Zeitschrift: Helvetica Physica Acta

Band: 67 (1994)

Heft: 1

Artikel: Correlation functions of general observables in dipole-type system. I, Accurate upper bounds

Autor: Brydges, David C. / Keller, Georg

DOI: https://doi.org/10.5169/seals-116641

Helv Phys Acta
Vol. 67 (1994)

0018-0238/94/010043-74$1.50+0.20/0

Correlation Functions of General Observables in Dipole-Type Systems I: Accurate Upper Bounds

By David C. Brydges[1] and Georg Keller[2]

Dept. of Mathematics, University of Virginia
Charlottesville VA 22903, USA

(24. I. 1994)

1.1 Background and examples

This is the first of two papers in which we will prove accurate upper and lower bounds on the decay at large separation of truncated correlation functions of very general observables. The context includes the equilibrium statistical mechanics of dilute dipole gases at equilibrium. We believe that it will extend to many other systems (the Kosterlitz-Thouless phase of two-dimensional Coulomb systems, critical ϕ_d^4, $d \geq 4$, for example).

There have already been a number of studies of truncated correlations for these problems [16, 1, 2, 11, 12, 14, 9, 15], but these methods give rather weak bounds for correlations of observables which are composite in the sine-Gordon field. Consider the following example.

Example 1.1.1. A dipole is described by its position x in a d-dimensional container $\Lambda \subset \mathbf{R}^d$ and a direction given by a unit vector $\hat{e} \in \mathbf{R}^d$. The potential energy, including self energy, of N dipoles, has the form

$$\frac{1}{2} \sum_{1 \leq 1,\, j \leq N} (\ell \hat{e}_i \cdot \partial_{x_i})(\ell \hat{e}_j \cdot \partial_{x_j}) C(x_i - x_j), \tag{1.1.1}$$

where ℓ is a length that characterizes the strength of the dipole and $C(x - y)$ is the potential energy of two charges at x, y. We shall take this to be the Coulomb energy

[1]Research partially supported by NSF Grant DMS-9102584.
[2]Supported by the Swiss NSF.

modified at distances of order 1 or less so that there is no singularity at $x = y$ but C is still positive-definite. Thus the Fourier transform has the form

$$C(p) = \frac{\chi(p^2)}{\sigma p^2}, \quad \text{if } p \neq 0, \tag{1.1.2}$$

where $\chi \in C^\infty$ and decays rapidly as $p^2 \to \infty$ (see Section 1.4). For example $\chi(x)$ can be e^{-x}. We assume that the container Λ is a hypercube whose side has length = side(Λ) and impose periodic boundary conditions on $C(x)$ at the boundary of Λ. We define $C(p = 0) = 0$ to remove the zero mode.

By the sine-Gordon transformation [13, 17], which is reviewed in many places including [3], we can write the grand canonical partition function for these dipoles at activity $z \geq 0$ and inverse temperature β, as

$$Z(\Lambda) := \int d\mu_C(\phi) e^{-V(\Lambda, \partial\phi)}, \tag{1.1.3}$$

where $d\mu_C(\phi)$ is a Gaussian measure on the space of functions $\phi(x)$, which can be taken to be $C^\infty(\Lambda)$ if $\chi(p^2)$ decays faster than any inverse power of p^2.

$$V(\Lambda, \partial\phi) := -2z \int_\Lambda dx \int_{S^{d-1}} d\sigma(\hat{e}) \cos(\sqrt{\beta}\ell\hat{e} \cdot \partial\phi(x)), \tag{1.1.4}$$

where $d\sigma$ is normalized surface measure on the sphere S^{d-1}. A local observable, under the sine-Gordon transformation, is mapped into a functional of ϕ. For example, the density $n(x)$ of dipoles at x becomes

$$\tilde{n}(x) := 2z \int_{S^{d-1}} d\sigma(\hat{e}) \cos(\sqrt{\beta}\ell\hat{e} \cdot \partial\phi(x)). \tag{1.1.5}$$

The expectation of a product of such observables at non-coincident points, $x_\alpha \in \Lambda$, is equal to

$$\langle \prod_\alpha \tilde{n}(x_\alpha) \rangle = \frac{1}{Z(\Lambda)} \int d\mu_C(\phi) e^{-V(\Lambda,\partial\phi)} \prod_\alpha \tilde{n}(x_\alpha). \tag{1.1.6}$$

The following theorem is an immediate consequence of our general result described in the next section.

Theorem 1.1.2. Let $d \geq 1$. Let $\delta > 0$, $\beta\ell^2 \geq 0$. There exists[3] L, $z(L, \beta\ell^2)$, $C(\delta, L, \beta\ell^2)$ such that for all $z \in [0, z(\delta, \beta\ell^2)]$, all x_1, x_2, Λ with side $(\Lambda) \in \{L^N : N \in \mathbf{N}\}$ and side $(\Lambda) \geq L \cdot \|x_1 - x_2\| \geq L$,

$$|\langle \tilde{n}(x_1)\tilde{n}(x_2)\rangle - \langle \tilde{n}(x_1)\rangle\langle \tilde{n}(x_2)\rangle| \leq C(\delta, L, \beta\ell^2) z^2 \|x_1 - x_2\|^{-2(d-\delta)}. \tag{1.1.7}$$

We will see that under the same hypotheses, the same upper bound holds for very general even functions of $\partial\phi$.

[3]We do not show dependence on d and χ.

Let $f(x_1 - x_2) :=$ (left-hand side of (1.1.7)) $\cdot \|x_1 - x_2\|^{2d}$. This paper accomplishes most of the work required to prove that when z is small depending on $\beta\ell^2$, L and x_1, x_2, Λ are as in the theorem, then there exists $c > 0$ such that $f(x_1 - x_2) \geq cz^2\beta^2\ell^4$, but we defer these results to the next paper.

We became interested in this type of problem because the quantum non-relativistic Coulomb plasma at equilibrium at long distances becomes a theory of effective dipoles. This is argued (non-rigorously) in [5] where we produce a simplified model that is claimed to capture this failure of screening. We produce this model by approximations which include integrating out short distance structure. The result is that the original charge density observables become complicated even functionals of $\partial\phi$ so the results of the next section are designed to include this model. We will give the proof that there is no screening within this approximation in the next paper.

The previous work on correlations [12, 9, 15] would at best bound $|\langle\tilde{n}(x_1)\tilde{n}(x_2)\rangle - \langle\tilde{n}(x_1)\rangle\langle\tilde{n}(x_2)\rangle|$ by $\|x_1 - x_2\|^{-d}$ and would give no lower bound.

Example 1.1.3. Consider the observable $(\partial\phi)^4(x) := (\partial\phi(x) \cdot \partial\phi(x))^2$. Under the same hypotheses as Theorem 1.1.2, we can obtain, as a corollary to the theorem in the next section, an upper bound by $O\left(\|x_1 - x_2\|^{-2(d-\delta)}\right)$. This will turn out to be a sharp upper bound despite the dimension of $(\partial\phi)^4$ being $(\text{Length})^{-2d}$ so that two of them would naively decay as $(\text{Length})^{-4d}$. One[4] reason is that $(\partial\phi)^4$ couples to the interaction $\cos(\sqrt{\beta}\ell\hat{e}\cdot\partial\phi)$ to generate a renormalized observable containing $(\partial\phi)^2$. In principle by our methods one can construct a polynomial of 4^{th} degree in $\partial\phi$, $N_4(\partial\phi(x))$, such that the truncated expectation of two of them will decay as $\|x_1 - x_2\|^{-4d}$.

1.2 The main result

Our notation is almost the same as that of the previous papers [6, 8, 10, 7, 9] of this genre. Precise definitions of our terms are provided in Section 1.4. The following discussion covers the main points, informally.

As in the earlier papers we start with a Gaussian measure $d\mu_C(\phi)$ defined on the Sobolev space of functions $\mathcal{H}_s(\Lambda)/\{\text{constants}\}$. We choose an integer $P \geq 2$ and then choose $s \geq 2$ such that $\mathcal{H}_s(\Lambda) \supset C^P(\Lambda)$. The measure $d\mu_C$ is characterized by its covariance whose Fourier transform is

$$C(p) = \frac{\chi(p^2)}{\sigma p^2} \quad \text{for} \quad p \neq 0, \tag{1.2.1}$$

χ is specified further in Section 1.4. At this stage, without loss of generality, we assume $\sigma = 1$. Λ is a torus such that side $(\Lambda) = L^N$ for some $N \in \mathbf{N}$.

[4] $(\partial\phi)^4$ also generates $(\partial\phi)^2$ by a diagram with a tadpole.

We consider perturbations of $d\mu_C(\phi)$ which can be written as a polymer expansion

$$\sum \frac{1}{N!} \sum_{\substack{X_1,\ldots,X_N \subset \Lambda \\ \text{disjoint}}} \prod_{j=1}^{N} \tilde{K}(X_j, \Psi^\phi). \tag{1.2.2}$$

$\tilde{K}$ is defined on a class of sets called complexes which are described in Section 1.4. The argument Ψ^ϕ abbreviates the collection of derivatives $(\partial^\alpha \phi(x))$ for all multi-indices α with $1 \leq |\alpha| \leq P$. Ψ is a set of fields also specified by (x, α), $1 \leq |\alpha| \leq P$, but not necessarily obtained as derivatives of a single field ϕ. We require that $\tilde{K}$ be defined for all Ψ in a neighborhood of Ψ^ϕ for some ϕ. As a functional on $C(\Lambda \times \{\text{indices}\})$, $\tilde{K}(X, \Psi)$ is C^∞. A derivative of order n is a signed measure on $(\Lambda \times \{\text{indices}\})^{\times n}$. We require that as a measure it be supported in $(X \times \{\text{indices}\})^{\times n}$. Thus $\tilde{K}(X, \Psi^\phi)$ is "independent of $\phi(x)$" for $x \notin X$. If $\tilde{K}(X, \Psi)$ has these properties it is said to be *regular* and *local.*

As summarized in Section 1.4, the polymer gas (1.2.2) is an exponential relative to the product $(A \circ B)(X) := \sum_{Y \subset X} A(X \sim Y) B(Y)$. For this reason it is denoted by $\mathcal{E}^{\Box + \tilde{K}} \equiv \mathcal{E}xp[\Box + \tilde{K}]$. $\Box$ is a special function on complexes independent of Ψ. See Section 1.4.

We can think of $\log \int d\mu_C \mathcal{E}xp[\Box + \tilde{K}]$ as a generating function as follows: suppose

$$\tilde{K} = \mu K + \lambda_1 O_1 + \lambda_2 O_2 + \lambda_1 \lambda_2 O_{12}, \tag{1.2.3}$$

where K, O_1, O_2, O_{12} are regular and local, $\mu, \lambda_1, \lambda_2 \in \mathbf{C}$. Then we generate "insertions" by derivatives with respect to λ_1, λ_2, e.g., at $\mu = 1$, with $\tau_{12} := \frac{\partial^2}{(\partial\lambda_1 \partial\lambda_2)}|_{\lambda=0}$,

$$\begin{aligned} \tau_{12}\left\{\log \int d\mu_C \mathcal{E}^{\Box+\tilde{K}}(\Lambda)\right\}|_{\mu=1} &= \frac{1}{Z(\Lambda)} \int d\mu_C \mathcal{E}^{\Box+K} \circ \{O_1 \circ O_2 + O_{12}\}(\Lambda) \\ &\quad - \left(\frac{1}{Z(\Lambda)} \int d\mu_C \mathcal{E}^{\Box+K} \circ O_1(\Lambda)\right) \cdot \\ \cdot \left(\frac{1}{Z(\Lambda)} \int d\mu_C \mathcal{E}^{\Box+K} \circ O_2(\Lambda)\right), \end{aligned} \tag{1.2.4}$$

where $Z(\Lambda) := \int d\mu_C \mathcal{E}xp[\Box + K](\Lambda)$. Define

$$\langle O_1; O_2 | O_{12} \rangle_{\Lambda;K,C} := \text{R.H.S. of (1.2.4)}. \tag{1.2.5}$$

(We will have hypotheses that imply $Z(\Lambda) \neq 0$.)

We say that K is I-type (interaction type) iff K is regular, local, even, Euclidean invariant and real. We say O_1, O_2, O_{12} are O-type (observable type) if they are regular, local, even[5] and $\bar{\alpha}$*-pinned.* Pinning is the important property that encodes the idea that

[5]Results are also easily obtained for observables which are odd, or functionals of ϕ as well as $\partial^\alpha \phi$, $1 \leq |\alpha| \leq P$.

$O_{\bar{\alpha}}$, $\bar{\alpha} \in \{1, 2, (12)\}$, is a local observable. $O_{\bar{\alpha}}$ is $\bar{\alpha}$-pinned iff there exist two points x_α, $\alpha \in \{1,2\}$, such that $\forall X$, $O_\alpha(X) = 0$ if $X - \partial X \not\ni x_\alpha$ and $O_{12}(X) = 0$ if $X - \partial X \not\supset \{x_1, x_2\}$.

Norms. K and $O_{\bar{\alpha}}$ are required to be finite in a norm $\| \quad \|_{G,\Gamma,H}$. For the precise definition please refer to Section 1.4. To understand the content of our results it is helpful to know that when $\|J\|_{G,\Gamma,H}$ is finite, then for all $n \in \mathbf{N}_0$, X, $\phi \in C^P$,

$$\Bigg\| \underbrace{\frac{\partial}{\partial \psi} \cdots \frac{\partial}{\partial \psi}}_{n} J(X, \Psi^\phi) \Bigg\|_{\text{var}} \leq \left(\frac{n!}{H^n} G(X,\phi) \frac{1}{\Gamma(X)} \right) \|J\|_{G,\Gamma,H}, \tag{1.2.6}$$

where $\| \quad \|_{\text{var}}$ is the variation norm of the measure taken pointwise in X, Ψ^ϕ. $G(X, \phi)$ is a weight that specifies how rapidly J is permitted to grow when ϕ (actually $\nabla\phi$) is large. It is given by

$$G(X,\phi) := \exp\left\{ \kappa \sum_{1 \leq |\alpha| \leq s} \int_X |\partial^\alpha \phi|^2 + \frac{\kappa}{c} \int_{\partial X} |\partial \phi|^2 \right\} \tag{1.2.7}$$

for some $\kappa \geq 0$, $c > 0$. $\Gamma(X)$ is a weight which becomes very large when the set X is either large in volume or highly disconnected. It is specified by two parameters A, Q, of which the important one is A. When A is large, $\Gamma(X)$ grows rapidly as the volume and "disconnectedness" of X grow. See Section 1.4. For simplicity we set $Q = 1$.

Parameters. First there are the parameters: $d \geq 1$ (dimension of space), $P \equiv N^0$ of derivatives on which K, $O_{\bar{\alpha}}$ are permitted to depend, $c \equiv$ a constant in the function $G(X, \phi)$, $s =$ index of Sobolev space $\mathcal{H}_s(\Lambda)/\{\text{constants}\}$ in which ϕ lives, χ the cutoff in the covariance. We suppose these have been chosen and we do not show how constants depend on them. The remaining parameters are $\delta \in (0, \frac{1}{2})$, see Theorem 1.1.2, $L \in \mathbf{N}$ which specifies that side $(\Lambda) = L^N$ for $N \in \mathbf{N}$, A, $Q = 1$, specifying Γ, κ specifying G, and H. G, Γ and H are parameters in the norm $\| \cdot \|_{G,\Gamma,H}$.

Choice of parameters. $\forall\, \delta \in (0, \frac{1}{2})$, $\forall\, L \geq L(\delta)$, $\forall\, A \geq A(L)$, $\forall \kappa \in (0, \kappa(L)]$, $\forall\, H \geq H(L, A, \kappa)$, $\forall I$-type K with $\|K\|_{G,\Gamma,H} \leq \rho(L, \kappa, H)$, $\forall\, O$-type $O_{\bar{\alpha}}$ with $\|O_{\bar{\alpha}}\|_{G,\Gamma,H} < \infty$, where $\kappa(L) > 0$, $\rho(L, \kappa, H) > 0$.

Theorem 1.2.1. There exists a choice of parameters as above and $C(L, \delta)$ such that for all Λ with side $(\Lambda) \in \{L^N : N \in \mathbf{N}\}$, all $x_1, x_2 \in \Lambda$ with $\|x_1 - x_2\| \geq 1$ and with side $(\Lambda) \geq \|x_1 - x_2\| \cdot L$:

$$\begin{aligned} |\langle O_1; O_2 | O_{12} \rangle_{\Lambda;K,C}| \;\leq\; & \|x_1 - x_2\|^{-2(d-\delta)} \cdot C(L,\delta) \\ & \cdot \left\{ \|x_1 - x_2\|^{-\frac{1}{2}} \|O_{12}\|_{G,\Gamma,H} + \prod_{\alpha=1}^{2} \|O_\alpha\|_{G,\Gamma,H} \right\}. \end{aligned} \tag{1.2.8}$$

The power $\|x_1 - x_2\|^{-\frac{1}{2}}$ could be replaced by $\|x_1 - x_2\|^{-r}$ for any $r \geq 0$, fixed before the parameters are chosen.

A similar theorem holds if O_1, O_2 are odd functionals of Ψ, but $\|x_1 - x_2\|^{-2(d-\delta)}$ should be replaced by $\|x_1 - x_2\|^{-(d-\delta)}$. Also observables can be permitted to depend on ϕ as well as $\partial^\alpha \phi$, $1 \leq |\alpha| \leq P$, in which case the decay becomes $\|x_1 - x_2\|^{-2(d-2-\delta)}$ for even observables, $\|x_1 - x_2\|^{-(d-2-\delta)}$ for odd observables.

