \mapsto (U(g)\psi)_i(x) &= \sqrt{j(\varphi)(x)}\Phi_g(\varphi_g^{-1}(x))_i^j \psi_j(\varphi_g^{-1}(x)) \end{aligned}$$

where $j(\varphi)$ denotes the jacobian of φ.

3 Nonlinear representations of the state space

B is also a functor, but from the category of fibered bundles into the category of Banach manifolds. Thus, if

$$\Phi \; : \; R^3 \times C^{2s+1} \to R^3 \times M \; ; \; (x;z) \mapsto (x;\Phi(z))$$

is a diffeomorphism (between real manifolds), then

$$B(\Phi) \; : \; B(R^3, C^{2s+1}) \to B(R^3, M)$$

is a diffeomorphism. This means that one can construct local representations of the state space by simply exhibiting local representations of the symplectic manifold $(C^{2s+1}, \sum_j idz_j \wedge d\bar{z}_j)$.

i) spin 0.
The spinor space is C and the symplectic form $idz \wedge d\bar{z}$. C is diffeomorphic to the plane R^2, or again to $R_+ \times S^1 \cup \{0\}$ *). A local representative of the corresponding diffeomorphism (that is restricted to a chart)

$$c \; : \; R_+ \times S^1 \cup \{0\} \to C \; ; \; (\rho, w) \mapsto (z)$$

is defined by

$$z = \sqrt{\rho} e^{iw} \; .$$

One easily verifies that c is a symplectomorphism with respect to the symplectic form $d\rho \wedge dw$, that is

$$c^*(idz \wedge d\bar{z}) = d\rho \wedge dw \; .$$

ii) spin 1/2.
The spinor space is C^2, which is diffeomorphic to $R_+ \times S^3 \cup \{0\}$. A local diffeomorphism

$$c \; : \; R_+ \times S^3 \cup \{0\} \to C^2 \; ; \; (\rho, \tau, w, \nu) \mapsto (z_1, z_2)$$

is defined by

$$z_1 = \sqrt{\rho/2}\sqrt{1+\tau/\rho}\, e^{i(w+\nu)}$$
$$z_2 = \sqrt{\rho/2}\sqrt{1-\tau/\rho}\, e^{i(w-\nu)} \; .$$

Again the symplectic structure on $R_+ \times S^3 \cup \{0\}$, locally defined by $\omega = d\rho \wedge dw + d\tau \wedge d\nu$, is equivalent to the canonical symplectic structure on C^2, that is

$$c^*(idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2) = idc_1 \wedge d\bar{c}_1 + idc_2 \wedge d\bar{c}_2 = \omega \; .$$

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*) S^n denotes the n-dimensional sphere. S^1 is the circle.

The rotation group $SO(3)$ acts on C^2 by the spinor representation D. A basis for the generators of D is provided by the operators

$$(1/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},1/2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},1/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$$

which thus also represents the spin 1/2 observables.

Let s' denote the quadratic functions

$$s'_i \ : \ C^2 \to R \ ; \ (z) \mapsto (\bar{z})S_i(z) \ .$$

The corresponding functions s_i on $R_+ \times S^3 \cup \{0\}$ are

$$(s_i) = (s'_i \circ c) \ : \ R_+ \times S^3 \cup \{0\} \to R$$
$$(\rho, \tau, w, \nu) \mapsto (\rho/2\sqrt{1-\tau^2/\rho^2}\cos 2\nu \ , \ -\rho/2\sqrt{1-\tau^2/\rho^2}\sin 2\nu \ , \ \tau/2) \ .$$

By construction (s_i) transforms as a vector under rotations, whereas ρ is invariant.