Proof of Theorem 1.1.2. We deduce the result from Theorem 1.2.1 by writing

$$\tilde{n}(x_1)\tilde{n}(x_2)e^{-V(\Lambda,\partial\phi)} = \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}|_{\lambda_\alpha=0} \prod_{\Delta\subset\Lambda} e^{-\tilde{V}(\Delta,\partial\phi)}$$

$$\tilde{V}(\Delta,\partial\phi) := V(\Delta,\partial\phi) + \sum_\alpha \lambda_\alpha \tilde{n}(x_\alpha) \mathbf{1}_{\Delta\ni x_\alpha}, \tag{1.2.9}$$

where Δ is a unit block as defined in Section 1.4 and without loss of generality $x_\alpha \notin \partial\Delta$ for any $\Delta \subset \Lambda$; if not, move Λ. Expand

$$\begin{aligned}\prod_\Delta \left(e^{-\tilde{V}(\Delta,\partial\phi)} - 1 + 1\right) &= \sum \frac{1}{N!} \sum_{\substack{X_1,\ldots,X_N \subset \Lambda \\ \text{disjoint}}} \prod_{j=1}^N \tilde{K}(X_j,\partial\phi) \\ &= \mathcal{E}^{\square+\tilde{K}}(\Lambda), \end{aligned}\tag{1.2.10}$$

where $(X_1,\ldots,X_N)$ is summed over N-tuples of disjoint connected sets that are unions of blocks $\Delta \subset \Lambda$, and

$$\tilde{K}(X,\partial\phi) := \prod_{\Delta\subset X} \left(e^{-\tilde{V}(\Delta,\partial\phi)} - 1\right). \tag{1.2.11}$$

Then

$$\langle\tilde{n}(x_1)\tilde{n}(x_2)\rangle - \langle\tilde{n}(x_1)\rangle\langle\tilde{n}(x_2)\rangle = \langle 0_1; O_2 | O_{12}\rangle_{\Lambda;K,C}, \tag{1.2.12}$$

where $O_\alpha = \frac{\partial}{\partial\lambda_\alpha}|_{\lambda_\alpha=0}\tilde{K}$, $O_{12} = \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}|_{\lambda_\alpha=0}\tilde{K}$, $K = \tilde{K}|_{\lambda_\alpha=0}$. It is easy to prove that there is $(\beta\ell^2)(H) > 0$ such that for $\beta\ell^2 \in [0,(\beta\ell^2)(H)]$

$$z \to 0 \Rightarrow \|K\|_{G,\Gamma,H} = O(z) \text{ for } G,\Gamma,H$$

as in Theorem 1.2.1 and that $\|O_{\bar{\alpha}}\|_{G,\Gamma,H} = O(z)$ (for $\bar{\alpha} \in \{1,2\}$), $O(z^2)$ else. See, for example, the proof of Lemma 5.1 in [6]. It is clear that K is I-type, $O_{\bar{\alpha}}$ is O-type, therefore all hypotheses of Theorem 1.2.1 are implied by the hypotheses of Theorem 1.1.2, and we obtain Theorem 1.1.2 from (1.2.12) and (1.2.8). □

1.3 Bilinear Formulas, Summary of Proof

The proof of Theorem 1.2.1 is accomplished by controlling the renormalization group map RG. RG is a map, for $j \in \mathbf{N}_0$,

$$\left(\Lambda^{(j)}; K^{(j)}, O_{\bar{\alpha}}^{(j)}, C^{(j)}\right) \xrightarrow{RG} \left(\Lambda^{(j+1)}; K^{(j+1)}, O_{\bar{\alpha}}^{(j+1)}, C^{(j+1)}\right), \tag{1.3.1}$$

where $\Lambda^{(j)} := L^{-dj}\Lambda$, $C^{(j)}(p) = \chi(p^2)/(\sigma^{(j)}p^2)$ for $p \neq 0$, $K^{(j)}$ is I-type, $O_{\bar{\alpha}}^{(j)}$, $\bar{\alpha} \in \{1, 2, (12)\}$ are O-type for all j.

RG is induced by a map $\mathcal{T}$ using

$$\begin{aligned} K^{(j+1)} &= \tau_0 \mathcal{T}[\tilde{K}^{(j)}]_{\mu=1} \\ O_{\bar{\alpha}}^{(j+1)} &= \tau_{\bar{\alpha}} \mathcal{T}[\tilde{K}^{(j)}]_{\mu=1}, \end{aligned} \tag{1.3.2}$$

where τ acts on $f = f(\lambda_1, \lambda_2)$ by $\tau_0 f := f(0,0)$, $\tau_\alpha f := (\partial/(\partial\lambda_\alpha) f)(0,0)$, $\tau_{12} f := (\partial^2/(\partial\lambda_1\partial\lambda_2) f)(0,0)$, $\tilde{K}^{(j)} := \mu K^{(j)} + \sum \lambda_{\bar{\alpha}} O_{\bar{\alpha}}^{(j)}$. $\lambda_{12} := \lambda_1\lambda_2$. $\bar{\alpha} \in \{1, 2, (12)\}$.

$\mathcal{T}$ is designed so that

$$\int d\mu_{C^{(j)}} \mathcal{E}^{\square + \tilde{K}^{(j)}}(\Lambda^{(j)}) = N^{(j)} \cdot e^{\Omega_I^{(j)}(\Lambda^{(j)})} \int d\mu_{C^{(j+1)}} \mathcal{E}^{\square + \mathcal{T}[\tilde{K}^{(j)}]}(\Lambda^{(j+1)}), \tag{1.3.3}$$

where $N^{(j)}$ is independent of λ_α, $\Omega_I^{(j)}(\Lambda^{(j)})$ depends on μ, λ_α and will be discussed below.

We iterate (1.3.3), starting with $(\Lambda^{(0)}; K^{(0)}, O_{\bar{\alpha}}^{(0)}, C^{(0)}) = (\Lambda; K, O_{\bar{\alpha}}, C)$ as in Theorem 1.2.1, N times so that $\Lambda^{(N)}$ is a unit block. By setting $\mu = 1$ and applying τ_{12} to the resulting identity we obtain

$$\begin{aligned} \langle O_1; O_2 | O_{12} \rangle_{\Lambda; K, C} &= \sum_{j=0}^{N-1} \tau_{12} \Omega_I^{(j)}(\Lambda^{(j)})|_{\mu=1} \\ &\quad + \tau_{12} \log \int d\mu_{C^{(N)}} \mathcal{E}^{\square + \tilde{K}^{(N)}}(\Lambda^{(N)})|_{\mu=1}, \end{aligned} \tag{1.3.4}$$

which essentially reduces our task to analysis of $\sum \tau_{12} \Omega_I^{(j)}(\Lambda^{(j)})$.

$\mathcal{T}$ is constructed as a composition of maps $\mathcal{F}, S, \mathcal{B}, \mathcal{E}_{II}, \mathcal{E}_I$ whose action on a local regular polymer activity J is defined next.

$$\mathcal{F}: \quad \mathcal{E}^{\square + \mathcal{F}[J]} = \mu_{L,1} * \mathcal{E}^{\square + J}. \tag{1.3.5}$$

$\mu_{L,1}$ is convolution in field space by a Gaussian measure $d\mu_{L,1}$, to be further described below. We shall need the first and second derivatives $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ defined by

$$\begin{aligned} \mathcal{F}^{(1)}[J] &:= \frac{\partial}{\partial\lambda}|_{\lambda=0} \mathcal{F}[\lambda J] \\ \mathcal{F}^{(2)}[J_1, J_2] &:= \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}|_{\lambda=0} \mathcal{F}[\lambda_1 J_1 + \lambda_2 J_2]. \end{aligned} \tag{1.3.6}$$

They are given more explicitly by

$$\begin{aligned} \mathcal{F}^{(1)}[J] &= \mu_{L,1} * J \\ \mathcal{F}^{(2)}[J_2, J_2] &= \mu_{L,1} * (J_1 \circ J_2) - (\mu_{L,1} * J_1) \circ (\mu_{L,1} * J_2), \end{aligned} \tag{1.3.7}$$

see Section 6.

S is rescaling defined by

$$S: \quad S[J](X, \Psi) := J(L^d X, \Psi_L). \tag{1.3.8}$$

We define $(\partial^\alpha \phi)_L(x) := L^{1-d/2-|\alpha|} \partial^\alpha \phi\left(\frac{x}{L}\right)$. This defines $(\Psi^\phi)_L$ and $(\Psi)_L$ in an obvious way.

$\mathcal{B}$ reorganizes a polymer expansion on (scale 1)-polymers into a polymer expansion on (scale L)-polymers. Thus it is defined by

$$\begin{aligned} \mathcal{B}: \quad \mathcal{E}xp_L[\Box_L + \mathcal{B}[J]](X) &= \mathcal{E}xp_1[\Box_1 + J](X) \\ &\quad \forall \text{ (L-scale)-polymers } X. \end{aligned} \tag{1.3.9}$$

$\Box_L$ and $\mathcal{E}xp_L$ are defined as were $\mathcal{E}xp \equiv \mathcal{E}xp_1$ and $\Box \equiv \Box_1$ but with (scale L)-cells (see Section 1.4) replacing (scale 1)-cells. We shall need $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ defined as first and second derivatives of $J = 0$ in analogy to $\mathcal{F}^{(i)}$ above. They are given by (see Section 4),

$$\begin{aligned} \mathcal{B}^{(1)}[J](U) &:= \sum_{X:\bar{X}=U} J(X) \\ \mathcal{B}^{(2)}[J_1, J_2](U) &:= \sum_{X_1, X_2} J_1(X_1) J_2(X_2), \end{aligned} \tag{1.3.10}$$

where $\bar{X}_1 \cup \bar{X}_2 = U$, $\bar{X}_1 \cap \bar{X}_2 \neq \phi$, $X_1 \cap X_2 = \phi$, U is a (scale L)-polymer and for any (scale 1)-polymer X, $\bar{X}$ is the smallest (scale L)-polymer such that $\bar{X} \supset X$.

The operations $\mathcal{E}_I$, $\mathcal{E}_{II}$ remove relevant operators from polymer activities and are designed so that

$$\mathcal{E}_\#: \quad \mathcal{E}^{\Box + J} \doteq e^{\Omega_\#[J]} \mathcal{E}^{\Box + \mathcal{E}_\#[J]} \tag{1.3.11}$$

with $\# = I$ or II together with: $\Omega_I[J]$ is a field independent polymer activity chosen such that

$$\mathcal{E}_I[J](X, \Psi = 0) = 0, \quad \forall X, \tag{1.3.12}$$

$\Omega_{II}[J]$ is a polymer activity of the form

$$\Omega_{II}[J](\Lambda, \Psi^\phi) = -\frac{1}{2} \delta\sigma \int_\Lambda dz (\partial\phi)^2(z). \tag{1.3.13}$$

$\delta\sigma \in \mathbf{R}$ can be chosen so that certain low-dimensional parts[6] of J are cancelled on small sets (a class of polymers defined in Section 1.4).

[6] $\mathcal{E}_{II}$ is applied to $J := \mathcal{E}_I[\tilde{K}^{(j)}]$, which is analytic in μ, λ_α. $\delta\sigma$ is chosen to cancel $(\partial\phi)^2$ in $\tau_0 J$; hence $\delta\sigma$ is independent of λ_α, analytic in μ.

The $\Omega_I^{(j)}(\Lambda^{(j)})$ in (1.3.3), (1.3.4) is given by

$$\Omega_I^{(j)}(\Lambda^{(j)}) = \Omega_I[\tilde{K}^{(j)}](\Lambda^{(j)}) \tag{1.3.14}$$

$\Omega_{II}[J]$ in (1.3.11) with $J = \mathcal{E}_I[K^{(j)}]$ is absorbed into a shift in the covariance $C^{(j)} \to SC^{(j)}$ and change of normalization of $d\mu_{C^{(j)}}$ in (1.3.3). This is why there is $N^{(j)}$ in (1.3.3). Then the covariance $C_{L,1}$ of the Gaussian measure $\mu_{L,1}$ in $\mathcal{F}$ is chosen so that when $\sigma^{(j+1)} := \sigma^{(j)} + \delta\sigma^{(j)}$,

$$SC^{(j)}(p) = C_{L,1}(p) + \frac{\chi(p^2)}{\sigma^{(j+1)}p^2}. \tag{1.3.15}$$

Then $\mathcal{T} := \mathcal{F} \circ S \circ \mathcal{B} \circ \mathcal{E}_{II} \circ \mathcal{E}_I$ satisfies (1.3.3) as can easily be verified.

The main difference in this procedure compared with earlier versions, e.g., [6, 8, 10, 7, 9] is that $\mathcal{E}_I$ is designed so that $\mathcal{E}_I[J](X,0)$ is identically *zero* for *all* X, not just $\mathcal{O}(J^2)$ on small sets. This gives some neater formulas and is not much more difficult to handle (but we could carry out the procedure with the original type of extraction as well).

From these formulas straightforward calculations permit us to calculate the image $\mathcal{T}[\tilde{K}^{(j)}]$, $\mathcal{T} := \mathcal{F} \circ S \circ \mathcal{B} \circ \mathcal{E}_{II} \circ \mathcal{E}_I$, to order $O(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2)$. The corresponding formulas will be useful in our next paper in which we obtain lower bounds so we summarize them here. Given any analytic $J = J(\mu, \lambda_\alpha)$, e.g., $J = \mathcal{T}[\tilde{K}^{(j)}]$, we define μ, λ_α independent coefficients $(J)_\mu$, $(J)_{\bar{\alpha}}$ by

$$J = \mu(J)_\mu + \sum \lambda_{\bar{\alpha}}(J)_{\bar{\alpha}} + O(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2).$$

The next proposition refers to a linear projection $\mathcal{R}^{(d)}$ discussed in Section 3. It projects a polymer activity $J(X, \Psi)$ onto a new polymer activity with no parts of dimension less than or equal to d.

We find, by easy calculations, that

Proposition 1.3.1.

$$\begin{aligned}
(\mathcal{T}[\tilde{K}^{(j)}])_\# &= \mathcal{F}^{(1)}[(\tilde{J})_\#], \quad \# = \mu \text{ or } \alpha, \\
(\mathcal{T}[\tilde{K}^{(j)}])_{12} &= \mathcal{F}^{(1)}[(\tilde{J})_{12}] + \mathcal{F}^{(2)}[(\tilde{J})_1, (\tilde{J})_2],
\end{aligned}$$

where, upon defining $\delta O(\cdot, \Psi) := O(\cdot, \Psi) - O(\cdot, 0)$ and $\delta K(\cdot, \Psi) := K(\cdot, \Psi) - K(\cdot, 0)$, $\tilde{J}$ satisfies

$$\begin{aligned}
(\tilde{J})_\mu(\cdot, \Psi) &:= S \circ \mathcal{B}^{(1)} \circ \mathcal{R}^{(d)}[\delta K^{(j)}(\cdot, \Psi)] \\
(\tilde{J})_\alpha(\cdot, \Psi) &:= S \circ \mathcal{B}^{(1)}[\delta O_\alpha^{(j)}(\cdot, \Psi)] \\
(\tilde{J})_{12}(\cdot, \Psi) &:= S \circ \mathcal{B}^{(1)}[\delta O_{12}^{(j)}(\cdot, \Psi)] \\
&\quad - S \circ \mathcal{B}^{(1)}\left[\sum_{\substack{X_1 \cup X_2 = \cdot \\ X_1 \cap X_2 \neq \phi}} (\delta O_1^{(j)}(X_1, \Psi) \cdot O_2^{(j)}(X_2, 0) + (O_1 \leftrightarrow O_2))\right] \\
&\quad + S \circ \mathcal{B}^{(2)}\left[\delta O_1^{(j)}(\cdot, \Psi), \delta O_2^{(j)}(\cdot, \Psi)\right].
\end{aligned}$$

Proposition 1.3.2.

$$\tau_{12}\Omega_I^{(j)}(\Lambda^{(j)}) = \sum_{X \subset \Lambda^{(j)}} \tau_{12}\omega_I[\tilde{K}^{(j)}](X),$$

where $\tau_{12} = \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}|_{\lambda_\alpha=0}$, and

$$\begin{aligned} \omega_I[\tilde{K}^{(j)}](X) \quad := \quad & \mu K^{(j)}(X,0) + \sum \lambda_{\bar{\alpha}} O_{\bar{\alpha}}^{(j)}(X,0) \\ & -\lambda_1\lambda_2 \sum_{\substack{X_1 \cup X_2 = X \\ X_1 \cap X_2 \neq \phi}} O_1^{(j)}(X_1,0) O_2^{(j)}(X_2,0) \\ & + O(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2). \end{aligned}$$

See Section 5.3.

The first step in proving Theorem 1.2.1 is to repeat the arguments of [6, 8, 10, 7, 9] to establish a crude bound

$$\|\mathcal{T}[\tilde{K}^{(j)}]\|_{j+1} \leq O(L^d)\|\tilde{K}^{(j)}\|_j,$$

where $\|\cdot\|_j := \|\cdot\|_{G^{(j)},\Gamma,H}$ and $G^{(j)}$ is determined by $\kappa^{(j)} = \kappa \sum_0^j 2^{-2n}$ with κ, Γ, H as in Theorem 1.2.1. We use this crude estimate in conjunction with good bounds on bilinear approximations to get better bounds. For example, starting with $O_\alpha^{(j+1)} := \tau_\alpha \mathcal{T}[\tilde{K}^{(j)}]_{\mu=1}$ and expanding about $\mu = 0$, we obtain

$$\begin{aligned} \|O_\alpha^{(j+1)}\|_{j+1} \quad \leq \quad & \|(\mathcal{T}[\tilde{K}^{(j)}])_\alpha\|_{j+1} \\ + \quad & \frac{1}{2\pi}\oint \frac{|d\lambda_\alpha|}{|\lambda_\alpha^2|}\frac{1}{2\pi}\oint \frac{|d\mu|}{|\mu(\mu-1)|}\|\mathcal{T}[\tilde{K}^{(j)}]\|_{j+1}|_{\lambda_\beta=0,\ \beta\neq\alpha}. \end{aligned}$$

By choosing a large μ contour and using the crude estimate we see that the second term is $O(\|K^{(j)}\|_j\|O_\alpha^{(j)}\|_j)$. Easy estimates on the explicit formula for the first term show that $\|(\mathcal{T}[\tilde{K}^{(j)}])_\alpha\|_{j+1} \leq O(L^{-d})\|O^{(j)}\|_j$. Since we choose $\|K^{(0)}\|_0$ small after fixing L and since this will imply that $\|K^{(j)}\|_j$ is small compared to L^{-d}, the second term is negligible in comparison with the bound on the first term. This type of argument, introduced in [7], also permits all $O(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2)$ terms in Proposition 1.3.2 to be ignored and reduces upper (and lower) bounds to calculations within the bilinear approximations.

The introduction of the bilinear $\lambda_1\lambda_2 O_{12}$ into the generating function is an important ingredient in the success of this procedure. Good estimates on correlations of more observables would require higher order terms but we see no obstacles other than notational ones. It would also be possible to analyze other types of observables, e.g. odd functionals, or perhaps functionals of ϕ or $\partial^\alpha\phi$ for some $|\alpha| \neq 1$. The only new issue, other than changing some dimensions, would be whether perturbation theory couples the observable to new operators of lower dimension than the original observable (see Example 1.1.3).

Organization: Sections 2 – 4 introduce the operations $\mathcal{E}_I$, $\mathcal{E}_{II}$, $\mathcal{B}$. This is largely already in previous papers but we have been more complete. Section 5 gives bounds on the composition $S \circ \mathcal{B} \circ \mathcal{E}_{II} \circ \mathcal{E}_I$. Section 6 covers the $\mathcal{F}$ operation. Section 7 puts it all together to get our main results. We have begun each section with a synopsis and generally the synopsis is enough to obtain a good understanding of the rest of the paper.