Now $(n_i) = (2s_i/\rho)$ is a unit vector, thus it is meaningful to consider the spin S_n in the "direction" (n_i),

$$S_n = \sum_i n_i S_i = 1/2 \begin{pmatrix} \tau/\rho & \sqrt{1-\tau^2/\rho^2}\, e^{i2\nu} \\ \sqrt{1-\tau^2/\rho^2}\, e^{-i2\nu} & -\tau/\rho \end{pmatrix} \ .$$

Using c we can parametrize the spinors $(z) \in C^2$ by $R_+ \times S^3 \cup \{0\}$:

$$(z) \circ c = \sqrt{\rho/2}\begin{pmatrix} \sqrt{1+\tau/\rho} & e^{i\nu} \\ \sqrt{1-\tau/\rho} & e^{-i\nu} \end{pmatrix} e^{iw} \ .$$

We observe that the spinors are eigenvectors with eigenvalue 1/2 of the spin in the direction (n_i). Accordingly, a rotation by an angle φ around the axis (n_i) simply changes the global phase by $\varphi/2$.

The unitary group of C^2 endowed with the hermitean scalar product is $SU(2)$ which is also the universal covering group of the rotation group $SO(3)$. The action of $SU(2)$ on C^2 foliates $C^2 \simeq R^4$ into a collection of submanifolds, leading to the decomposition $C^2 \simeq R_+ \times S^3 \cup \{0\}$. In fact, $\rho = z_1\bar{z}_1 + z_2\bar{z}_2$ is the "only" invariant function on C^2. Moreover, the stability subgroups are $SU(2)$ for the origin and $\{e\}$ otherwise. The surfaces of transitivity are therefore $SU(2)/SU(2) = \{0\}$ and $SU(2)/\{e\} = S^3$. The canonical action of $SU(2)$ on S^3 is compatible with the Hopf fibration of S^3 over S^2 (with fiber S^1). This follows from the fact that $U(1)$ is a subgroup of $SU(2)$ and $SU(2)/U(1) = S^2$.

Returning to the results of the local study, we note that (τ, ν) parametrizes the base space S^2 in the fibration $S^3 \to S^2$ and w the fibers, and that the spin degrees of freedom therefore are entirely parametrized by S^2, that is

the spin is a direction.

Finally, we note that the functions (s_i) form a basis for the generators of the rotations in the Poisson bracket formalism defined by the symplectic form ω; that is

$$\begin{aligned}\{s_i, s_j\} &= \partial_\rho s_i \partial_w s_j + \partial_\tau s_i \partial_\nu s_j - \partial_\rho s_j \partial_w s_i - \partial_\tau s_j \partial_\nu s_i = \\ &= \epsilon_{ijk}\, s_k \;.\end{aligned}$$

The definition of any spin $s > 1/2$ can be modified so as to conform with the hypothesis that spin is a direction *).

<u>Proposition 1</u> : Let $s = 1/2, 1, 3/2, \ldots$ and let

$$M = \begin{cases} R_+ \times S^3 \cup \{0\} \text{ for } s = 1/2, 3/2, \ldots \\ R_+ \times \tilde{S}^3 \cup \{0\} \text{ for } s = 1, 2, \ldots \end{cases}$$

($\tilde{S}^3$ is the projective space associated with S^3). Then there exists embeddings $c : M \to C^{2s+1}$, such that :

i) $S_n c = sc$, $S_n = \sum_i n_i S_i$, $n_i = s_i/\rho s$ where $s_i : M \to R$ is defined by

$$\begin{aligned} s_1(.) &= s\rho\sqrt{1-\tau^2/\rho^2}\cos(\nu/s) \\ s_2(.) &= -s\rho\sqrt{1-\tau^2/\rho^2}\sin(\nu/s) \\ s_3(.) &= s\tau \end{aligned}$$

and S_i is a basis for the generators of D on C^{2s+1}.

ii) $s_i = \bar{c} S_i c$.

<u>Proof</u> : By computation. We note that in the standard basis on C^{2s+1} diagonalizing S_3, the components c_m of c are given by the "spherical harmonics"

$$c_m = (\frac{1}{2})^s (\binom{2s}{s-m})^{1/2} (1+\tau/\rho)^{1/2(s+m)} (1-\tau/\rho)^{1/2(s-m)} e^{im\nu/s} e^{iw} \sqrt{\rho}$$

where () denotes the binominal coefficients and $m = -s, -s+1, \cdots, s$. c is normalized, that is

$$\begin{aligned} |c|^2/\rho &= \sum_{m=-s}^{s} (\frac{1}{2})^{2s} \binom{2s}{s-m} (1+\tau/\rho)^{s+m} (1-\tau/\rho)^{s-m} \\ &= ([(1+\tau/\rho) + (1-\tau/\rho)]/2)^{2s} = 1. \end{aligned}$$