1.4 Appendix on Notation

The order of topics is (with some cross references):

- Objects connected with geometry in $\mathbf{R}^d$, particularly polymers;
- functions on polymers, polymer activities;
- functions on $\mathbf{R}^d$, particularly fields;
- functionals of fields;
- notation connected with observables;
- parameters and constants.

- Norms on $\mathbf{R}^d$:

$$|x| := \max_{1\leq\mu\leq d} |x_\mu|;$$

$$\|x\| := \left(\sum_1^d x_\mu^2\right)^{\frac{1}{2}}.$$

- Cells: To each point $\hat{\alpha} \in \mathbf{Z}^d$ we associate a *d-cell* α which is the open hypercube centered on $\hat{\alpha}$

$$\alpha := \left\{x \in \mathbf{R}^d : |x - \hat{\alpha}| < \frac{1}{2}\right\}.$$

A *$(d-1)$-cell* is an open face of a d-cell. Similarly there are $(d-2)$-cells,…,1-cells, 0-cells where 1-cells are edges of d-cells and 0-cells are vertices.

- ℓ-scale cells: Given $\ell \in \mathbf{N}$, we can replace $\mathbf{Z}$ by $\ell\mathbf{Z}$ in the above construction. The resulting r-cells, $r = 0, \ldots, d$ are by definition ℓ-scale cells. Cells are 1-scale unless we state to the contrary.

- Complexes: A *complex* X is an empty or non-empty union of cells.
- Blocks, (Δ): The closure of a d-cell is called a *block*. We generally denote blocks by the letter Δ.
- Complex Activity: A $\mathbf{C}$-valued function $J(X)$ defined on all complexes X is, by definition, a *Complex Activity*.
- Some special Complex Activities, $(1, \Box)$:

$$
\begin{aligned}
1(X) &:= \begin{cases} 1 & \text{if } X = \emptyset \\ 0 & \text{otherwise.} \end{cases} \\
\Box(X) &:= \begin{cases} 1 & \text{if } X \text{ is a cell} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

- Circle Product, $(\circ)$: Let J_1 and J_2 be complex activities, we define a commutative product
$$(J_1 \circ J_2)(X) := \sum_{Y \subset X} J_1(Y) J_2(X - Y).$$
Under this product the set of complex activities forms an algebra with identity $1(X)$ (defined above).
- The Exponential, $(\mathcal{E}^J)$: Let $J(X)$ be any complex activity such that $J(\emptyset) = 0$. Define
$$\mathcal{E}^J(X) \equiv \mathcal{E}xp[J](X) := 1(X) + J(X) + \frac{1}{2!} J \circ J(X) + \cdots.$$
This series terminates after a finite number of terms determined by X.
- Polymers, X: X is a *polymer* if either $X = \emptyset$ or X is a union of blocks. We denote polymers by $X, Y, \ldots$. The reader should assume these letters represent polymers unless informed otherwise.
- $|X| := |\{\Delta : \Delta \subset X\}|$.
- Some special polymers: $\Lambda := \{x \in \mathbf{R}^d : |x| \leq \frac{1}{2}\mathrm{side}(\Lambda)\}$, where $\mathrm{side}(\Lambda) = L^N$ for parameters $L, N \in \mathbf{N}$. All fields (see below) will be continuous functions on Λ with periodic boundary conditions on $\partial\Lambda$. Therefore we refer to Λ as a *torus*.
- Rescaled Λ, $(\Lambda^{(j)})$: We set, for $j \in \{0, 1, \ldots, \log_L \mathrm{side}(\Lambda)\}$,
$$\Lambda^{(j)} := L^{-jd} \Lambda^{(0)}; \quad \Lambda^{(0)} := \Lambda$$

We give the remaining notation for the case $\Lambda^{(j=0)} \equiv \Lambda$ except for the discussion of pinning of observables where the generalization involves more than replacing Λ by $\Lambda^{(j)}$ everywhere.

- Special polymers (continued), small sets, ($\mathcal{S}$): We say X is a small set, $X \in \mathcal{S}$ iff

$$|X| \leq 2^d \text{ and } X \text{ is connected.}$$

- The overlap graph Given a set $\{X_1, \ldots, X_n\}$ of polymers, the *overlap graph* is the graph whose vertices are $1, \ldots, n$ and whose lines are those pairs $\{i, j\}$ such that $X_i \cap X_j \neq \emptyset$.

- Polymer Activities, $(J(X), K(X))$: A complex activity $J(X)$ is said to be a *polymer activity* iff $J(\emptyset) = 0$ and $J(X) = 0$ whenever X is not a polymer.

- Fields Ψ: Fix an integer $P \geq 1$. Let $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_p)$ where $1 \leq p \leq P$ and $\mu_j \in \{1, \ldots, d\}$. Let $|\boldsymbol{\mu}| := p$. $\Psi_{\boldsymbol{\mu}}(x)$ is a collection, indexed by $\boldsymbol{\mu}$, of $C(\Lambda)$ functions, which are periodic on Λ. We write

$$\begin{aligned} \Psi(\xi) &:= \Psi_{\boldsymbol{\mu}}(x) \quad \text{where } \xi := (x; \boldsymbol{\mu}) \\ \Psi &:= (\Psi_{\boldsymbol{\mu}}(x)), \end{aligned}$$

where $\boldsymbol{\mu}$ and $|\boldsymbol{\mu}|$ range over all their domains. The point of this notation is partly explained by considering the special Ψ's defined by: let $\phi \in C^P(\Lambda)$ be periodic. Then define

$$\Psi^{\phi}(\xi) := \partial_{\mu_1} \cdots \partial_{\mu_p} \phi(x) \quad \text{when } \xi = (x; \boldsymbol{\mu}).$$

Given p, we define $\int d\mu(\xi) f(\xi) := \int dx \sum_{\boldsymbol{\mu}:|\boldsymbol{\mu}|=p} f_{\boldsymbol{\mu}}(x)$.

- Norms on fields:

$$\begin{aligned} \|\Psi\| &:= \max_{p:1 \leq p \leq P} \max_{\boldsymbol{\mu}:|\boldsymbol{\mu}|=p} \|\Psi_{\boldsymbol{\mu}}\| \\ \|\Psi_{\boldsymbol{\mu}}\| &:= \max_{x \in \Lambda} |\Psi_{\boldsymbol{\mu}}(x)| \\ \operatorname{dist}(\Psi) &:= \inf_{\phi \in \mathcal{H}(\Lambda)} \|\Psi - \Psi^{\phi}\| \end{aligned}$$

$\mathcal{H}_s(\Lambda)$ is a Sobolev space. $s = s(P, d)$ is chosen sufficiently large that $\mathcal{H}_s \supset C^P(\Lambda)$.

- Functionals on Fields: K is *regular* iff (a) $K = K(X, \Psi)$ is a polymer activity. (b) K is C^∞ with respect to $\Psi_{\boldsymbol{\mu}} \in C(\Lambda)$, viewed as a Banach space with supremum norm (see below).

- Functional Derivatives: Since we require existence of (Fréchet) derivatives with respect to $\Psi_{\boldsymbol{\mu}}$, the derivatives of a regular $K(X, \Psi)$ are regular Borel measures on $(\Lambda \times \text{indices})^{\times \#}$ where $\#$ = the number of derivatives. Let $\mathbf{n} = (n_1, \ldots, n_P)$, $n_p \in \mathbf{N}_0$, signify n_p derivatives with respect ot $(\Psi_{\boldsymbol{\mu}}(x))$, $|\boldsymbol{\mu}| = p$, $x \in \Lambda$, $p = 1, \ldots, P$. We write

$$D(\mathbf{n})K(X, \Psi; \xi_{1,1}, \ldots, \xi_{1,n_1}; \ldots; \xi_{P,1}, \ldots, \xi_{P,n_P})$$

for the regular Borel measure which is formally

$$\prod_{p=1}^{P} \frac{1}{n_p!} \left(\prod_{\ell=1}^{n_p} \frac{\delta}{\delta\Psi(\xi_{p,\ell})} \right) K(X,\Psi) d^d x_{1,1} \cdots d^d x_{P,n_P},$$

and

$$|n| := \prod_{p=1}^{P} n_p.$$

- Weak Equality, ($\doteq$). Let K_1 and K_2 be regular. We say

$$K_1 \doteq K_2 \Longleftrightarrow K_1(X,\Psi^\phi) = K_2(X,\Psi^\phi)$$

for all X and all $\phi \in \mathcal{H}_s(\Lambda)$ (periodic).

- Local: $K(X,\Psi)$ is *local* iff (a) it is regular, (b) $\forall\, \mathbf{n}$, $D(\mathbf{n})K(X,\Psi)$ is supported (as regards the x part of the measure) in $(X)^{\times|\mathbf{n}|}$.

- Even: $K(X,\Psi) = K(X,-\Psi)$, $\forall\, X$.

- Euclidean Invariance: For all $E \in IO(d)$ with $E(\mathbf{Z}^d) = \mathbf{Z}^d$ we set $x_E = Ex$ mod periodicity of Λ. Then for any polymer $X, X_E := \{x_E : x \in X\}$ is a polymer. E decomposes into a translation and $R \in O(d)$. We define

$$(\Psi_E)_{\mu_1,\ldots,\mu_p}(x_E) := R_{\mu_1,\mu_1'} \cdots R_{\mu_p,\mu_p'} \Psi_{\mu_1',\ldots,\mu_p'}(x).$$

A regular polymer activity K is said to the *Euclidean Invariant* iff $K(X_E,\Psi_E) = K(X,\Psi)\ \forall\, X,\ \forall\, E$.

- Large Field Regulators, (G,g): A *large field regulator* $g(X,\phi)$ is a function defined for all polymers and all $\phi \in \mathcal{H}_s(\Lambda)$ such that $g(X,\phi) > 0\ \forall\, X$. Further properties are imposed and listed at the beginning of each section. The following specific regulator

$$G_\kappa(X,\phi) := \exp\left(\kappa \sum_{\alpha:1\le|\alpha|\le s} \int_X |\partial^\alpha\phi|^2 + \frac{\kappa}{c}\int_{\partial X} |\partial\phi|^2 \right)$$

obeys

- **Lemma 1.4.1.** [[9],Appendix A] If $s \ge s_0(d), c \ge c_0(d)$, then $\forall\, \kappa \ge 0$

  (a) $G_\kappa(X,\phi) \ge G_\kappa(X',\phi) \quad \forall\, X \supset X'$

  (b) $G_{\tau\kappa}\left(\bigcup_1^r X_j,\phi\right) \ge \prod_j G_\kappa(X_j,\phi)$

  where $\tau := \sup_{x\in\Lambda} |\{j : X_j \ni x\}|$

  (c) $G_\kappa \ge 1,\ G_\kappa(X,0) = 1.$

[The parameters c and s can be set so that this regulator is consistent with all conditions imposed in the rest of the paper.]

- Large Set Regulators, (γ, Γ): A *large set regulator* $\gamma(X)$ is a function defined on all polymers X such that $\gamma(X) > 0$. The following specific large set regulator Γ appears throughout:

$$\Gamma(X) := A^{|X|}\theta_A(X), \qquad \text{some } A \geq 1,$$

where

$$\theta_A(X) := \inf_{\text{trees } \tau \text{ on } X} \prod_{b \in \tau} \theta_A(|b|),$$

where τ on X means that τ is a tree graph whose vertices are the centers of blocks Δ in X, whose edges are denoted by $b \in \tau$. If $b = xy$, then $|b| := |x - y|$. θ_A is a function on $\mathbf{N}_0$ such that

$$\theta_A(0) = \theta_A(1) = 1,$$
$$\theta_A(s) \cdot A^{-Q} \leq \theta_A\left(\left\{\frac{s}{L}\right\}\right) \leq \theta_A(s)A^{-1}$$

for some $Q \geq 1$, where $\{x\}$:= smallest integer larger than or equal to x. The following lemma is easy to prove:

- **Lemma 1.4.2.** Let $\text{dist}(X, Y) := \inf\{|x - y| : x \in X, \ y \in Y\}$. Then

  (i) $\Gamma \geq 1$;

  (ii) $\Gamma(X \cup Y) \leq \Gamma(X)\Gamma(Y)\theta_A(\text{dist}(X, Y))$.

- h and $\mathbf{h}$: These will be used to measure radii of analyticity of functionals of Ψ. We set $\mathbf{h} := (h_1, \ldots, h_P)$, $h_p \geq 0 \ \forall\, p = 1, \ldots, P$. $\mathbf{h}^{\mathbf{n}} := \prod_p h_p^{n_p}$. When $h_p = h$, $\forall\, p$, we write h instead of $\mathbf{h}$.

- Norms on $K(X, \Psi)$: for any polymer activity $J(X)$

$$\|J\|_\gamma := \sup_\Delta \sum_{X \supset \Delta} J(X)\gamma(X).$$

For any regular polymer activity, $K(X, \Psi)$,

$$\|D(\mathbf{n})K(X, \Psi)\| := \ \text{var}\ D(\mathbf{n})K(X, \Psi),$$

where var is the variation norm of the measure.

$$\|(D(\mathbf{n})K(X)\|_g := \sum_{\mathbf{\Delta}} \sup_\phi (g(X, \phi))^{-1} \|D(\mathbf{n})K(X, \Psi^\phi)1_{\mathbf{\Delta}}\|,$$

where $\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_{|\mathbf{n}|})$ and $1_{\mathbf{\Delta}}(\xi_1, \ldots, \xi_{|\mathbf{n}|}) = 1 \iff \xi_i = (x_i; \boldsymbol{\mu}_i)$ with $x_i \in \Delta_i$, $i = 1, \ldots, |n|$.

$$\begin{aligned}
\|D(\mathbf{n})K\|_{g,\gamma} &:= \|(\|D(\mathbf{n})K(\cdot)\|_g)\|_\Gamma \\
&\equiv \operatorname{Sup}_\Delta \sum_{X \supset \Delta} \gamma(X)\|D(\mathbf{n})K(X)\|_g \\
\|K\|_{g,\gamma,\mathbf{h}} &:= \sum_{\mathbf{n}} \mathbf{h}^{\mathbf{n}} \|D(\mathbf{n})K\|_{g,\gamma}.
\end{aligned}$$

We write $\|K\|_{g,\gamma,h}$ when $\mathbf{h} = (h, \ldots, h)$.

- $\mathcal{K}(g, \gamma, h) :=$ Banach spaces of all regular local polymer activities K with the norm $\|K\|_{g,\gamma,h} < \infty$.

- I-type: A polymer activity $K(X, \Psi)$ is said to be *I-type* (I for *i*nteraction) iff K is regular, local, even, Euclidean invariant, and real.

- O-type: See below, under notation for observables.

- Pinning: A polymer activity $K(X)$ is said to be *pinned* at $y_1, \ldots, y_r \in \Lambda \sim \bigcup_\Delta \partial\Delta$ if

$$K(X) \neq 0 \Rightarrow X \ni y_i, \quad \forall\, X,\ \forall\, i = 1, \ldots, r.$$

- $\bar{\alpha}$-pinning: $\alpha \in \{1, 2\}$, $\bar{\alpha} \in \{1, 2, (12)\}$. Similarly $y_\alpha \in \{y_1, y_2\}$ where $y_1, y_2 \in \Lambda \sim \left(\bigcup_\Delta \partial\Delta\right)$ and $y_{12} = (y_1, y_2)$. $K_{\bar{\alpha}}(X)$ is said to be *$\bar{\alpha}$-pinned* iff $K_{\bar{\alpha}}$ is pinned at $y_{\bar{\alpha}}$ for some y_1, y_2.

  After j iterations of the Renormalization Group: $K_{\bar{\alpha}}^{(j)}(X)$ is defined on $X \subset \Lambda^{(j)}$. Given $x_1, x_2 \in \Lambda^{(0)} \sim \bigcup_\Delta \partial\Delta$ we say $K_{\bar{\alpha}}^{(j)}$ is *pinned* iff $K_{\bar{\alpha}}^{(j)}$ is $\bar{\alpha}$-pinned at $y_{\bar{\alpha}} := x_{\bar{\alpha}}^{(j)} := L^{-j} x_{\bar{\alpha}}$.

- For $\lambda_1, \lambda_2 \in \mathbf{C}$, set $\lambda_{(12)} := \lambda_1 \lambda_2$, and if F is a sufficiently differentiable function of λ_1, λ_2, then

$$\begin{aligned}
\tau_0 F &:= F(\lambda_1 = 0,\ \lambda_2 = 0)\\
\tau_\alpha F &:= \left(\frac{\partial}{\partial \lambda_\alpha} F\right)(\lambda_1 = 0,\ \lambda_2 = 0)\\
\tau_{12} F &:= \left(\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} F\right)(\lambda_1 = 0,\ \lambda_2 = 0)
\end{aligned}$$

- O-type: A polymer activity $K_{\bar{\alpha}}$ is *O-type* (O for *o*bservable) iff $K_{\bar{\alpha}}$ is regular, local, even, $\bar{\alpha}$-pinned.

- **Parameters:** Choose and fix once and for all

  (a) $d \geq 1$
  $P \geq 2$
  $c \geq c_0(d)$ (in G_κ)
  $s \geq \max\{s_0(d), s(P, d)\}$

  (b) $\eta > e$ (will appear in large set regulators).

  (c) $\chi \in C^\infty[0, \infty)$ such that (1) $0 \leq \chi \leq 1$; (2) for $x > 0$, $\frac{d}{dx}\chi(x) \leq -\frac{1}{x}\chi(x)(1 - \chi(x))$; (3) $\int_0^\infty dx\, x^n \left|\left(\frac{\partial}{\partial x}\right)^m \chi(x)\right| < \infty$, $\forall\, n, m \geq 0$.

[This χ is the high momentum cutoff in the covariance

$$C(p) = \chi(p^2)/(p^2\sigma)\ \mathbf{1}_{p\neq 0}(p).$$

A possible choice is $\chi(x) = e^{-x}$.]

See below for conventions concerning constants.

Choose $x_1, x_2 \in \mathbf{R}^d \sim \bigcup_{\Delta} \partial\Delta$ with $|x_1 - x_2| \geq 1$ (without loss of generality).

- Additional parameters set in the proof.

  $L \in \{2, 3, \ldots\}$

  A, Q in Γ

  κ in G_κ

  H (specific instance of h in $\|\ \ \|_{G,\Gamma,h}$)

  $J \in \mathbf{N}_0$ defined by $L^J \leq |x_1 - x_2| < L^{J+1}$.