Thus

$$0 = \partial_\tau |c|^2 = \frac{1}{\rho} \sum_m 2|c_m|^2 \frac{m - s\tau/\rho}{1-\tau^2/\rho^2}$$

$$0 = \partial_\tau^2 |c|^2 = \frac{2}{\rho^2(1-\tau^2/\rho^2)} \sum_m |c_m|^2 (2m^2 - s - (4ms - 2m)\tau/\rho + (2s^2 - s)\tau^2/\rho^2)$$

*) The morphisms c and the functions s_i and n_i depend on s.

or

$$\sum_m |c_m|^2 m = s\tau \text{ and } \sum_m |c_m|^2 m^2 = \rho s/2 + (s^2 - s/2)\tau^2/\rho \ .$$

By means of these relations one easily proves the statements of this and the following propositions.

Proposition 2 : $c : M \to C^{2s+1}$ is a symplectic map,

$$c^*(\sum_m i dz_m \wedge d\bar{z}_m) = \sum_m i dc_m \wedge d\bar{c}_m = \omega \ .$$

Proof : By computation.

Corollary 3 : The spin functions (s_i) define a basis for a representation of the Lie algebra of the rotation group on (M, ω), where

$$\{s_i, s_j\} = \in_{ijk} s_k \ .$$

Proposition 4 : The restriction is compatible with the action of the rotation group, that is the following diagram commutes

$$\begin{tikzcd} C^{2s+1} \arrow[r, "D_g"] & C^{2s+1} \\ M \arrow[u, "c"] \arrow[r, "\sigma_g"'] & M \arrow[u, "c"'] \end{tikzcd}$$

for all $g \in SO(3)$. Here σ denotes the action generated by (s_i).

Proof : The criteria for the commutativity of the diagram can be expressed by the local conditions

$$\{s_j, c\} = iS_j c \quad j = 1, 2, 3 \ .$$

This can be verified by computation.

Remark : Let $b : C^2 \to M$ be the symplectomorphism locally defined by

$$\begin{aligned} \rho &= |z_1|^2 + |z_2|^2 \\ \tau &= |z_1|^2 - |z_2|^2 \\ w &= \frac{s}{2i} \ell n(z_1 z_2 / \bar{z}_1 \bar{z}_2) \\ \nu &= \frac{s}{2i} \ell n(z_1 \bar{z}_2 / \bar{z}_1 z_2) \end{aligned}$$

that is

$$b^*\omega = i2s(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \ .$$

The spin s is represented by the functions

$$s_i \circ b = s(\bar{z}\sigma_i z)$$

and, in particular, we can compose with c to obtain a globally defined representation

$$c \circ b \ : \ C^2 \to C^{2s+1}$$

defined by

$$c_m \circ b = (|z_1|^2 + |z_2|^2)^{-s+\frac{1}{2}} \begin{pmatrix} 2s \\ s-m \end{pmatrix}^{\frac{1}{2}} (z_1)^{s+m}(z_2)^{s-m} \ .$$

We note that for $s = n + 1/2$, $n = 0, 1, 2, \cdots$, $c \circ b$ is injective, that is isomorphic to its image. For $s = n$, however, $c \circ b$ is a double cover of the image since $c \circ b(z_1, z_2) = c \circ b(-z_1, -z_2)$.

The states discussed in this paragraph are known as the coherent spin states [4]. One could use the parametrization $c \circ b$, however, the parametrization c has some merits and will be applied throughout this paper.