  $N \in \mathbf{N}$, $N > (J+1)$, determines $\Lambda = \Lambda^{(0)}$, side $(\Lambda) = L^N$ (thus N large enough so that $x_\alpha \in \Lambda^{(0)}$).

- Constants: Constants occurring in the proof are denoted by $C(\cdot)$ where $\cdot$ is a list of parameters which must be fixed before $C(\cdot)$ can be considered "constant." We do not include in this list parameters occurring under (a), (b), and (c) above.

2 Extraction I ($\mathcal{E}_I$)

The purpose of this section is to define and study the operation $K \to \mathcal{E}_I[K]$ discussed in Section 1.3. See Lemmas 2.1 and 2.2.

Lemma 2.1. Let K be a regular polymer activity on Λ. Assume that there are g, γ, h, with $a_\gamma > e$, such that

$$\|K\|_{g,\gamma,h} < \epsilon_\gamma \equiv \frac{\log(a_\gamma) - 1}{\log(a_\gamma)}. \tag{2.1}$$

Then there exists a regular polymer activity $\Omega_I[K]$ satisfying

$$e^{\Omega_I[K](X)} = \mathcal{E}^{\Box + K}(X, \Psi = 0), \quad \forall\, X \subset \Lambda. \tag{2.2}$$

$\Omega_I[K]$ has the following additional properties:

(a) $\Omega_I[K]$ is analytic in K on $U_{\mathcal{E}_\gamma}(0) \subset \mathcal{K}(g, \gamma, h)$.

(b) $\delta_\Psi \Omega_I[K] \equiv 0$.

(c) $K(\cdot, \Psi = 0) \in \mathbf{R} \Rightarrow \Omega_I[K] \in \mathbf{R}$.

(d) If K is invariant, then so is $\Omega_I[K]$.

Proof. Under the hypotheses of the lemma, the convergence of the Mayer expansion for $\mathcal{E}^{\square+K}(X,\Psi=0)$ is an immediate consequence of Theorem 3.4 in [4]; indeed (cf. (3.10a) in [4])

$$\begin{aligned}
Q_X &:= \sum_{d\geq 0}\frac{1}{d!}\sup_{\Delta\subset X}\left(\sum_{\substack{Y\subset X\\ Y\supset\Delta}}|K(Y,\Psi=0)|\cdot|Y|^d\right)\\
&\leq Q_\Lambda \leq \sum_{d\geq 0}\frac{1}{d!}\sup_{\Delta\subset\Lambda}\left(\sum_{Y\supset\Delta}\|D(0)K(Y)\|_g\cdot|Y|^d\right)\\
&\leq \sum_{d\geq 0}\frac{1}{d!}\sup_{\Delta\subset\Lambda}\left(\sum_{Y\supset\Delta}\|D(0)K(Y)\|_g\cdot\gamma(Y)\cdot\sup_{|Y|}\left(a_\gamma^{-|Y|}|Y|^d\right)\right)\\
&\leq \|K\|_{g,\gamma,h}\cdot\sum_{d\geq 0}\left(\frac{d^d}{d!}e^{-d}\right)\cdot(\log(a_\gamma))^{-d}<1,\quad \forall X\subset\Lambda,
\end{aligned}$$

where the last inequality follows from $\frac{d^d}{d!}\leq e^d$, from $a_\gamma>e$ and (2.1). Therefore, we define $\Omega_I[K]$ by the Mayer series

$$\Omega_I[K](X) := \sum_{n\geq 1}\frac{1}{n!}\sum_{X_1,\ldots,X_n\subset X}\left(\prod_{\ell=1}^{n}K(X_\ell,\Psi=0)\right)\cdot\Psi_c(X_1,\ldots,X_n) \tag{2.3}$$

(where $\Psi_c(X_1,\ldots,X_n)\in\mathbf{R}$ is defined in [4]). $\Omega_I[K](\phi)=0$, $\delta_\Psi\Omega_I[K]\equiv 0$, and so $\Omega_I[K]$ is a regular polymer activity; by its very definition $\Omega_I[K]$ obeys (2.2); and finally, properties (a), (c), and (d) are easily inferred from (2.3). □

Lemma 2.2. Let K be a regular polymer activity on Λ obeying the assumptions in Lemma 2.1. Then there exists a unique regular polymer activity $\mathcal{E}_I[K]$ such that

$$\mathcal{E}^{\square+K}(X) = e^{\Omega_I[K](X)}\mathcal{E}^{\square+\mathcal{E}_I[K]}(X). \tag{2.4}$$

$\mathcal{E}_I[K]$ also satisfies

(a) $\mathcal{E}_I[K]$ is analytic in K on $U_{\varepsilon_\gamma}(0)\subset\mathcal{K}(g,\gamma,h)$.

(b) $\mathcal{E}_I[K](X,\Psi=0)=0$.

(c) $K\in\mathbf{R}\Rightarrow\mathcal{E}_I[K]\in\mathbf{R}$.

(d) If K is local, resp. even, resp. invariant, then also $\mathcal{E}_I[K]$ is so.

Proof. In order to prove (2.4) (and simultaneously (a)-(d)), we use induction in $|X|$.

$|X| = 0$: According to our conventions, if we want $\mathcal{E}_I[K]$ to be a polymer activity we need to define $\mathcal{E}_I[K](\phi) := 0$. Luckily, this definition is consistent with (2.4) at $X = \phi$ and with (a)-(d).

$|X| > 0$: The induction hypothesis is that we have found a unique regular $\mathcal{E}_I[K]$ on all polymers Y with $|Y| < |X|$ such that (2.4) and (a)-(d) hold on all these Y's. Induction step: Since $\mathcal{E}^{\Box+\mathcal{E}_I[K]}(X) \equiv 1 + \mathcal{E}_I[K](X) + \cdots$, where the omitted terms, indicated by $\cdots$, are (by the induction hypothesis) well under control, there is evidently a unique regular $\mathcal{E}_I[K](X)$ such that

$$\mathcal{E}^{\Box+\mathcal{E}_I[K]}(X) = e^{-\Omega_I[K](X)}\mathcal{E}^{\Box+K}(X);$$

hence (2.4) holds at X. At the same time this argument tells us that $\mathcal{E}_I[K](X)$ obeys (a)-(d) because $\Omega_I[K]$ satisfies (a)-(d) in Lemma 2.1. □

3 Localization, Extraction II ($\mathcal{L}^{(r)}$, $\mathcal{R}^{(r)}$, $\mathcal{E}_{II}$)

3.1 Summary

The first purpose of this chapter is to construct, for each $r \in \frac{1}{2}\mathbf{N}_0$, linear complementary projections $\mathcal{L}^{(r)}$, $\mathcal{R}^{(r)}$ defined on any polymer activity J such that $\mathcal{L}^{(r)}[J](X, \psi^\phi)$ is the integral over $x \in X$ of a polynomial in derivatives of $\phi(x)$. $\mathcal{L}^{(r)}[J](X)$ vanishes if X is not a small set; $\mathcal{L}^{(r)}[J]$ has scaling dimension less than or equal to r. $\mathcal{R}^{(r)}[J]$ under rescaling, $x \mapsto x/L$, decreases in norm as $L^{-r-\frac{1}{2}}$ or better. These properties are summarized in Theorem 3.1.1 below.

We have given the construction of $\mathcal{L}^{(r)}$ and $\mathcal{R}^{(r)}$ in some detail because previous discussions, e.g. [3], contain "errors of omission." The main point is that the remainder $\mathcal{R}^{(r)}$ is constructed by integrating derivatives of fields along paths: to construct $\mathcal{R}^{(r)}[J](X, \psi)$ such that (1) dependence on ψ is localized in the set X (2) lattice Euclidean invariance is preserved requires some care. Furthermore, the projection properties of $\mathcal{L}^{(r)}$ and $\mathcal{R}^{(r)}$ are new results.

The second purpose of this chapter is to define an extraction $\mathcal{E}_{II}$ with the properties that

$$\mathcal{E}^{\Box+K}(\Lambda, \psi^\phi) = e^{\Omega_{II}[K](\Lambda,\psi^\phi)}\mathcal{E}^{\Box+\mathcal{E}_{II}[K]}(\Lambda, \psi^\phi),$$

where $\mathcal{L}^{(d)}[\mathcal{E}_{II}[K]] = 0$, and $\Omega_{II}(\Lambda, \psi) = \sum_{X \subset \Lambda} \mathcal{L}^{(d)}[K](X, \psi)$. In particular, if K is lattice Euclidean invariant and has no field independent part (because of the $\mathcal{E}_I$ extraction), then $\Omega_{II}(\Lambda, \psi^\phi) = \frac{1}{2}\delta\sigma \int_\Lambda (\partial\phi)^2(x) d^d x$ for some $\delta\sigma$.

In order to read the rest of the paper the essential parts of this chapter are given in Theorem 3.1.1 and Section 3.4, where $\mathcal{E}_{II}$, Ω_{II} are defined and their important properties are listed.

Theorem 3.1.1. For each $r \in \frac{1}{2}\mathbf{N}_0$ there exist linear projections $\mathcal{L}^{(r)}$, $\mathcal{R}^{(r)} : \mathcal{K}(g_\epsilon, \gamma, h) \to \mathcal{K}(g_\epsilon, \gamma, h)$ for all $\epsilon > 0$, γ, $h > 0$ such that

(i) $\mathcal{L}^{(r)}$, $\mathcal{R}^{(r)}$ are complementary projections: $\mathcal{L}^{(r)} + \mathcal{R}^{(r)} \doteq Id$, $\mathcal{L}^{(r)}\mathcal{R}^{(r)} = \mathcal{R}^{(r)}\mathcal{L}^{(r)} = 0$.

(ii) Given $J \in \mathcal{K}(g_\epsilon, \gamma, h)$ there exist coefficients $a_{\boldsymbol{\mu}}[J](X)$ such that

$$\mathcal{L}^{(r)}[J](X, \psi^\phi) = \sum_m \sum_{\boldsymbol{\mu}=(\mu_1,\ldots,\mu_m)} a_{\boldsymbol{\mu}}[J](X) \int_X d^d x \partial^{\mu_1}\phi(x) \cdots \partial^{\mu_m}\phi(x),$$
$$a_{\boldsymbol{\mu}}[J](X) = 0 \text{ if } X \text{ is not a small set},$$

where[7] $1 \leq |\boldsymbol{\mu}_1| \leq |\boldsymbol{\mu}_2| \leq \cdots \leq |\boldsymbol{\mu}_m|$ and the sum is over all $\boldsymbol{\mu}$ with

$$\sum_{j=1}^m \left(\frac{d}{2} - 1 + |\boldsymbol{\mu}_j|\right) \leq r.$$

(iii) Suppose $J(X) = 0$ whenever $X \notin \mathcal{S}$. For all $\epsilon > 0$, γ, $h > 0$, $\ell \geq 1$,

$$\|\mathcal{R}^{(r)}[J]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell} \leq C(r)\ell^{-r-\frac{1}{2}} \left(1 + \frac{\ell^{\frac{d}{2}+P-1}C_s}{\sqrt{\epsilon h}}\right)^{C(r)} \|J\|_{g_\epsilon,\gamma,h}$$

where

$$\mathbf{h}_\ell := h \cdot \left(\ell^{-\frac{d}{2}}, \ell^{-\frac{d}{2}-1}, \ldots, \ell^{-\frac{d}{2}-P+1}\right).$$

(iv) If J is lattice Euclidean invariant, then so are $\mathcal{L}^{(r)}[J]$ and $\mathcal{R}^{(r)}[J]$.

3.2 Localization ($\mathcal{L}_r$, $\mathcal{R}_r$)

We localize a polymer activity which is a polynomial in the fields by "moving" the fields to a common point which is then averaged over X, the support of the polymer activity. The difference between the polymer activity and its localized form is written in terms of integrals along paths joining the positions of the fields to the common point. These paths are chosen carefully so that

(1) the localization operation is a projection;

(2) the formula for the difference is Euclidean invariant and still supported in the same set X.

[7]The $m = 0$ term, being equal to $a_\phi[J](X)|X|$, is included in the sum over m.

We now turn to the construction of these paths although only one endpoint of the path plays a role in the localized polymer activity which will be the first topic.

Construction of paths on small sets: For each small set X, and for each pair $\{\alpha, \beta\}$ of cells in X, consider the set of continuous paths lying inside X connecting the centers $\hat{\alpha}, \hat{\beta}$ of α and β. The minimal length paths of this set of paths are polygonal; let $\gamma_i(\{\alpha, \beta\}; X)$, $i = 1, \ldots, n(X, \alpha, \beta)$ be the set of them. Next, for each pair of points $\{x, y\}$ in X we define the polygonal path $\gamma_i(\{x, y\}; X) := [x, \widehat{\alpha(x)}] \cup \gamma_i(\{\alpha(x), \alpha(y)\}; X) \cup [\widehat{\alpha(y)}, y]$, where $\alpha(x)$ is the cell containing x and where $[x, y]$ is the straight line connecting x to y.

Definition 3.2.1. For each small set X and each pair of points $x, y \in X$, let $\gamma_{X,x,y}(t)$, $t \in [0, 1]$, be the parametrized and piecewise differentiable chain[8]

$$\gamma_{X,x,y} := (n(X, \alpha(x), \alpha(y)))^{-1} \sum_{i=1,\ldots,n(X,\alpha(x),\alpha(y))} \gamma_{X,x,y,i}(t)$$

where

(a) $\{\gamma_{X,x,y,i}(t) : t \in [0, 1]\} = \gamma_i(\{x, y\}; X)$;

(b) $\gamma_{X,x,y,i}(t = 0) = x$, $\gamma_{X,x,y,i}(t = 1) = y$;

(c) if, for given t, $\gamma_{X,x,y,i}(t)$ is within a linear segment of $\gamma_i(\{x, y\}; X)$, then the velocity $|\dot{\gamma}_{X,x,y,i}(t)|$ is given by

$$|\dot{\gamma}_{X,x,y,i}(t)| = \text{length of } \gamma_i(\{x, y\}; X).$$

We note that according to this definition

$$\gamma_{X,x,y}(t) = \gamma_{X,y,x}(1 - t). \tag{3.2.1}$$

Definition 3.2.2. We say that the regular polymer activity M is a *local monomial* of *degree* $\deg(M) = m \geq 1$ if there are $p_1, \ldots, p_m \in \mathbf{N}$ with $1 \leq p_1 \leq p_2 \leq \cdots \leq p_m$ such that

$$M(X, \Psi) = \begin{cases} \int d\mathcal{M}(X, \boldsymbol{\xi}) \Psi(\xi_1) \cdots \Psi(\xi_m), & X \in \mathcal{S}, \\ 0, & X \notin \mathcal{S}, \end{cases} \tag{3.2.2}$$

where $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_m)$ with $|\boldsymbol{\mu}_i| = p_i$, and where the x-part of the support of the regular Borel measure $d\mathcal{M}(X, \boldsymbol{\xi})$ obeys

$$x - \text{supp } d\mathcal{M}(X, \boldsymbol{\xi}) \subset (X)^m, \quad X \in \mathcal{S}. \tag{3.2.3}$$

[8]A chain is a formal sum of parametrized paths.

The *dimension of M*, $\dim(M)$, is defined as

$$\dim(M) := \sum_{r=1}^{m} \left(\frac{d}{2} - 1 + p_r \right). \tag{3.2.4}$$

A local monomial M of degree 0 is by definition a Ψ-independent polymer activity supported on small sets; moreover, $\dim(M) := 0$.

Any polymer activity P which has the form $P = \sum_{r=1}^{n} M_r$, each M_r a local monomial, is called a *local polynomial.*

Definition 3.2.3. The local monomials M, M' are called *equivalent*, $M \sim M'$, iff

(a) $\deg(M) = \deg(M') = m$,

(b) if $m \geq 1$ ($M^{\#}$ has $p_1^{\#}, \ldots, p_m^{\#}$) : $p_i = p_i'$, $\forall\, i$.

We will write E_m for an equivalence class of local monomials of degree m.

Fix E_m; assume that $M_i \in E_m$, $i \in I$, $|I| < \infty$; then $M := \sum_{i\in I} a_i M_i$, $a_i \in \mathbf{C}$, belongs to E_m as well, and, if $m \geq 1$, then $d\mathcal{M} = \sum_{i\in I} a_i d\mathcal{M}_i$. Every local polynomial $P_{\mathbf{M}'} = \sum_{i\in I'} a_i' M_i'$, $\mathbf{M}' \equiv (M_i')_{i\in I'}$, can thus be written $P = \sum_{m\geq 0} \sum_{E_m} (P_{\mathbf{M}'})_{E_m}$ where $(P_{\mathbf{M}'})_{E_m} = \sum_{\substack{i\in I' \\ M_i' \in E_m}} M_i' a_i' \in E_m$. If we have another decomposition of $P_{\mathbf{M}'}$ as $P_{\mathbf{M}'} = P_{\mathbf{M}''} \equiv \sum_{i\in I''} a_i'' M_i''$, then obviously $\sum_{m\geq 0} \sum_{E_m} (P_{\mathbf{M}'})_{E_m} = \sum_{m\geq 0} \sum_{E_m} (P_{\mathbf{M}''})_{E_m}$. However, since

$$\sum_{m\geq 0} \sum_{E_m} M_{E_m} \equiv 0 \Rightarrow M_{E_m} \equiv 0, \ \forall\, E_m,$$

we see that $(P_{\mathbf{M}'})_{E_m} \equiv (P_{\mathbf{M}''})_{E_m}$. Moreover, $M \equiv 0 \Rightarrow d\mathcal{M} = 0$.

Definition 3.2.4. Let M be a local monomial. We define the *localizing operator* L by

$$L[M] := M, \text{ if } \deg(M) = 0; \tag{3.2.5}$$

$$L[M](X, \Psi) := \begin{cases} 0, \text{ if } \deg(M) \geq 1 \wedge X \notin \mathcal{S}, \\ \frac{1}{|X|} \int_X d^d z M(X, \Psi \circ \gamma_{X,z,(\cdot)}(t=0)) \\ \equiv \int d\mathcal{M}(X, \xi) \frac{1}{|X|} \int_X d^d z \Psi_{\mu_1}(z) \cdots \Psi_{\mu_m}(z), \\ \text{if } \deg(M) \geq 1 \wedge X \in \mathcal{S}. \end{cases} \tag{3.2.6}$$

Corollary 3.2.5. (a) $L[M]$ is a regular, local polymer activity, supported on small sets; in particular, $L[M]$ is a local monomial.

(b) If $M \in E_m$ and $M = \sum_{i\in I} a_i M_i$, $M_i \in E_m$, $a_i \in \mathbf{C}$, then $L[M] = \sum_{i\in I} a_i L[M_i]$.