4 Reduced representation of the Schrödinger equations

The energy function for a "free" quantum particle of spin s is

$$\mathcal{H}'(\psi, \bar{\psi}) = \int h'(\psi, \bar{\psi}, \nabla_i \psi, \cdots) d^3x \ ,$$

where the energy density h' is defined by *)

$$h'(\cdot) = -Re(\bar{\psi}\frac{1}{2m}\nabla^2\psi) \simeq \frac{1}{2m}|\frac{1}{i}\nabla\psi|^2$$

for $\psi \in L^2(R^3, C^{2s+1}; d^3x)$. Similarly the momentum density is given by

$$\pi'_j(\cdot) = Re(\bar{\psi}\frac{1}{i}\nabla_j\psi) \ .$$

i) spin 0.
The pullbacks of h' and π'_j under c are easily determined,

$$h(\cdot) = \frac{1}{2m\rho}(\rho\nabla w)^2 + \frac{1}{2m}(\nabla\sqrt{\rho})^2$$
$$\pi_i(\cdot) = \rho\nabla_i w \ .$$

*) There exist different equivalent forms : $a \sim b$ iff $a - b = \nabla_i f^i$.

We note that the energy density function consists of two terms. It is natural to interpret the first as the kinetic energy density and to identify the second term with an internal energy density. Since c is a symplectomorphism transforming the conjugate variables $(\psi, \bar{\psi})$ into the conjugate variables (ρ, w), the Hamilton equations *)

$$\begin{aligned} \partial_t \rho &= \nabla_w h = -(\rho \nabla w/m) \\ \partial_t w &= -\nabla_\rho h = -(\frac{1}{2m}\nabla w^2 + \frac{1}{2m}\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}) \end{aligned}$$

are equivalent to the Schrödinger equations. This representation of the Schrödinger equations is known as the Madelung equations.

ii) spin $s > 0$.
The pullbacks of h' and π'_j under c are

$$\begin{aligned} h(\cdot) &= \frac{1}{2m\rho}(\rho\nabla w + \tau\nabla\nu)^2 + \frac{1}{2m}(\nabla\sqrt{\rho})^2 + \frac{1}{2m}\rho f(\tau/\rho, \nabla(\tau/\rho), \nabla\nu) \\ \pi_i(\cdot) &= \rho\nabla_i w + \tau\nabla_i\nu \end{aligned}$$

where

$$\begin{aligned} f(\cdot) &= \frac{1}{2s}(1 - \tau^2/\rho^2)(\nabla\nu^2 + s^2(\frac{\nabla(\tau/\rho)}{1 - \tau^2/\rho^2})^2) \\ &= \frac{\rho}{2s}\sum_{ij}\nabla_i(s_j/\rho)\nabla_i(s_j/\rho) = \frac{1}{2s\rho}(\sum_{ij}\nabla_i s_j \nabla_i s_j) - 2s(\nabla\sqrt{\rho})^2 \ . \end{aligned}$$

The energy density now consists of three terms, two of the form found already for spin 0 and a third one that it is natural to associate with the internal energy density of the spin degrees of freedom.

Also the pullbacks for hamiltonians defining more complicated models are easily computed. Thus for energy operators of the form

$$H = \frac{1}{2m}\sum_i(\frac{1}{i}\nabla_i - A_i)^2 + V$$

where A and V are functions on R^3, the pullback is

$$\frac{1}{2m\rho}\sum_i(\rho\nabla_i w + \tau\nabla_i\nu - \rho A_i)^2 + \rho V + \frac{1}{2m}(\nabla\sqrt{\rho})^2 + \frac{1}{2m}\rho f \ .$$

For all of these the "Schrödinger equations" read

$$\partial_t\rho = \nabla_w h, \qquad \partial_t\tau = \nabla_\nu h, \qquad \partial_t w = -\nabla_\rho h \text{ and } \partial_t\nu = -\nabla_\tau h \ .$$

*) Note that $\nabla_y \equiv \partial_y - \nabla_i\partial_{\nabla_i y} + \nabla_i\nabla_j\partial_{\nabla_i\nabla_j y}$.

5 Systems of two particles of spin 0: standard description

A system of two particles of spin 0 is associated with the Hilbert space

$$L^2(R^3, C; d^3x_1) \otimes L^2(R^3, C; d^3x_2) \simeq L^2(R^3, C; d^3x_1 d^3x_2) .$$

The rotations act by the unitary transformations

$$(U(R)\psi)(x_1, x_2) = \psi(R^{-1}x_1, R^{-1}x_2) .$$

For a system of two particles of masses m_1 and m_2, one can separate the center of mass and the internal system by means of the barycentric coordinates. This separation is defined by the unitary transformation ($M = m_1 + m_2$)

$$L^2(R^6, C; d^3x_1 d^3x_2) \to L^2(R^6, C; d^3X d^3x) ,$$

$$\psi(x_1, x_2) \mapsto \varphi(X, x) = \psi(X - \frac{m_2}{M}x, X + \frac{m_1}{M}x) .$$

The observables associated with this separation are the momentum P and position Q of the center of mass and the momentum p and position q of the internal system.

A model of an isolated system of two particles in interaction is defined by the (self-adjoint) energy operator H which by assumption is of the form

$$H = \frac{P^2}{2M} + h(p, q)$$

where h is invariant under rotation. The form of the energy operator justifies the separation of variables and permits the independent investigation of the spectral properties of $h_{CM} = \frac{P^2}{2M}$ and h, thus simplifying the determination of the "stationary" solutions of the Schrödinger equation.

For an important class of models, h (considered as an operator on the internal space $H_i = L^2(R^3, C; d^3x)$) has a negative discrete spectrum and a positive continuous spectrum, that is $H_i = H_i^c \oplus H_i^d$.

Of special interest are the solutions for which the internal system is stationary, but for which no particular condition is imposed on the motion of the center of mass. These solutions are of the form

$$\Psi_t(X, x) = \sum_m C_{tm}(X)\psi_m(x)$$

where (ψ_m) denotes a basis of eigenvectors of h for a spectral subspace in H_i, and $C_t \in L^2(R^3, H_i; d^3X)$ satisfies

$$\partial_t C_t = \frac{1}{i} H_{CM} C_t .$$

Due to the rotation invariance of h, its spectral subspaces in H_i each carry a representation of the rotation group. The corresponding decomposition thus typically looks like (without accidental degeneracy)

$$H_i^d = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1} C^{2\ell+1} .$$

Moreover, for each (n,ℓ) there is associated a spectral value for h, the internal energy $e_{n\ell}$. The solutions for which the internal system has the internal energy $e_{n\ell}$ are thus of the form

$$\Psi_{n\ell}(\vec{X},\vec{x}) = \sum_{m=-\ell}^{\ell} C_{tm}(\vec{X})R_{n\ell}(|\vec{x}|)Y_m^\ell(\frac{\vec{x}}{|\vec{x}|})$$

that is they constitute the space $L^2(R^3, C^{2\ell+1}; d^3X)$. Accordingly, the center of mass appears as a particle of mass M and spin ℓ. In fact, the internal spin density for given (n,ℓ) is

$$s_{ti} = \int \Psi_{tn\ell}(X,x)L_i\Psi_{tn\ell}(X,x)d^3x = (\bar{C}_t(X)S_iC_t(X))$$

where S_i denote the generators of the rotations on $C^{2\ell+1}$ in a basis diagonalizing S_3, and $L_i = \in_{ijk}\; x_j\frac{1}{i}\nabla_k$ denotes the generators of the rotations on H_i. This shows that the standard definition of spin is compatible with the hypothesis that spin is some sort of internal angular momentum. In order to preserve an analoguous compatibility for the description of spin presented in the last paragraph however, one must assume that only coefficients of the form $c\;:\;R_+\times\tilde{S}^3\cup\{0\}\to C^{2\ell+1}$ are admissible;

$$\Psi_{tn\ell}(\vec{X},\vec{x}) = \sum_m c_m^\ell(\rho_t(\vec{X}),\tau_t(\vec{X}),w_t(\vec{X}),\nu_t(\vec{X}))R_{n\ell}(|\vec{x}|)Y_m^\ell(\frac{\vec{x}}{|\vec{x}|}) .$$

In other words, one must assume that the states of a system of two particles constitute only a submanifold of the "state space" associated with the tensor product.