Definition 3.2.6. Let P be a local polynomial. Decompose $P \equiv P_{\mathbf{M}'} = \sum_{i\in I'} a_i' M_i'$. Define

$$L_{\mathbf{M}'}[P] := \sum_{i\in I'} a_i' \cdot L[M_i']. \tag{3.2.7}$$

Corollary 3.2.7. Let $\mathbf{M}'$, $\mathbf{M}''$ be any two decompositions of P. Then $L_{\mathbf{M}'}[P] \equiv L_{\mathbf{M}''}[P]$.

Proof. Use Corollary 3.2.5(b) and the remarks after Definition 3.2.3:

$$\begin{aligned}
L_{\mathbf{M}'}[P] &= \sum_{i\in I'} a_i' \cdot L[M_i] = \sum_{m\geq 0}\sum_{E_m} \sum_{\substack{i\in I' \\ M_i'\in E_m}} a_i' \cdot L[M_i'] \\
&= \sum_{m\geq 0}\sum_{E_m} L\left[\sum_{\substack{i\in I' \\ M_i'\in E_m}} a_i' \cdot M_i'\right] \\
&= \sum_{m\geq 0}\sum_{E_m} L\left[(P_{\mathbf{M}'})_{E_m}\right] \\
&= \sum_{m\geq 0}\sum_{E_m} L\left[(P_{\mathbf{M}''})_{E_m}\right] = \cdots = L_{\mathbf{M}''}[P].
\end{aligned}$$

□

Definition 3.2.8. Let P be a local polynomial, and let $\mathbf{M}$ be any decomposition of P. Then

$$L[P] := L_{\mathbf{M}}[P]. \tag{3.2.8}$$

Corollary 3.2.9. (a) $L[P]$ is a regular, local polymer activity supported on small sets; $L[P]$ is a local polynomial.

(b) L is a linear operator.

(c) P is even/real/invariant $\Rightarrow$ so is $L[P]$.

If $m = \deg(M)$ equals 0, then $L[M] \equiv M$; if $m \geq 1$, however, we have, for $X \in \mathcal{S}$,

$$\begin{aligned} M(X,\Psi) - L[M](X,\Psi) &= \frac{1}{|X|}\int_X d^dz \Big\{ M(X, \Psi \circ \gamma_{X,z,(\cdot)}(t=1)) \\ &\quad - M(X, \Psi \circ \gamma_{X,z,(\cdot)}(t=0)) \Big\} \\ &= \frac{1}{|X|}\int_X d^dz \int_0^1 dt \frac{d}{dt} M\left(X, \Psi \circ \gamma_{X,z,(\cdot)}(t)\right) \qquad (3.2.9) \\ &= \sum_{i=1}^m M^{(i)}(X, \Psi^{(i)}), \end{aligned}$$

where[9] $\Psi^{(i)} \equiv (\Psi(\xi_1), \ldots, \Psi(\xi_{i-1}),\ \partial_\nu \Psi_{\boldsymbol{\mu}_i}(x_i),\ \Psi(\xi_{i+1}), \ldots, \Psi(\xi_m))$, and

$$\begin{aligned} M^{(i)}(X, \Psi^{(i)}) &= \frac{1}{|X|}\int_X d^dz \int_0^1 dt \int d\mathcal{M}(X, \boldsymbol{\xi}) \sum_\nu (\dot\gamma_{X,z,x_i}(t))_\nu \\ &\quad \cdot \left(\prod_{r \neq i} \Psi_{\boldsymbol{\mu}_r}(\gamma_{X,z,x_r}(t)) \right) \cdot \partial_\nu \Psi_{\boldsymbol{\mu}_i}(\gamma_{X,z,x_i}(t)) \,. \qquad (3.2.10) \end{aligned}$$

Define now, for $X \in \mathcal{S}$, $M^{(i)}(X, \Psi)$ by

$$\begin{aligned} M^{(i)}(X, \Psi) &:= \int d\mathcal{M}(X, \boldsymbol{\xi}) \frac{1}{|X|}\int_X d^dz \int_0^1 dt \sum_\nu (\dot\gamma_{X,z,x_i}(t))_\nu \\ &\quad \cdot \left(\prod_{r \neq i} \Psi_{\boldsymbol{\mu}_r}(\gamma_{X,z,x_r}(t)) \right) \cdot \Psi_{\boldsymbol{\mu}_i,\nu}(\gamma_{X,z,x_i}(t)) \,. \qquad (3.2.11) \end{aligned}$$

In passing from (3.2.10) to (3.2.11) we have replaced $\partial_\nu \Psi_{\boldsymbol{\mu}_i}$ by $\Psi_{\boldsymbol{\mu}_i,\nu}$; these are only equal when $\Psi = \Psi^\phi$, which is the reason for the appearance of $\doteq$ in Corollary 3.2.11 below.

It follows that $M^{(i)}(X, \cdot)$ is a bounded linear functional on $C((X^m) \times$ (index space)). Hence, by the Riesz representation theorem, there exists a regular Borel measure $d\mathcal{M}^{(i)}(X, \boldsymbol{\xi}')$, supported in X^m, such that

$$M^{(i)}(X, \Psi) = \int d\mathcal{M}^{(i)}(X, \boldsymbol{\xi}') \quad \Psi(\xi_1') \cdots \Psi(\xi_m'), \qquad (3.2.12)$$

and where $\boldsymbol{\xi}' \equiv (\xi_1', \ldots, \xi_m')$, having $|\boldsymbol{\mu}_\ell'| \leq |\boldsymbol{\mu}_{\ell+1}'|$, is obtained from

$$(\xi_1, \ldots, \xi_{i-1},\ (x_i; (\boldsymbol{\mu}_i, \nu)),\ \xi_{i+1}, \ldots, \xi_m)$$

by a permutation $\pi \in S_m$ to put the arguments in the order imposed in Definition 3.2.2.

$M^{(i)}(X, \Psi)$ thus is a local monomial, of $\deg(M^{(i)}) = m$ and $\dim(M^{(i)}) = \dim(M) + 1$. Finally, we define the local polynomial $R[M]$ by

$$R[M](X, \Psi) := \begin{cases} 0, & X \notin \mathcal{S} \text{ or } \deg(M) = 0 \\ \sum_{i=1}^m M^{(i)}(X, \Psi), & X \in \mathcal{S} \text{ and } \deg(M) \geq 1. \end{cases} \qquad (3.2.13)$$

[9] For γ a chain, $\gamma := \Sigma a_i \gamma_i$, we define $\int_0^1 f(\gamma(t))dt := \Sigma a_i \int_0^1 f(\gamma_i(t))dt$.

Corollary 3.2.10. (a) If $M \in E_m$ and $M = \sum_{i \in I} a_i M_i$, $M_i \in E_m$ and $a_i \in \mathbf{C}$, then $R[M] = \sum_{i \in I} a_i R[M_i]$.

(b) M even/real/invariant $\Rightarrow R[M]$ enjoys the same properties.

Just as we did for L, we now define $R_{\mathbf{M}}[P]$ and show that Corollary 3.2.10(a) implies $R_{\mathbf{M}'}[P] = R_{\mathbf{M}''}[P]$; hence

$$R[P] := R_{\mathbf{M}}[P] \equiv \sum_{i \in I} a_i R[M_i]. \tag{3.2.14}$$

Corollary 3.2.11. (a) $R[P]$ is a local polynomial.

(b) R is linear.

(c) $M \doteq L[M] + R[M]$.

Definition 3.2.12. The *localization operator*, $\mathcal{L}_r$, $r \in \frac{\mathbf{N}_0}{2}$, on local polynomials is defined as follows. For a local monomial M

$$\mathcal{L}_r[M] := \sum_{k=0}^{[r-\dim(M)]} L[R^k[M]], \tag{3.2.15}$$

where, for $x \in \mathbf{R}$, $[x] := \max\{z \in \mathbf{Z} : z \leq x\}$ (and hence $\mathcal{L}_r[M] = 0$, if $\dim(M) > r$); and for the general local polynomial P

$$\mathcal{L}_r[P] := \mathcal{L}_{r_{\mathbf{M}}}[P] \equiv \sum_{i \in I} a_i \mathcal{L}_r[M_i]; \tag{3.2.16}$$

as indicated (and as readily checked) $\mathcal{L}_{r_{\mathbf{M}'}}[P] \equiv \mathcal{L}_{r_{\mathbf{M}''}}[P]$, and so the definition (3.2.16) does make sense.

Corollary 3.2.13. (a) $\mathcal{L}_r[P]$ is a regular, local polymer activity supported on small sets; $\mathcal{L}_r[P]$ is a local polynomial.

(b) $\mathcal{L}_r$ is a linear operator.

(c) If P is even/real/invariant, then the same holds for $\mathcal{L}_r[P]$.

Definition 3.2.14. M will be called *localized* if there is a local monomial M' with $M = L[M']$; similarly for localized polynomials.

Lemma 3.2.15. If M is localized, then $L[M] = M$ and $M^{(i)} \equiv 0,\ 1 \le i \le m$. Thus $R[M] = 0$.

Proof. If M is localized, then M has the form $(X \in \mathcal{S})$

$$M(X,\Psi) = \sum_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_m} \mathcal{M}(X,\boldsymbol{\mu}) \cdot \int_X d^dx \Psi_{\boldsymbol{\mu}_1}(x) \cdots \Psi_{\boldsymbol{\mu}_m}(x), \tag{3.2.17}$$

as can be seen in (3.2.6). This immediately implies $M = L[M]$, so let's turn to the second assertion.

Fix $X \in \mathcal{S}$. From (3.2.17) and (3.2.11) we infer that $(\boldsymbol{\mu} \equiv (\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_m))$

$$M^{(i)}(X,f) = \sum_{\nu,\boldsymbol{\mu}} \frac{1}{|X|} \mathcal{M}(X,\boldsymbol{\mu}) \cdot F(X,f,\boldsymbol{\mu},\nu), \tag{3.2.18}$$

where

$$\begin{aligned} F(X,f,\boldsymbol{\mu},\nu) \quad &= \quad \int_0^1 dt \int_{X^2} d^dz d^dx \left(\dot{\gamma}_{X,z,x}(t)\right)_\nu \\ &\quad \cdot f_{\boldsymbol{\mu}_1,\dots\boldsymbol{\mu}_{i-1},(\boldsymbol{\mu}_i,\nu),\boldsymbol{\mu}_{i+1},\dots\boldsymbol{\mu}_m}\left(\gamma_{X,z,x}(t),\dots,\gamma_{X,z,x}(t)\right). \end{aligned} \tag{3.2.19}$$

Performing the change of variables $t' := 1-t,\ x' := z,\ z' := x$ on (3.2.19), and using (3.2.1) we see that $F(X,f,\boldsymbol{\mu},\nu) = -F(X,f,\boldsymbol{\mu},\nu)$. □

Proposition 3.2.16. The localization operator $\mathcal{L}_r$ is a projection operator, i.e.,

$$(\mathcal{L}_r)^2 = \mathcal{L}_r. \tag{3.2.20}$$

Proof. Use equations (3.2.15), (3.2.16), together with Lemma 3.2.15 and the linearity of L, R. □

Remark. According to our conventions in Section 1.4 we restricted ourselves to fields $\Psi_{\boldsymbol{\mu}}$ with $|\boldsymbol{\mu}| \le P$. When defining $M^{(i)}(X,\Psi)$ in terms of $M^{(i)}(X,\Psi^{(i)})$ (cf. (3.2.9)-(3.2.11)) we have of course to assume that $\Psi_{\boldsymbol{\mu}_i,\nu}$ still belongs to our set of Ψ's, i.e., that $|\boldsymbol{\mu}_i| + 1 \le P$. As a result, the remainder $R[M]$ can only be defined if $p_m \le P-1$ (cf. (3.2.2)).

It follows from (3.2.15) and Corollary 3.2.11(c) that, for $r \ge \dim(M)$,

$$(1-\mathcal{L}_r)[M] \quad \equiv \quad M - L[M] - \sum_{k=1}^{[r-\dim(M)]} L[R^k[M]]$$

$$\begin{aligned}
&\doteq R[M] - L[R[M]] - \sum_{k=2}^{[r-\dim(M)]} L[R^k[M]] \\
&\doteq R^2[M] - L[R^2[M]] - \sum_{k=3}^{[r-\dim(M)]} \cdots \\
&\vdots \\
&\doteq R^{1+[r-\dim(M)]}[M].
\end{aligned} \tag{3.2.21}$$

Definition 3.2.17. For $r \in \frac{\mathbf{N}_0}{2}$ we define $\mathcal{R}_r$ on local monomials as

$$\mathcal{R}_r[M] := \begin{cases} R^{1+[r-\dim(M)]}[M], & r \geq \dim(M) \\ M, & r < \dim(M), \end{cases} \tag{3.2.22}$$

and extend to local polynomials by

$$\mathcal{R}_r[P] := \sum_{i \in I} a_i \mathcal{R}_r[M_i]. \tag{3.2.23}$$

Note that (3.2.23) is in fact independent of the decomposition of P. Clearly, $\mathcal{R}_r$ is a linear operator, $\mathcal{R}_r[P]$ is a regular local polymer activity supported on small sets, and $\mathcal{R}_r[P]$ is even, resp. real, resp. invariant, whenever P has such properties.

Due to (3.2.21)-(3.2.23) and the linearity of $(1 - \mathcal{L}_r)$ we obtain

Corollary 3.2.18. For any local polynomial P

$$(1 - \mathcal{L}_r)[P] \doteq \mathcal{R}_r[P]; \tag{3.2.24}$$

moreover, due to Lemma 3.2.15,

$$\mathcal{L}_r \mathcal{R}_r = \mathcal{R}_r \mathcal{L}_r = 0. \tag{3.2.25}$$

3.3 Bounds on $\mathcal{R}$

Proposition 3.3.1. Let M be a local monomial of degree $\deg(M)$ and dimension $\dim(M)$ with measure $d\mathcal{M}(X, \boldsymbol{\xi})$. Define

$$\mathbf{h}_\ell := h \cdot (\ell^{-d/2}, \ell^{-d/2-1}, \ldots, \ell^{-d/2-P+1}) \tag{3.3.1}$$

$$\|dM\|_\gamma := \sup_{\Delta} \sum_{X \supset \Delta} \gamma(X) \ \mathrm{Var}(d\mathcal{M}(X)). \tag{3.3.2}$$

Then, for any $\epsilon > 0$, γ, $h > 0$, $\ell \geq 1$, and $r \geq 0$,

$$\|\mathcal{R}_r[M]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell} \leq \ell^{-D} \cdot C \cdot \left(1 + \frac{\ell^{d-2+2P}}{\epsilon h^2}\right)^D h^{\deg(M)} \|dM\|_\gamma, \tag{3.3.3}$$

where C depends only on r, d, $\deg(M)$, $\dim(M)$, and $D := \dim(M)$ if $r < \dim(M)$, $r+1$ if $r - \dim(M) \in \mathbf{N}_0$, $r + \frac{1}{2}$ otherwise.

Proof. For $\mathbf{n} = (n_1, \ldots, n_P)$, $n_j \in \mathbf{N}_0$, we put

$$\dim(\mathbf{n}) := \sum_{j=1}^{P} n_j \left(\frac{d}{2} + j - 1\right),$$

$$\|J\|_{g,\gamma,\mathbf{h};\mathbf{n}} := \|D(\mathbf{n})J\|_{g,\gamma}\mathbf{h}^{\mathbf{n}}, \quad \|J\|_{g,\gamma,h;D} := \sum_{\mathbf{n}:\dim(\mathbf{n})=D} \|J\|_{g,\gamma,h;\mathbf{n}}.$$

By construction of $\mathcal{R}_r$, $(D(\mathbf{n}, \boldsymbol{\xi})\ \mathcal{R}_r[M])(X, \Psi = 0) = 0$, $\forall\, \mathbf{n}$ with $\dim(\mathbf{n}) < D$, therefore, by Lemma 4.3 of [6],

$$\|\mathcal{R}_r[M]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell} \leq C_D \left(1 + \frac{\ell^q}{\epsilon h^2}\right)^D \|\mathcal{R}_r[M]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell;D}, \tag{3.3.4}$$

where $q = d + 2P - 2$. By the construction of $\mathcal{R}_r$, $\mathcal{R}_r[M]$ is a sum of monomials of dimension D, therefore (c.f. (3.3.2))

$$\begin{aligned} &\equiv C_D \left(1 + \frac{\ell^q}{\epsilon h^2}\right)^D \sum_{\mathbf{n}:\dim(\mathbf{n})=D} \|D(\mathbf{n})\mathcal{R}_r[M]\|_\gamma \mathbf{h}_\ell^{\mathbf{n}} \\ &= \ell^{-D} C_D \left(1 + \frac{\ell^q}{e h^2}\right)^D \sum_{\mathbf{n}:\dim(\mathbf{n})=D} \|D(\mathbf{n})\mathcal{R}_r[M]\|_\gamma h^{|\mathbf{n}|}. \end{aligned} \tag{3.3.5}$$

By the definition of $\mathcal{R}_r$, (3.2.22), (3.2.13), $\|D(\mathbf{n})\mathcal{R}_r[M]\|_\gamma \leq C\|dM\|_\gamma$. Since $|\mathbf{n}| = \deg(M)$, we obtain (3.3.3) by combining these estimates with (3.3.5). □

Definition 3.3.2. Let K be a regular, local polymer activity. Then, the regular polymer activity $T_{s'}[K]$ and the local polynomial $T_{s'}^{\mathcal{S}}[K]$ are defined, for $s' \in \mathbf{N}_0$, by

$$T_{s'}[K](X, \Psi) := \sum_{j=0}^{s'} \frac{1}{j!} \left(\frac{d}{dt}\right)^j K(X, t\Psi)|_{t=0} \tag{3.3.6}$$

$$T_{s'}^{\mathcal{S}}[K](X, \Psi) := T_{s'}[K](X, \Psi) \cdot 1(X \in \mathcal{S}). \tag{3.3.7}$$

The next proposition is similar in proof to Lemma 4.3 of [6].

Proposition 3.3.3. Let J be any regular polymer activity. Then, for all $\epsilon > 0$, $\gamma, h > 0, \alpha \in (0,1]$ and $s' \geq 0$,

$$\|(1-T_{s'})[J]\|_{g_\epsilon,\gamma,\alpha\cdot h} \leq 3 \cdot \alpha^{s'+1}\sqrt{(s'+1)!}\left(1+\frac{C_s}{\sqrt{\epsilon}\alpha h}\right)^{s'+1}\|J\|_{g_\epsilon,\gamma,h}. \tag{3.3.8}$$

Proof. It is easy to verify that $D(\mathbf{n})(1-T_{s'})[J] = (1-T_{s'-|\mathbf{n}|})[D(\mathbf{n})J]$, where $T_{s'} := 0$ if $s' < 0$. For $s' \geq |\mathbf{n}|$,

$$\left(1-T_{s'-|\mathbf{n}|}\right)[D(\mathbf{n})J] = \int_0^1 dt \frac{(1-t)^{s'-|\mathbf{n}|}}{(s'-|\mathbf{n}|)!}\left(\frac{d}{dt}\right)^{s'-|\mathbf{n}|+1}(D(\mathbf{n})J)_t,$$

where $(F)_t(\Psi) := F(t\Psi)$,

$$= \int_0^1 dt\, \frac{(1-t)^{s'-|\mathbf{n}|}}{(s'-|\mathbf{n}|)!} \sum_{\substack{\mathbf{m}\geq\mathbf{n}\\ |\mathbf{m}|=s'+1}} \frac{\mathbf{m}!}{\mathbf{n}!}\int (D(\mathbf{m})J)_t(\boldsymbol{\xi}) \prod_{j=1}^{|\mathbf{m}-\mathbf{n}|}\Psi(\xi_j).$$

We take the $\|\ \ \|_{g_\epsilon,\gamma}$ norm of both sides using $\|(F)_t\|_{g_{\epsilon t^2}} = \|F\|_{g_\epsilon}$,

$$|\Pi\Psi(\xi_j)| \leq \sqrt{|\mathbf{m}-\mathbf{n}|!}(C_s/\sqrt{\epsilon})^{|\mathbf{m}-\mathbf{n}|}(1-t^2)^{-|\mathbf{m}-\mathbf{n}|/2}g_{\epsilon(1-t^2)}$$

and $g_\epsilon = g_{\epsilon t^2}g_{\epsilon(1-t^2)}$, obtaining

$$\begin{aligned}\|D(\mathbf{n})(1-T_{s'})[J]\|_{g_\epsilon,\gamma} \leq & \sum_{\substack{\mathbf{m}\geq\mathbf{n}\\ |\mathbf{m}|=s'+1}} \frac{\mathbf{m}!}{\mathbf{n}!}\|D(\mathbf{m})J\|_{g_\epsilon,\gamma} \\ & \cdot\sqrt{|\mathbf{m}-\mathbf{n}|!}\left(\frac{C_s}{\sqrt{\epsilon}}\right)^{|\mathbf{m}-\mathbf{n}|}\int_0^1 dt \frac{(1-t)^{s'-|\mathbf{n}|}(1-t^2)^{-|\mathbf{m}-\mathbf{n}|/2}}{(s'-|\mathbf{n}|)!}.\end{aligned}$$

We estimate the integral by $2(|\mathbf{m}-\mathbf{n}|!)^{-1}$ using $1-t^2 \leq 1-t$, $s'-|\mathbf{n}| = |\mathbf{m}-\mathbf{n}|-1$. Then apply $\sum_{|\mathbf{n}|\leq s'}(\alpha h)^{\mathbf{n}}$ to both sides obtaining

$$\begin{aligned}&\sum_{\mathbf{n}:|\mathbf{n}|\leq s'}(\alpha h)^{\mathbf{n}}\|D(\mathbf{n})(1-T_{s'})[J]\|_{g_\epsilon,\gamma} \\ &\leq 2\sum_{\mathbf{m}:|\mathbf{m}|=s'+1}\left\{\sum_{\mathbf{n}\leq\mathbf{m}}\frac{\mathbf{m}!}{\mathbf{n}!}(|\mathbf{m}-\mathbf{n}|!)^{-\frac{1}{2}}\left(\frac{C_s}{\sqrt{\epsilon}}\right)^{|\mathbf{m}-\mathbf{n}|}(\alpha h)^{\mathbf{n}}\right\}\|D(\mathbf{m})J\|_{g_\epsilon,\gamma}.\end{aligned}$$

We use $(|\mathbf{m}-\mathbf{n}|!)^{-\frac{1}{2}} \leq (|\mathbf{m}-\mathbf{n}|!)^{-1}\sqrt{(s'+1)!}$ and the binomial theorem to continue with

$$\begin{aligned}&\leq 2\sqrt{(s'+1)!}\alpha^{s'+1}\left(1+\frac{C_s}{\sqrt{\epsilon}\alpha h}\right)^{s'+1}\sum_{\mathbf{m}:|\mathbf{m}|=s'+1}h^{\mathbf{m}}\|D(\mathbf{m})J\|_{g_\epsilon,\gamma} \\ &\leq 2\sqrt{(s'+1)!}\alpha^{s'+1}\left(1+\frac{C_s}{\sqrt{\epsilon}\alpha h}\right)^{s'+1}\|J\|_{g_\epsilon,\gamma,h}.\end{aligned}$$

We combine this estimate with

$$\sum_{\mathbf{n}:|\mathbf{n}|>s'} (\alpha h)^{\mathbf{n}} \|D(\mathbf{n})(1-T_{s'})[J]\|_{g_\epsilon,\gamma} = \sum_{\mathbf{n}:|\mathbf{n}|>s'} (\alpha h)^{\mathbf{n}} \|D(\mathbf{n})J\|_{g_\epsilon,\gamma}$$
$$\leq \alpha^{s'+1}\|J\|_{g_\epsilon,\gamma,h}$$

to obtain the result of the Proposition. □

Definition 3.3.4. Let K be a regular, local polymer activity. We define, for $r \in \frac{\mathbf{N}_0}{2}$, the *localized part of dimension* r, $\mathcal{L}^{(r)}[K]$, of K by

$$\mathcal{L}^{(r)}[K] := \mathcal{L}_r[T^{\mathcal{S}}_{[2r/d]}[K]]. \tag{3.3.9}$$

The *remainder of dimension* $> r$, $\mathcal{R}^{(r)}[K]$, of K is defined by

$$\mathcal{R}^{(r)}[K] := \left(1 - T^{\mathcal{S}}_{[2r/d]}\right)[K] + \mathcal{R}_r[T^{\mathcal{S}}_{[2r/d]}[K]]. \tag{3.3.10}$$

Corollary 3.3.5. $\mathcal{L}^{(r)}$ and $\mathcal{R}^{(r)}$ obey

$$(1-\mathcal{L}^{(r)})[K] \doteq \mathcal{R}^{(r)}[K] \tag{3.3.11}$$

$$(\mathcal{L}^{(r)})^2 = \mathcal{L}^{(r)} \tag{3.3.12}$$

$$\mathcal{L}^{(r)}\mathcal{R}^{(r)} = \mathcal{R}^{(r)}\mathcal{L}^{(r)} = 0. \tag{3.3.13}$$

Proof. Use (3.3.9), (3.3.10), and (3.2.24) to verify (3.3.11). (3.3.12) follows from

$$\mathcal{L}_r[T^{\mathcal{S}}_{s'}[\mathcal{L}_r[T^{\mathcal{S}}_{s'}[\,\cdot\,]]]] = (\mathcal{L}_r)^2[T^{\mathcal{S}}_{s'}[\,\cdot\,]]$$

and from (3.2.20). Use (3.3.9), (3.3.10), and (3.2.25) to prove (3.3.13). □

Proposition 3.3.6. Let J be a regular, local polymer activity such that $J(X) = 0$ whenever $X \notin \mathcal{S}$. Let $\mathbf{h}_\ell$ be as in (3.3.1). Then, for $r \in \frac{\mathbf{N}_0}{2}$, there exists $C(r)$ such that for all $\epsilon > 0$, γ, $h > 0$ and $\ell \geq 1$

$$\|\mathcal{R}^{(r)}[J]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell} \leq \ell^{-r-\frac{1}{2}} \cdot C(r) \cdot \left(1 + \frac{\ell^{\frac{d}{2}+P-1}C_s}{\sqrt{\epsilon}\cdot h}\right)^{C(r)} \cdot \|J\|_{g_\epsilon,\gamma,h}. \tag{3.3.14}$$

If e.g., J is even and $r \in \frac{d}{2}\mathbf{N}_0 \cap \mathbf{N}$, $d \geq 2$, then the factor $\ell^{-r-\frac{1}{2}}$ improves to ℓ^{-r-1}.

Proof. According to (3.3.10), and because $(\mathbf{h}_\ell)_j \equiv h \cdot \ell^{-\frac{d}{2}+1-j} \leq h \cdot \ell^{-\frac{d}{2}}$ for $\ell \geq 1$ and since $T^S_{s'}[J] \equiv T_{s'}[J]$, we have

$$\begin{aligned} \|\mathcal{R}^{(r)}[J]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell} \leq\; & \|(1 - T_{[2r/d]})[J]\|_{g_\epsilon,\gamma,\ell^{-\frac{d}{2}}h} \\ & + \|\mathcal{R}_r[T^S_{[2r/d]}[J]]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell}. \end{aligned} \tag{3.3.15}$$

Due to (3.3.8) we get

$$\begin{aligned} \|(1 - T_{[2r/d]})[J]\|_{g_\epsilon,\gamma,\ell^{-\frac{d}{2}}h} \leq\; & \ell^{-\frac{d}{2}([2r/d]+1)} \cdot C'(r) \\ & \cdot \left(1 + \frac{\ell^{\frac{d}{2}} C_s}{\sqrt{\epsilon} \cdot h}\right)^{[2r/d]+1} \cdot \|J\|_{g_\epsilon,\gamma,h}. \end{aligned} \tag{3.3.16}$$

It is easy to check that, for $r \in \frac{\mathbf{N}_0}{2}$, $\frac{d}{2}\left(\left[\frac{2r}{d}\right] + 1\right) \geq r + \frac{1}{2}$; obviously, for $r \in \frac{d}{2} \cdot \mathbf{N}_0$, $\frac{d}{2}\left(\left[\frac{2r}{d}\right] + 1\right) = r + \frac{d}{2}$. Next, we decompose $T^S_{[2r/d]}[J]$ into inequivalent (c.f. Definition 3.2.3) monomials to each of which we apply (3.3.3). By the triangle inequality we obtain

$$\|\mathcal{R}_r[T^S_{[2r/d]}[J]]\|_{g_\epsilon,\gamma,\mathbf{h}_\ell} \leq \ell^{-r-\frac{1}{2}} \cdot C''(r) \cdot \left(1 + \frac{\ell^{d-2+2P}}{\epsilon \cdot h^2}\right)^{C'''(r)} \cdot \|J\|_{g_\epsilon,\gamma,h}. \tag{3.3.17}$$

This, together with (3.3.15) and (3.3.16) leads to (3.3.14). □

3.4 Extraction II ($\mathcal{E}_{II}$)

In this section, $\tilde{K}$ will stand for a regular, local polymer activity which is analytic in $\mu, \lambda_1, \lambda_2$ in a neighborhood of 0, and $\tau_0 \tilde{K}$ is supposed to be even.

Definition 3.4.1. Assume that $P \geq 2$. The regular, local, even polymer activities $\omega_{II}[\tilde{K}]$, $\Omega_{II}[\tilde{K}]$ are defined by

$$\omega_{II}[\tilde{K}] := \mathcal{L}^{(d)}[\tau_0 \tilde{K}] \qquad \text{(cf. (3.3.9))} \tag{3.4.1}$$

and

$$\Omega_{II}[\tilde{K}](X) := \sum_{Y \subset X} \omega_{II}[\tilde{K}](Y). \tag{3.4.2}$$

Obviously, $\omega_{II}[\tilde{K}]$ and $\Omega_{II}[\tilde{K}]$ are analytic in $\mu, \lambda_1, \lambda_2$ around 0.

Corollary 3.4.2. If $\tau_0 \tilde{K}$ is real, resp. invariant, then the same holds for $\omega_{II}[\tilde{K}]$, $\Omega_{II}[\tilde{K}]$. Moreover, $\mathcal{L}^{(d)}[\omega_{II}[\tilde{K}]] = \omega_{II}[\tilde{K}]$. $\tau_0 \tilde{K}(\,\cdot\,, \Psi = 0) = 0 \Rightarrow \omega_{II}[\tilde{K}](\,\cdot\,, \Psi = 0) = \Omega_{II}[\tilde{K}](\,\cdot\,, \Psi = 0) = 0$.

Proof. Use (3.3.9), the fact that $T^S_{s'}$ preserves reality/invariance, and Corollary 3.2.13(c) to check the first part. The second part follows from (3.3.12). The third part is obvious. □

Lemma 3.4.3. There exists a unique, regular, local polymer activity $\mathcal{E}'_{II}[\tilde{K}]$ such that for all $X \subset \Lambda$

$$\mathcal{E}^{\Box+\tilde{K}}(X) = e^{\Omega_{II}[\tilde{K}](X)}\mathcal{E}^{\Box+\mathcal{E}'_{II}[\tilde{K}]}(X). \tag{3.4.3}$$

$\mathcal{E}'_{II}[\tilde{K}]$ enjoys the following properties:

(a) $\mathcal{E}'_{II}[\tilde{K}]$ is analytic in $\tilde{K}$; in particular, it is analytic in $\mu, \lambda_1, \lambda_2$ in a neighborhood of 0.

(b) If $\tilde{K}$ is even, then $\mathcal{E}'_{II}[\tilde{K}]$ is also even. If $\tau_0\tilde{K}$ is I-type, then so is $\tau_0\mathcal{E}'_{II}[\tilde{K}]$. If $\tau_{\bar{\alpha}}\tilde{K}$ are $\mathcal{O}$-type, then so are $\tau_{\bar{\alpha}}\mathcal{E}'_{II}[\tilde{K}]$.

(c) If $\tilde{K}(\,\cdot\,, \Psi = 0) = 0$, then $\mathcal{E}'_{II}[\tilde{K}](\,\cdot\,, \Psi = 0) = 0$.

(d) $\tau_0\mathcal{E}'_{II}[\tilde{K}] = \mathcal{E}'_{II}[\tau_0\tilde{K}]$.

(e) If $\tilde{K}$ is even and obeys $\tilde{K}(\,\cdot\,, \Psi = 0) = 0$, then

$$T_2[\mathcal{E}'_{II}[\tilde{K}]] = T_2[\tilde{K}] - \omega_{II}[\tilde{K}], \tag{3.4.4}$$

and therefore

$$\mathcal{L}^{(d)}[\tau_0\mathcal{E}'_{II}[\tilde{K}]] = 0. \tag{3.4.5}$$

Proof. (a)-(d): cf. proof of Lemma 2.2 and use Corollary 3.4.2.

(e): Act with T_2 on (3.4.3); use evenness of $\tilde{K}$, Ω_{II} and of $\mathcal{E}'_{II}$ and $\tilde{K}(\,\cdot\,, \Psi = 0) = \Omega_{II}(\,\cdot\,, \Psi = 0) = \mathcal{E}'_{II}(\,\cdot\,, \Psi = 0) = 0$; employ (3.4.2) and induction in $|X|$ to obtain (3.4.4). (3.4.5) follows from (3.4.4) and from (3.4.1) and Corollary 3.4.2. □

Definition 3.4.4. The regular, local polymer activity $\mathcal{E}_{II}[\tilde{K}]$ is defined as (cf. (3.3.10))

$$\mathcal{E}_{II}[\tilde{K}] := (1 - \tau_0)\mathcal{E}'_{II}[\tilde{K}] + \mathcal{R}^{(d)}[\tau_0\mathcal{E}'_{II}[\tilde{K}]]. \tag{3.4.6}$$

Lemma 3.4.5.

(a) $\mathcal{E}_{II}[\tilde{K}]$ is analytic in $\tilde{K}$; in particular, $\mathcal{E}_{II}[\tilde{K}]$ is analytic in μ, λ_1, λ_2 in a neighborhood of 0.

(b) If $\tilde{K}$ is even, then $\mathcal{E}_{II}[\tilde{K}]$ is so.

(c) If $\tau_0\tilde{K}$ is I-type, then so is $\tau_0\mathcal{E}_{II}[\tilde{K}]$. $\tau_0\mathcal{E}_{II}[\tilde{K}] = \mathcal{E}_{II}[\tau_0\tilde{K}]$.

(d) $\tau_{\bar{\alpha}}\mathcal{E}_{II}[\tilde{K}] = \tau_{\bar{\alpha}}\mathcal{E}'_{II}[\tilde{K}]$; hence, if $\tau_{\bar{\alpha}}\tilde{K}$ are $\mathcal{O}$-type, then so are $\tau_{\bar{\alpha}}\mathcal{E}_{II}[\tilde{K}]$.

(e) $\tilde{K}(\,\cdot\,,\Psi=0)=0 \Rightarrow \mathcal{E}_{II}[\tilde{K}](\,\cdot\,,\Psi=0)=0$.

(f) If $\tilde{K}$ is even with $\tilde{K}(\,\cdot\,,\Psi=0)=0$, then

$$\mathcal{E}_{II}[\tilde{K}] \doteq \mathcal{E}'_{II}[\tilde{K}]. \tag{3.4.7}$$

Proof. For (a)-(e) use Lemma 3.4.3. (f) is verified by applying (3.3.11) and (3.4.5) to (3.4.6). □

Remark. It is possible to find for any r (assuming P sufficiently large) a pair of polymer activities ω_{II}, $\mathcal{E}'_{II}$ such that (3.4.3) holds with $\mathcal{L}^{(r)}[\tau_0\mathcal{E}'_{II}]=0$ and ω_{II} a local polynomial. The definition of $\mathcal{E}_{II}$ (3.4.6) carries over directly to this more general case. Hence, instead of taking $r=d$ as we did, we could have chosen $r>d$ which, however, would have resulted in "oversubtractions," i.e., in extractions which are technically not really necessary for the purposes of this paper.

4 Extraction and Reblocking Lemmas ($\mathcal{B}$)

In this section we have collected some basic estimates used to control the extraction of small sets and reblocking of polymer expansions. Similar results have appeared in [6, 8, 10, 7, 9]. In particular, the simplifications arising from the use of analyticity originate in [7].