6 Characteristics of the rotation invariant "stationary" solutions

Proposition 5 : Let $\vec{u}$ and $\vec{v}$ be the maps $R_+\times\tilde{S}^3\cup\{0\}\to R^3$ defined by $u_j + iv_j = e^{iw/\ell}(a_j+ib_j) = w_j$, with

$$(a_i) = (-\tau/\rho\cos\nu/\ell, \tau/\rho\sin\nu/\ell, \sqrt{1-\tau^2/\rho^2})$$
$$(b_i) = (-\sin\nu/\ell, -\cos\nu/\ell, 0)$$

Then $\vec{u}$, $\vec{v}$ and $\vec{n}$ form an orthonormal basis of R^3,

$$\vec{u}\cdot\vec{v} = 0 \text{ and } \vec{u}\wedge\vec{v} = \vec{n} .$$

Moreover, $\vec{u}$, $\vec{v}$ and $\vec{n}$ transform as vectors under the given action of the rotation group,

$$\{s_i, u_j\} = \in_{ijk} u_k \text{ etc. .}$$

Proof : By computation.

Proposition 6 : The internal angular dependence can be given in the form

$$\psi^\ell(\frac{\vec{x}}{|\vec{x}|}) = \sum_m c^\ell_m Y^\ell_m(\frac{\vec{x}}{|\vec{x}|}) = \sqrt{\rho d_\ell}(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|})^\ell,$$

where d_ℓ is the normalization coefficient

$$d_\ell = 2\pi^{3/2}\frac{\ell!}{\Gamma(\ell+1+1/2)} .$$

Proof : Let $\vec{L} = \frac{1}{i}\vec{x}\wedge\nabla$, then by duality

$$\vec{n}\cdot\vec{L}\sum_m c^\ell_m Y^\ell_m = \sum_m (\vec{n}\cdot\vec{S}c)_m Y^\ell_m = \ell\sum_m c^\ell_m Y^\ell_m .$$

Since this is the unique eigenfunction of $\vec{n}\cdot\vec{L}$ with eigenvalue ℓ normalized to ρ, it is sufficient to prove that $\sqrt{d_\ell}(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|})^\ell$ is an eigenfunction of $\vec{n}\cdot\vec{L}$ and that it is normalized to 1. The first part of this statement is proved by direct computation, using the fact that $i\vec{n}\wedge\vec{w} = \vec{w}$; the normalization is easily computed by chosing spherical coordinates for $\vec{x}$ with respect to a cartesian coordinate system with basis $\vec{u}$, $\vec{v}$ and $\vec{n}$. Then,

$$d_\ell\int|\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|}|^{2\ell}d\Omega = d_\ell\int_0^{2\pi}\int_0^{\pi}\sin^{2\ell+1}\vartheta d\vartheta d\varphi = 1 .$$

Remark : The same methodology applies to systems of particles with spin 1/2.

Proposition 7 : For a system composed of a particle with spin 0 and a particle with spin 1/2, the internal angular momentum eigenfunctions ψ^j satisfying the conditions

$$\begin{aligned}\vec{L}^2\psi^j &= \ell(\ell+1)\psi^j\\ \vec{n}\cdot\vec{J}\psi^j &= \vec{n}\cdot(\vec{L}+\vec{S})\psi^j = j\psi^j\\ \vec{J}^2\psi^j &= j(j+1)\psi^j\end{aligned}$$

are of the form

$j=\ell+\frac{1}{2}$:

$$\begin{aligned}\psi^j &= \sqrt{\rho d_{j-\frac{1}{2}}}(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|})^{j-\frac{1}{2}}c\\ \vec{L}\cdot\vec{S}\psi^j &= \frac{1}{2}(j-\frac{1}{2})\psi^j\end{aligned}$$

$j=\ell-\frac{1}{2}>0$:

$$\begin{aligned}\psi^j &= \sqrt{\rho d_{j-\frac{1}{2}}}(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|})^{j-\frac{1}{2}}\frac{\vec{x}\cdot\vec{\sigma}}{|\vec{x}|}c\\ \vec{L}\cdot\vec{S}\psi^j &= -\frac{1}{2}(j+3/2)\psi^j\end{aligned}$$