Notation. The polymers and complexes in this section are assumed to be unions of cells associated with a lattice of lattice constant 1 or L, distinguished with the epithet 1-scale or L-scale when both are in use. The default is 1-scale. The norms refer to large set regulators $\gamma, \gamma_\eta, \gamma_L$ given by

$$\begin{aligned} \gamma(X) &:= a^{|X|}\theta_a(X), \quad a\ge 1;\\ \gamma_\eta(X) &:= \eta^{-|X|}\gamma(X), \quad \eta\ge 1. \end{aligned}$$

γ_L is not γ_η with $\eta=L$, but instead denotes the L-scale version of γ; its argument is only permitted to be an L-scale polymer X

$$\gamma_L(X) := a^{|X|_L}\theta_{a,L}(X),$$

where $|\ |_L := N^o$ of L-scale blocks in X and $\theta_{a,L}(X)$ measures the length of the tree graph on scale L. The large field regulators $g(X,\phi), f(X,\phi)$ in this section can be arbitrary functions defined on polymers such that

$$\begin{aligned} &g(X,\phi)\ge g(Y,\phi), \quad \text{if } X\supset Y;\\ &\Big(g\big(\bigcup_j X_j,\phi\big)\Big)^\tau \ge \prod_j g(X_j,\phi), \quad \text{where } \tau := \sup_x |\{j : X_j \ni x\}|;\\ &g(X,\phi)>0, \ \forall\, X,\phi, \end{aligned}$$

and the same properties are supposed to hold for $f(X,\phi)$.

Definition 4.1. Let $J_1,\ldots,J_p$ be polymer activities. Define, for non-negative integers $r_1,r_2,\ldots,r_p$, a new polymer activity

$$V[J_1^{r_1},\ldots,J_p^{r_p}](X) := \frac{1}{r_1!}\cdots\frac{1}{r_p!}\sum_{g^{(r_1,\ldots,r_p)}(X)}\prod_{k=1}^{p}\prod_{j=1}^{r_k} J_k(X_{k,j}), \tag{4.1}$$

where $g^{(r_1,\ldots,r_p)}(X) = \{(X_{1,1},\ldots,X_{1,r_1};X_{2,1},\ldots,X_{2,r_2};\ldots;X_{p,1},\ldots,X_{p,r_p})$ such that (a), (b), (c)$\}$ where

(a) $\cup X_{k,j} = X$.

(b) The overlap graph on $(X_{1,1},\ldots,X_{p,r_p})$ is connected.

(c) Further restrictions on which sets can overlap, to be described below.

(d) If $r_j = 0$, for some j with $1 \le j \le p$, then we interpret $V\left[J_1^{r_1},\ldots,J_p^{r_p}\right]$ as $V\left[J_1^{r_1},\ldots,J_{j-1}^{r_{j-1}},J_{j+1}^{r_{j+1}},\ldots J_p^{r_p}\right]$.

Easy consequences of (4.1) are:

$$V[J^0] = 0, \quad V[J] = J. \tag{4.2}$$

Proposition 4.2. Let J,ω be any polymer activities, and let $\mathcal{E}[J]$ be defined[10] by

$$\mathcal{E}^{\Box+\mathcal{E}[J]} = e^{-\Omega}\mathcal{E}^{\Box+J}, \tag{4.3}$$

$$\Omega(X) := \sum_{Y\subset X}\omega(Y). \tag{4.4}$$

Then

$$\mathcal{E}[J] \equiv \mathcal{E}[J,R] = \sum_{r,s\ge 0} V[J^r,R^s], \tag{4.5}$$

where

$$R(X) := e^{-\omega(X)} - 1, \tag{4.6}$$

and condition (c) in Definition 4.1 is that the J-polymers $X_{1,1},\ldots,X_{1,r}$ are disjoint and the R-polymers $X_{2,1},\ldots,X_{2,s}$ are distinct (i.e., $X_{2,j}\neq X_{2,k}$ if $j\neq k$).

This proposition is implicitly within [8, 6] so we sketch the proof.

[10] $\mathcal{E}[J]$ depends on ω as well but we do not make this explicit.

Sketch of Proof. Following [6], Section 2,

$$\begin{aligned} e^{-\Omega(X)} &= \prod_{Y\subset X} e^{-\omega(Y)} = \prod_{Y\subset X}(1+R(Y)) \\ &= \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{Y_1,\dots,Y_N\subset X}\prod_{j=1}^{N}R(Y_j), \end{aligned}$$

where $(Y_1,\dots,Y_N)$ are *distinct* polymers. Therefore

$$\begin{aligned} e^{-\Omega(X)}\cdot\mathcal{E}^{\Box+J}(X) &= \sum_{N,M}\frac{1}{N!}\frac{1}{M!}\sum_{Y_1,\dots,Y_N\subset X}\;\sum_{Z_1,\dots,Z_M\subset X} \\ &\quad\cdot\prod_{j=1}^{N}R(Y_j)\prod_{k=1}^{M}J(Z_k), \end{aligned} \tag{4.7}$$

where $(Z_1,\dots,Z_M)$ are *disjoint* polymers. Now we perform the sum over $(Y_1,\dots,Y_N;$ $Z_1,\dots,Z_M)$ subject to the constraint that components with connected overlap graph of $(\cup Y_j)\cup(\cup Z_k)$ be $X_1,\dots,X_P$, and then sum over $(X_1,\dots,X_P)$ and P. Comparing the result with (4.3) we obtain

$$\mathcal{E}[J](X)=\sum_{r,s}\frac{1}{r!}\frac{1}{s!}\sum_{Y_1,\dots,Y_s;Z_1,\dots,Z_r}\prod_{j=1}^{s}R(Y_j)\prod_{k=1}^{r}J(Z_k), \tag{4.8}$$

where the sum is over all $(Y_1,\dots,Z_r)$ such that the union is X and the overlap graph is connected: the sum over $X_1,\dots,X_P$ and P becomes the polymer expansion for $\mathcal{E}^{\Box+\mathcal{E}[J]}$. Equation (4.8) is the same as the claims (4.5) and (4.6) of the proposition. □

Proposition 4.4. Let $\mathcal{E}[J,R]$ be as in (4.5). Given $\eta>1$ there exist constants $C_{4.1},C_{4.2}>0$ such that [11]

$$\|J\|_{g,\gamma,h},\ \|R\|_{f,\gamma,h}\le C_{4.1} \tag{4.9}$$

$\Rightarrow$

(i)

$$\|\mathcal{E}[J,R]\|_{gf^{\tau},\gamma_\eta,h}\le C_{4.2}\left\{\|J\|_{g,\gamma,h}+\|R\|_{f,\gamma,h}\right\}, \tag{4.10}$$

where g,f,γ are arbitrary regulators, $h\ge 0$,

$$\tau:=\sup_{x\in\Lambda}|\{X: X\ni x, R(X)\neq 0\}|. \tag{4.11}$$

(ii) The map $J,R\mapsto\mathcal{E}[J,R]$ is analytic as a map from $\mathcal{K}(g,\gamma,h)\times\mathcal{K}(f,\gamma,h)\to$ $\mathcal{K}(gf^{\tau},\gamma_\eta,h)$ on a neighborhood of the balls $\|J\|_{g,\gamma,h}$, $\|R\|_{f,\gamma,h}\le C_{4.1}$.

We omit the proof: the bound (4.10) is a simple variation on Lemma 5.1 in [6]. The essential point in the proof is that at most τ R-polymers can overlap with a given cube

[11] In Section 5 we are going to assume that $C_{4.1}\le 1$ in order to avoid distinguishing among several cases.

Δ and at most one J-polymer can simultaneously contain the same Δ. Therefore the worst growth in ϕ that can occur within any given cube Δ is less than $g(\Delta,\phi)f^{\tau}(\Delta,\phi)$, as claimed by the choice of regulator on the left-hand side of (4.10). The large set regulator $\gamma(X)$ has to become a little weaker $\gamma \rightarrow \gamma_\eta$ to control the combinatorics of the sum over all ways in which the overlaps of the R and J polymers in $\mathcal{E}[J,R]$ can occur.

To prove the analyticity, replace J and R by $J+\alpha J_1$ and $R+\beta R_1$ respectively and note that $C_{4.1}$ can be chosen so that the sum on the right-hand side of (4.5) is uniformly convergent for $|\alpha|$, $|\beta|$ sufficiently small. Each term under the sum is analytic in α,β, therefore the sum is analytic in α,β. This implies $\mathcal{E}[J,R]$ is analytic in J and R.

From the formula (4.5) and Proposition 4.4 we immediately deduce

Corollary 4.5. Suppose that, for μ,λ_α in a neighborhood of zero, J and R are analytic in μ,λ_α and $J\in\mathcal{K}(g,\gamma,h)$, $R\in\mathcal{K}(f,\gamma,h)$. Expand

$$\begin{aligned} J &= \mu\cdot J_\mu+\sum_{\bar\alpha}\lambda_{\bar\alpha}\cdot J_{\bar\alpha}+\mathcal{O}(\mu^2,\mu\lambda_\alpha,\lambda_\alpha^2)\\ R &= \mu\cdot R_\mu+\sum_{\bar\alpha}\lambda_{\bar\alpha}\cdot R_{\bar\alpha}+\mathcal{O}(\mu^2,\mu\lambda_\alpha,\lambda_\alpha^2). \end{aligned} \tag{4.12}$$

Then

$$\begin{aligned} \mathcal{E}[J,R](X) &= \mu\cdot(J_\mu+R_\mu)(X)+\sum_{\bar\alpha}\lambda_{\bar\alpha}(J_{\bar\alpha}+R_{\bar\alpha})(X)\\ &+\lambda_1\lambda_2\left\{\sum_{\substack{X_1\cup X_2=X\\ X_1\cap X_2\neq\phi\\ X_1\neq X_2}}R_1(X_1)R_2(X_2)+\sum_{\substack{X_1\cup X_2=X\\ X_1\cap X_2\neq\phi}}(J_1(X_1)R_2(X_2)+(1\longleftrightarrow 2))\right\}\\ &+\mathcal{O}(\mu^2,\mu\lambda_\alpha,\lambda_\alpha^2). \end{aligned} \tag{4.13}$$

Definition 4.6.

(i) For X any 1-scale polymer we define $\bar X :=$ smallest L-scale polymer containing X.

(ii) For J any polymer activity defined on 1-scale polymers, let $\mathcal{B}[J]$ be defined on L-scale polymer U by

$$\mathcal{B}[J](U) := \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{g_r(U)}\prod_{j=1}^{r}J(X_j), \tag{4.14}$$

where $g_r(U) := \{(X_1,\ldots,X_r) : \cup\bar X_j = U,\ X_i\cap X_j=\phi \text{ if } i\neq j, \text{ the overlap graph on } (\bar X_1,\ldots,\bar X_r) \text{ is connected}\}$.

(iii) With J, U as in (ii), let $\mathcal{B}^{(1)}[J](U)$ be the $r = 1$ term in (4.14), namely

$$\mathcal{B}^{(1)}[J](U) := \sum_{X:\bar{X}=U} J(X). \tag{4.15}$$

(iv) If J_1, J_2 are L-scale polymer activities, then (cf. (4.14))

$$\mathcal{B}^{(2)}[J_1, J_2](U) := \sum_{g_2(U)} \prod_{j=1}^{2} J_j(X_j). \tag{4.16}$$

Because of the pairwise exclusion (i.e., $X_i \cap X_j = \phi$, for $i \neq j$) enforced by $g_r(U)$, the sum over r in (4.14) runs at most up to $r = |U|_1$. Therefore:

(a) J is regular/local/even/analytic in μ, λ_α in a neighborhood of zero $\Rightarrow$ also for $\mathcal{B}[J]$.

(b) $\tau_0\mathcal{B}[J] = \mathcal{B}[\tau_0 J]$. $\tau_0 J$ is I-type/$\tau_{\bar{\alpha}} J$ are $\mathcal{O}$-type $\Rightarrow$ also for $\tau_0\mathcal{B}[J]/\tau_{\bar{\alpha}}\mathcal{B}[J]$.

The $\mathcal{B}$ is defined so that whenever X is an ℓ-scale complex for $\ell = 1$ and L,

$$\mathcal{E}^{\square_1+J}(X) = \mathcal{E}^{\square_L+\mathcal{B}[J]}(X), \tag{4.17}$$

where $\square_L(X) = 1$ if X is an L-scale cell, zero otherwise.

Proposition 4.7. Given $\eta > 1$, a regulator γ with $a \geq \eta^{(2^d)}$ and a polymer activity $J \in \mathcal{K}(g, \gamma_\eta, h)$, then there exist $C_{4.3}(L) > 0$ and $C_{4.4} > 0$ such that[12]

(i)
$$\|J\|_{g,\gamma_\eta,h} \leq C_{4.3}(L) \Rightarrow \|\mathcal{B}[J]\|_{g,\gamma_L,h} \leq C_{4.4} \cdot L^d \cdot \|J\|_{g,\gamma_\eta,h}, \tag{4.18}$$

(ii) $\mathcal{B} : \mathcal{K}(g, \gamma_\eta, h) \to \mathcal{K}(g, \gamma_L, h)$ is analytic on a neighborhood of $\{J : \|J\|_{g,\gamma_\eta,h} \leq C_{4.3}(L)\}$.

Given $\eta \geq 1$, a regulator γ with $a \geq \eta^{(2^d)}$ and $J, J_1, J_2 \in \mathcal{K}(g, \gamma_\eta, \mathbf{h})$, then

(iii)
$$\|\mathcal{B}^{(1)}[J]\|_{g,\gamma_L,\mathbf{h}} \leq \eta^{(2^{d+1})} L^d \|J\|_{g,\gamma_\eta,\mathbf{h}} \cdot \begin{cases} a^{-1}, & \text{if } J(X \in \mathcal{S}) = 0 \\ 1, & \text{else;} \end{cases} \tag{4.19}$$

(iv) if J is pinned at y, then

$$\|\mathcal{B}^{(1)}[J]\|_{g,\gamma_L,\mathbf{h}} \leq \eta^{(2^{d+1})} \|J\|_{g,\gamma_\eta,\mathbf{h}} \cdot \begin{cases} a^{-1}, & \text{if } J(X \in \mathcal{S}) = 0 \\ 1, & \text{else;} \end{cases} \tag{4.20}$$

[12] In Section 5 we will assume, for the sake of simplicity, that $C_{4.3}(L) \leq C_{4.1}$.

(v) if J_1 is pinned at y_1, J_2 pinned at y_2, then

$$\|\mathcal{B}^{(2)}[J_1, J_2]\|_{g,\gamma_L,\mathbf{h}} \leq \eta^{(2^{d+2})}\left(\prod_{\alpha=1}^{2}\|J_\alpha\|_{g,\gamma_\eta,\mathbf{h}}\right)\cdot\begin{cases} a^{-1}, & \text{if } |x_1 - x_2| \geq 4L \\ 1, & \text{else.}\end{cases} \tag{4.21}$$

The proof of this proposition requires the following lemma, a slight generalization of Lemma 3.2 in [6].

Lemma 4.8. Let $\eta \geq 1$ and $a \geq \eta^{(2^d)}$, and recall that $L > 2^d$. For all polymers X

$$\gamma_L(\bar{X}) \leq \eta^{(2^{d+1})}\gamma_\eta(X)\cdot\begin{cases} 1, & \text{if } X \in \mathcal{S} \\ a^{-1}, & \text{if } X \notin \mathcal{S}.\end{cases} \tag{4.22}$$

Proof of Proposition 4.7. We begin with (iii): for any functional derivative, denoted by subscript $\mathbf{n}$, we have, by (4.15),

$$\|D(\mathbf{n})\mathcal{B}^{(1)}[J]\|_{g,\gamma_L} \leq \sup_{\Delta_L}\sum_{U\ni\Delta_L}\sum_{X:\bar{X}=U}\gamma_L(\bar{X})\|D(\mathbf{n})J(X)\|_g,$$

where Δ_L is an L-scale block; by Lemma 4.8,

$$\begin{aligned}
&\leq \eta^{2^{d+1}}\sup_{\Delta_L}\sum_{X:\bar{X}\supset\Delta_L}\gamma_\eta(X)\|D(\mathbf{n})J(X)\|_g \\
&\leq \eta^{2^{d+1}}\sup_{\Delta_L}\sum_{\Delta\subset\Delta_L}\sum_{X\supset\Delta}\gamma_\eta(X)\|D(\mathbf{n})J(X)\|_g \\
&\leq \eta^{2^{d+1}}L^d\sup_{\Delta}\sum_{X\supset\Delta}\gamma_\eta(X)\|D(\mathbf{n})J(X)\|_g \equiv \eta^{2^{d+1}}L^d\|D(\mathbf{n})J\|_{g,\gamma_\eta}.
\end{aligned}$$

If J vanishes on small sets, then by Lemma 4.8 we can have an additional factor of a^{-1} on the right-hand side. Summing both sides over $\mathbf{n}$ times $\mathbf{h}^{\mathbf{n}}$ proves (iii). The proof of (iv), (v) is similar (in (v) we use $L > 2^d$). For part (i), note that, by the same arguments as in the proof of Lemma 5.1 of [6], the terms with $r \geq 2$ in (4.14) are bounded by $O(\|J\|^2_{g,\gamma_\eta,h})$, so (iii) $\Rightarrow$ (i). Part (ii) follows from (4.14) also. □

Proof of Lemma 4.8. (a) $X \in \mathcal{S}$: Then $\bar{X}$, X are connected, $|X| \leq 2^d$, hence $\gamma_L(\bar{X}) = a^{|\bar{X}|} \leq a^{|X|} \leq \eta^{(2^d)}(a\eta^{-1})^{|X|} \leq \eta^{(2^d)}\gamma_\eta(X)$.

(b) $X \notin \mathcal{S}$, X connected: Induction hypothesis: $\gamma_L(\bar{Y}) \leq \eta^{(2^d)}\gamma_\eta(Y)$, $\forall\, Y$ connected with $|Y| < |X|$. Induction step: since X connected, $|X| > 2^d$ and $L > 2^d$, there exists an L-scale block Δ_L with $|\Delta_L \cap X| \geq 2$; therefore we can write $X = X_1 \cup X_2$ where X_1, X_2 are connected, $|X_1 \cap \Delta_L| \geq 1$, $|X_2 \cap \Delta_L| \geq 1$, $\mathring{X}_1 \cap \mathring{X}_2 = \phi$; obviously $|X_1|$, $|X_2| < |X|$ (so we may apply the induction hypothesis), and $|\bar{X}_1| + |\bar{X}_2| \geq |\bar{X}| + 1$, thus $\gamma_L(\bar{X}) = a^{|\bar{X}|} \leq$

$a^{-1}a^{|\bar{X}_1|}a^{|\bar{X}_2|} = a^{-1}\gamma_L(\bar{X}_1)\gamma_L(\bar{X}_2) \leq a^{-1}\eta^{(2^d)(2)}\gamma_\eta(X_1)\gamma_\eta(X_2) = a^{-1}\eta^{(2^{d+1})}\gamma_\eta(X)$. This implies the inductive hypothesis for $|X|$, but note that it is also the bound (4.22).

(c) $X \notin \mathcal{S}$, X disconnected: this case is reduced to the cases (a), (b) as in [6]. □

From Definition 4.6 the following is immediate.