Proposition 8 : For a system of two particles of spin 1/2, the internal angular momentum eigenfunctions ψ^j satisfying the conditions

$$\begin{aligned} \vec{L}^2\psi^j &= \ell(\ell+1)\psi^j \\ \vec{n}\cdot\vec{J}\psi^j &= \vec{n}\cdot(\vec{L}+\vec{S}_1+\vec{S}_2)\psi^j = j\psi^j \\ \vec{J}^2\psi^j &= j(j+1)\psi^j \end{aligned}$$

are of the form

$j=\ell+1$:

$$\psi^j = \sqrt{\rho d_{j-1}}\left(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|}\right)^{j-1}\begin{pmatrix}\vec{w}\\0\end{pmatrix}$$

$$\vec{L}\cdot\vec{S}_1\psi^j = \vec{L}\cdot\vec{S}_2\psi^j = \frac{1}{2}(j-1)\psi^j$$

$j=\ell=0$:

$$\psi^j = \begin{pmatrix}\vec{n}\\0\end{pmatrix}$$

$$\vec{L}\cdot\vec{S}_1\psi^j = \vec{L}\cdot\vec{S}_2\psi^j = 0\psi^j$$

$j=\ell>0$:

$$\psi^j = \sqrt{\frac{\rho}{2}d_j}\left(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|}\right)^{j-1}\begin{pmatrix}\frac{\vec{x}\wedge\vec{w}}{|\vec{x}|}\\ i\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|}\end{pmatrix}$$

$$\vec{L}\cdot\vec{S}_1\psi^j = -\frac{1}{2}(\ell+1)\psi^j$$

$j=\ell-1\geq 0$:

$$\psi^j = \sqrt{\rho\frac{2j+1}{j+1}d_j}\left[\left(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|}\right)^j\begin{pmatrix}\frac{\vec{x}}{|\vec{x}|}\\0\end{pmatrix} - \frac{j}{2j+1}\left(\frac{\vec{x}\cdot\vec{w}}{|\vec{x}|}\right)^{j-1}\begin{pmatrix}\vec{w}\\0\end{pmatrix}\right]$$

$$\vec{L}\cdot\vec{S}_1\psi^j = \vec{L}\cdot\vec{S}_2\psi^j = \frac{1}{2}(j+2)\psi^j$$

in the representation where

$$\vec{S}_1 = \left[\frac{1}{2}\begin{pmatrix}0&0&0&-1\\0&0&-i&0\\0&i&0&0\\-1&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&i&0\\0&0&0&-1\\-i&0&0&0\\0&-1&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}\right]$$

$$\vec{S}_2 = \left[\frac{1}{2}\begin{pmatrix}0&0&0&1\\0&0&-i&0\\0&i&0&0\\1&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&i&0\\0&0&0&1\\-i&0&0&0\\0&1&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right].$$

It is of some interest to express the probability and the momentum and angular momentum densities of the states ψ^{ℓ} in terms of $\vec{u}$, $\vec{v}$ and $\vec{n}$:

$$\rho_{n\ell}(\vec{X},\vec{x}) = |\Psi_{n\ell}(\vec{X},\vec{x})|^2 = \rho_\ell(\vec{X})|R_{n\ell}(|\vec{x}|)|^2 d_\ell(\frac{\vec{x}\cdot\vec{u}^2 + \vec{x}\cdot\vec{v}^2}{|\vec{x}|^2})^\ell$$

$$\vec{\pi}_{n\ell}(\vec{X},\vec{x}) = Re(\bar{\Psi}_{n\ell}(\vec{X},\vec{x})\frac{1}{i}\partial_{\vec{x}}\Psi_{n\ell}(\vec{X},\vec{x}) = \ell\rho_{n\ell}(\vec{X},\vec{x})\frac{\vec{n}\wedge\vec{x}}{\vec{x}\cdot\vec{u}^2 + \vec{x}\cdot\vec{v}^2}$$

$$\vec{\ell}_{n\ell}(\vec{X},\vec{x}) = \vec{x}\wedge\vec{\pi}_{n\ell}(\vec{X},\vec{x}) = \ell\rho_{n\ell}(\vec{X},\vec{x})\frac{\vec{x}\wedge(\vec{n}\wedge\vec{x})}{\vec{x}\cdot\vec{u}^2 + \vec{x}\cdot\vec{v}^2} \; .$$

We also note that

$$\int \rho_{n\ell}(\vec{X},\vec{x})d^3x = \rho_\ell(\vec{X})$$

$$\int \vec{\pi}_{n\ell}(\vec{X},\vec{x})d^3x = \vec{0}$$

$$\int \vec{\ell}_{n\ell}(\vec{X},\vec{x})d^3x = \ell\rho_\ell(\vec{X})\vec{n}(\vec{X}) = \vec{s} \; .$$

The hypothesis that spin is some sort of internal angular momentum coincides with the definition of spin as a direction in space provided one restricts the admissible rotation invariant stationary solutions to the kind defined above.

7 Discussion

The restriction of the state space does not have dramatic consequences for the standard applications of quantum mechanics. For the prediction of energy spectra of isolated (rotation invariant) systems, the restriction appears as a limitation on the set of possible linear combinations of the stationary solutions of the Schrödinger equation. This has no consequence for the prediction the energy spectrum of the system, only for the degree of degeneracy. In fact, while the degeneracy of the "energy value" with total angular momentum is parameterised by S^4 in the standard theory, it is parameterised by S^2 in the restricted theory. A solution referred to the restricted theory thus contains much more information about the system.

The reduction of the degree of degeneracy could a priori seem to be in contradiction with the observation that for an atom in an external magnetic field, the degenerate energy levels are split in such a way as to remove the degeneracy (Zeeman effect). It is however possible to avoid this argument.

The predictions of atomic energy spectra are based on the assumption that the stationary solutions of the Schrödinger equation associated with the atom models describe "semistable or stable" motions of the system. The notion of stability has no place in quantum mechanics, and the assumption has no theoretical justification, but is justified by the results of experiments: it permits us to compute the energy values observed in the radiation from atomic systems.

In order to be able to reproduce the splitting due to the Zeeman effect on the restricted state space, we must accept to extend the class of motions that are supposed to be semistable. In particular, in the case of the Zeeman effect, we must assume that the $2j+1$ non-stationary solutions

$$c_t^{(m)} = c(\rho_0, \frac{m}{j}\rho_0, \omega_0, \nu_0 + j\omega t)$$

that satisfy

$$\partial_t c_t = \frac{1}{i}\mu \overrightarrow{B} \cdot \overrightarrow{S} c_t$$

are semistable. Here j denotes the total internal angular momentum of the energy level considered, and $\mu \overrightarrow{B} \cdot \overrightarrow{S}$ denotes the energy operator of the atom in the homogeneous magnetic field $\vec{B}$. The energy of the system for the solution $c_t^{(m)}$ is

$$\langle c_t|\mu \overrightarrow{B} \cdot \overrightarrow{S}|c_t\rangle = m\mu B = m\omega.$$

These solutions describe the spin as precessing around the direction of the magnetic field at the angle $\arccos(m/j)$ with frequency ω,

$$(s_{ti}) = (\langle S_i \rangle) = j\rho_0(\sqrt{1-m^2/j^2}\cos(\omega t + \nu_o/j), -\sqrt{1-m^2/j^2}\sin(\omega t + \nu_o/j), m/j)$$

The state space of the spin s admits an action of the group $SO(3,2)$ and thus of the Lorentz group $SO(3,1)$. In fact, one easily verifies by direct computation that the functions

$$\vec{q} = \rho\vec{u}\ ,\ \vec{r} = \rho\vec{v}\ ,\ \rho \text{ and } \vec{s}$$

is closed under the Poisson bracket and, moreover, constitutes a basis for the Lie algebra $so(3,2)$. This observation has been used to construct a Lorentz covariant description of the classical radiation field [5]. The results of this paper show that the spin 1 of the photon is a spin in the sense of a "direction in space". It might be premature to claim that this must be the case for massive particles.

Acknowledgements

I am indebted to C. Piron and D. Moore for numerous discussions on these subjects. I also thank members of the Département de Physique Théorique for warm hospitality, and the Fonds National Suisse for support.

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