Corollary 4.9. If J is analytic in μ, λ_α for μ, λ_α in a neighborhood of zero, and if we expand $J = \mu \cdot J_\mu + \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot J_{\bar{\alpha}} + \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2)$, then

$$\begin{aligned} \mathcal{B}[J] = \mu \cdot \mathcal{B}^{(1)}[J_\mu] \ &+ \ \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot \mathcal{B}^{(1)}[J_{\bar{\alpha}}] \\ &+ \ \lambda_1\lambda_2 \cdot \mathcal{B}^{(2)}[J_1, J_2] \\ &+ \ \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2). \end{aligned} \tag{4.23}$$

5 The *RG* step. Part I: Bounds on Extraction/Reblocking/Rescaling

5.1 Survey of results

For the benefit of the reader we start with

5.1.1 A brief recapitulation of relevant notation/conventions

For given "rescaling length" L, $L \in \{2, 3, \ldots\}$, given "separation parameter" J, $J \in \mathbf{N}_0$ such that $L^J \leq |x_1 - x_2| < L^{J+1}$ where x_1, x_2 are the points where the observables are pinned at, and given finite volume cutoff N, $N \in \mathbf{N}$ such that $N > (J+1)$, the d-torus $\Lambda^{(0)}$ is defined by $\Lambda^{(0)} := \left[\frac{-L^N}{2}, \frac{L^N}{2}\right]^d$. The j-times $(0 \leq j \leq N)$ rescaled volume $\Lambda^{(j)}$ is $\Lambda^{(j)} := \left[\frac{-L^{(N-j)}}{2}, \frac{L^{(N-j)}}{2}\right]^d$, and $x_\alpha^{(j)} := L^{-j}x_\alpha$ do, by assumption, not lie on unit block boundaries (because we make this hypothesis for $x_\alpha^{(0)} \equiv x_\alpha$).

For measuring the size of polymer activities on $\Lambda^{(j)}$ we will rely on the specific large field regulators G_δ and $G^{(j)}$,

$$G_\delta(X, \phi) \ := \ \exp\left(\delta\left\{\sum_{1\leq|\mu|\leq s} \|\partial^\mu\phi\|_X^2 + \frac{1}{c}\|\partial\phi\|_{\partial X}^2\right\}\right), \quad \delta \geq 0, \tag{5.1.1}$$

$$G^{(j)} \ := \ G_{\kappa^{(j)}}, \tag{5.1.2}$$

and on the large set regulator Γ_ρ,

$$\Gamma_\rho(X) := \rho^{-|X|}\Gamma(X) = \rho^{-|X|} \cdot A^{|X|}\theta_A(X), \quad A \geq 1, \tag{5.1.3}$$

($\theta_A(X)$ has been defined in Section 1.4, and we write

$$\Gamma_\rho(X) = (A_{\Gamma_\rho})^{|X|} \cdot \theta_A(X) \tag{5.1.4}$$

(so that $A_\Gamma \equiv A$).

The parameters d (dimension), P (# of fields Ψ_μ), c (the constant in G_δ), s (the "Sobolev index"), and the purely technical parameter η (to appear as Γ_η, Γ_{η^2}, Γ_{η^3}, $\Gamma_{\eta^{-1}}$) should be considered to be fixed once and for all. We impose only that

$$d \geq 1, \quad P \geq 2,$$

and

$$c \geq c_0(d), \quad s \geq s_0(d, P) \tag{5.1.5}$$

ensuring that G obeys properties listed in Lemma 1.4.1 and for technical reasons

$$\eta > e. \tag{5.1.6}$$

Our convention concerning "constants" is: Any dependence on parameters other than the fixed d, P, c, s, η will be indicated explicitly.

5.1.2 Results

For $K, \mathcal{O}$ polymer activities on Λ, and C a covariance, we write $\langle \mathcal{O} \rangle_{\Lambda_j;K,C}$,

$$\begin{aligned} \langle \mathcal{O} \rangle_{\Lambda;K,C} &:= (Z_{\Lambda;K,C})^{-1} \int d\mu_C(\phi)(\mathcal{E}^{\square+K} \circ \mathcal{O})(\Lambda, \Psi^\phi) \\ Z_{\Lambda;K,C} &:= \int d\mu_C(\phi)\mathcal{E}^{\square+K}(\Lambda, \Psi^\phi), \end{aligned} \tag{5.1.7}$$

for the expectation of $\mathcal{O}$ w.r.t. the interaction K. Also, the truncated expectation of $\mathcal{O}_1, \mathcal{O}_2$ "subject to the connected part $\mathcal{O}_{12}$", denoted by $\langle \mathcal{O}_1; \mathcal{O}_2 | \mathcal{O}_{12} \rangle_{\Lambda;K,C}$, is defined by

$$\langle \mathcal{O}_1; \mathcal{O}_2 | \mathcal{O}_{12} \rangle_{\Lambda;K,C} := \langle \mathcal{O}_1 \circ \mathcal{O}_2 + \mathcal{O}_{12} \rangle_{\Lambda;K,C} - \prod_{\alpha=1}^{2} \langle \mathcal{O}_\alpha \rangle_{\Lambda;K,C}. \tag{5.1.8}$$

The objective of Section 5 is to prove the following main result which summarizes Proposition 5.5.5 and Theorem 5.5.9.

Theorem 5.1.1. Let $\kappa^{(j)} > 0$, $\epsilon > 0$, and assume that L, A, H are large enough[13]. Let $K^{(j)}$ be an I-type polymer activity on $\Lambda^{(j)}$ whose norm $\|K^{(j)}\|_{G^{(j)},\Gamma,H}$ is small enough, and let $\mathcal{O}_{\bar{\alpha}}^{(j)}$ be $\mathcal{O}$-type polymer activities on $\Lambda^{(j)}$ (pinned at $x_{\bar{\alpha}}^{(j)}$) with $\|\mathcal{O}_{\bar{\alpha}}^{(j)}\|_{G^{(j)},\Gamma,H} < \infty$. Let $C^{(j)}$ be a covariance on $\mathcal{H}_s(\Lambda^{(j)})/\{\text{constants}\}$. Then, for $0 \leq j \leq N-1$:

[13] For more precision, see Theorem 5.5.9.

(i) There exist I-type resp. $\mathcal{O}$-type polymer activities $SK^{(j)}$ resp. $S\mathcal{O}_{\alpha}^{(j)}$ on $\Lambda^{(j+1)}$ (where $SK^{(j)}$ is defined by $SK^{(j)} := (rescaling) \circ (reblocking) \circ (extractionII) \circ (extractionI)[K^{(j)}])$ and there is a covariance $SC^{(j)}$ on $\mathcal{H}_s(\Lambda^{(j+1)})/\{\text{constants}\}$ such that, if $Z_{\Lambda^{(j)};K^{(j)},C^{(j)}} \neq 0$,

$$\langle \mathcal{O}_1^{(j)}; \mathcal{O}_2^{(j)} | \mathcal{O}_{12}^{(j)} \rangle_{\Lambda^{(j)};K^{(j)},C^{(j)}} = \langle S\mathcal{O}_1^{(j)}; S\mathcal{O}_2^{(j)} | S\mathcal{O}_{12}^{(j)} \rangle_{\Lambda^{(j+1)};SK^{(j)},SC^{(j)}} + \Omega_{12}^{(j)}(\Lambda^{(j)}), \quad (5.1.9)$$

where $\Omega_{12}^{(j)}$ is a polymer activity on $\Lambda^{(j)}$ pinned at $x_1^{(j)}$ and $x_2^{(j)}$.

(ii) Let $(G^{(j)}G_\epsilon)_{L^{-1}}$ be the large field regulator (5.5.15) (see below) on $\Lambda^{(j+1)}$. For any $\delta > 0$, if L is large enough, we have

$$\begin{aligned}
\|SK^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} &\leq L^{-\frac{1}{2}+\delta}\|K^{(j)}\|_{G^{(j)},\Gamma,H} \\
\|S\mathcal{O}_{\alpha}^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} &\leq L^{-d+\delta}\|\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,H} \\
\|S\mathcal{O}_{12}^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} &\leq \left\{\|\mathcal{O}_{12}^{(j)}\|_{G^{(j)},\Gamma,H} + \prod_{\alpha=1}^{2}\|\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,H}\right\} \\
&\quad \cdot \begin{cases} L^{-2d-\frac{1}{4}}, & 0 \leq j \leq J-2 \\ L^{-2d+\delta}, & j = J-1 \\ L^{-d+\delta}, & J \leq j \leq N-1. \end{cases}
\end{aligned}$$

In Section 5.2 we list, for easy reference, those hypotheses (on A, L, $\|K^{(j)}\|_{G^{(j)},\Gamma,H}, \ldots$) which are assumed to hold in Sections 5.3-5.5 where Theorem 5.1.1 (and much more) is proven.

5.2 Hypotheses needed in Sections 5.3-5.5

In all of Sections 5.3-5.5, the hypotheses (1)-(4) below will tacitly be assumed to hold; in particular, except from the statement of our main result (Theorem 5.5.9) they won't be mentioned in the statement of our lemmas, corollaries.

(1)
$$L > \max\{3, 2^d\}. \quad (5.2.1)$$

(2)
$$A \geq \max\left\{\eta^{2^{d+2}}, L^{d+\frac{1}{2}}\right\}. \quad (5.2.2)$$

(3) $K^{(j)}$ is an I-type polymer activity on $\Lambda^{(j)}$ whose norm $\|K^{(j)}\|_{G^{(j)},\Gamma,H}$ is "small enough." By this we mean that there exists an upper bound $UB^{(j)}$ such that

$$\|K^{(j)}\|_{G^{(j)},\Gamma,H} \leq \frac{1}{6}UB^{(j)}, \quad (5.2.3)$$

with

$$UB^{(j)} \leq C_{5.0}(L) := \min\left\{\epsilon_\Gamma, \frac{1}{2}\left(\frac{\log\eta - 1}{\log\eta}\right), \frac{C_{4.1}}{C_{5.1}}e^{-C_{5.1}},\right.$$

$$\frac{C_{4.1}}{C_{5.2}}, \frac{C_{4.3}(L)}{C_{5.5}}, (L^{2d+\frac{1}{2}} \cdot C_{5.7}(L))^{-1} \Big\} . \tag{5.2.4}$$

The constant ϵ_Γ has been defined in (2.1). The other constants may be looked up according to their subscripts.

(4) $\mathcal{O}_{\bar{\alpha}}^{(j)}$ are $\mathcal{O}$-type polymer activities on $\Lambda^{(j)}$ (pinned at $x_{\bar{\alpha}}^{(j)}$) with finite norm, i.e., $\|\mathcal{O}_{\bar{\alpha}}^{(j)}\|_{G^{(j)},\Gamma,H} < \infty$.

Definition 5.2.1. The regular, local, even polymer activity $\tilde{K}^{(j)}$ is defined by

$$\tilde{K}^{(j)} := \mu \cdot K^{(j)} + \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot \mathcal{O}_{\bar{\alpha}}^{(j)}, \text{ for } \mu, \lambda_\alpha \in \mathbf{C}. \tag{5.2.5}$$

Remark. As mentioned in Section 4, we will assume w.l.g. that the constants $C_{4.1}$ and $C_{4.3}(L)$ obey

$$C_{4.3}(L) \leq C_{4.1} \leq 1. \tag{5.2.6}$$

5.3 Extraction I ($\mathcal{E}_I$)

5.3.1 Existence

Recall the definition of $\tilde{K}^{(j)}$ (5.2.5) and of the upper bound $UB^{(j)}$ in (5.2.3,5.2.4). The triangle inequality tells us that, if

$$|\mu| \leq \frac{UB^{(j)}/5}{\|K^{(j)}\|_{G^{(j)},\Gamma,H}}, \quad |\lambda_\alpha| \leq \frac{UB^{(j)}/5}{\|\mathcal{O}_\alpha^{(j)}\|_{G^{(j)},\Gamma,H}},$$
$$|\lambda_1|\,|\lambda_2| \leq \frac{UB^{(j)}/5}{\|\mathcal{O}_{12}^{(j)}\|_{G^{(j)},\Gamma,H}}, \tag{5.3.1}$$

we have $\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} \leq \frac{4}{5} UB^{(j)}$. Because of our hypothesis (5.2.3) on $K^{(j)}$, the disk of μ's specified by (5.3.1) contains the point $\mu = 1$. According to (5.2.4) we have $UB^{(j)} \leq \epsilon_\Gamma$, ϵ_Γ defined in (2.1), and obviously $A_\Gamma > e$ (cf. (5.1.6) and (5.2.2)). Hence we may apply Lemmas 2.1 and 2.2 to the regular, local, even polymer activity $\tilde{K}^{(j)}$ and obtain

Lemma 5.3.1. For μ, λ_α as in (5.3.1) the polymer activities $\Omega_I[\tilde{K}^{(j)}]$ and $\mathcal{E}_I[\tilde{K}^{(j)}]$ exist, are regular, local, even, and analytic in μ, λ_α. Moreover

(a) $\tau_0 \Omega_I[\tilde{K}^{(j)}] = \Omega_I[\mu K^{(j)}]$ and $\tau_0 \mathcal{E}_I[\tilde{K}^{(j)}] = \mathcal{E}_I[\mu K^{(j)}]$; these polymer activities are, for $\mu \in \mathbf{R}$, of I-type (cf. Section 1.4).

(b) $\tau_{\bar{\alpha}}\Omega_I[\tilde{K}^{(j)}]$ and $\tau_{\bar{\alpha}}\mathcal{E}_I[\tilde{K}^{(j)}]$ are of $\mathcal{O}$-type (cf. Section 1.4).

Proof. Properties (a), (b) for Ω_I follow from (2.3) and from the fact that $\tau_0\tilde{K}^{(j)} \equiv \mu K^{(j)}$ is I-type for real μ, and that $\tau_{\bar{\alpha}}\tilde{K}^{(j)} \equiv \mathcal{O}_{\bar{\alpha}}^{(j)}$ is $\mathcal{O}$-type. And now we use induction in $|X|$ to check (a), (b) for $\mathcal{E}_I$. □

Definition 5.3.2. The regular, local, even polymer activity $\omega_I[\tilde{K}^{(j)}]$ is recursively defined by

$$\omega_I[\tilde{K}^{(j)}](X) := \Omega_I[\tilde{K}^{(j)}](X) - \sum_{\substack{Y\subset X\\ \neq}} \omega_I[\tilde{K}^{(j)}](Y). \tag{5.3.2}$$

Combining (2.3) and (5.3.2) one sees that $\omega_I[\tilde{K}^{(j)}]$ has the Mayer expansion

$$\omega_I[\tilde{K}^{(j)}](X) = \sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{X_1,\ldots,X_n:\\ \cup X_r=X}}\left(\prod_{r=1}^{n}\tilde{K}^{(j)}(X_r,\Psi=0)\right)\Psi_c(X_1,\ldots,X_n). \tag{5.3.3}$$

5.3.2 Bounds

Lemma 5.3.3. For μ, λ_α as in (5.3.1) $\omega_I[\tilde{K}^{(j)}]$ is analytic in μ, λ_α and obeys

$$\begin{aligned}\omega_I[\tilde{K}^{(j)}](X) &= \mu\cdot K^{(j)}(X,0) + \sum_{\bar{\alpha}}\lambda_{\bar{\alpha}}\mathcal{O}_{\bar{\alpha}}^{(j)}(X,0)\\ &\quad -\lambda_1\lambda_2\sum_{\substack{X_1\cup X_2=X\\ X_1\cap X_2\neq\emptyset}}\mathcal{O}_1^{(j)}(X_1,0)\mathcal{O}_2^{(j)}(X_2,0)\\ &\quad +\mathcal{O}(\mu^2,\mu\lambda_\alpha,\lambda_\alpha^2), \end{aligned}\tag{5.3.4}$$

$$\|\omega_I[\tilde{K}^{(j)}]\|_{g,\Gamma_\eta,\mathbf{h}} \leq C_{5.1}\cdot\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H},\ \forall\, g\geq 1,\ \mathbf{h}\geq 0.\text{[14]} \tag{5.3.5}$$

Proof. (5.3.4) is obtained from (5.3.3).

Concerning (5.3.5) we first note that $\Gamma_\eta(\cup X_r) \leq \prod_r \Gamma_\eta(X_r)$ if the overlap graph on $(X_1,\ldots,X_n)$ is connected, because Γ_η has a parameter $A_{\Gamma_\eta}\geq 1$ (cf. (5.2.2)). Now, for any $g\geq 1$, $\mathbf{h}\geq 0$,

$$\begin{aligned}\|\omega_I[\tilde{K}^{(j)}]\|_{g,\Gamma_\eta,\mathbf{h}} &= \|D(0)\omega_I[\tilde{K}^{(j)}]\|_{g,\Gamma_\eta}\\ &= \sup_{\Delta}\sum_{X\supset\Delta}\Gamma_\eta(X)|\omega_I[\tilde{K}^{(j)}](X)| \leq \sum_{n\geq 1} I_n,\end{aligned}$$

[14] $g\geq 1$ means: $g(X,\phi)\geq 1$, $\inf_\phi g(X,\phi) = 1$. W.l.g. we assume that $C_{5.1}\geq 1$.

where

$$I_n = \frac{1}{n!} \sup_{\Delta} \sum_{X \supset \Delta} \sum_{\substack{X_1,\ldots,X_n: \\ \cup X_r = X}} \left(\prod_r |\tilde{K}^{(j)}(X_r, \Psi = 0)| \cdot \Gamma_\eta(X_r) \right) \cdot |\Psi_c(X_1, \ldots, X_n)|,$$

where we used (5.3.3), the fact that $\Psi_c(X_1, \ldots, X_n) = 0$ unless the overlap graph on $(X_1, \ldots, X_n)$ is connected, and the submultiplicativity of Γ_η mentioned before. The proof of Theorem 3.4 in [B1] reveals that $I_n \leq \frac{1}{n} Q^n$ with

$$\begin{aligned} Q &= \sum_{d \geq 0} \frac{1}{d!} \sup_{\Delta} \left(\sum_{X \supset \Delta} |\tilde{K}^{(j)}(X, \Psi = 0)| \cdot \Gamma_\eta(X) \cdot |X|^d \right) \\ &\leq \|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} \cdot \sum_{d \geq 0} (\log \eta)^{-d} \leq \frac{1}{2}, \end{aligned}$$

because, according to (5.3.1), $\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} \leq \frac{4}{5} UB^{(j)}$ with $UB^{(j)} \leq \frac{1}{2}(\log \eta - 1)/\log \eta$ (cf. (5.2.4)); cf. also the proof of Lemma 2.1. □

Lemma 5.3.4. For $\mathcal{E}_I[\tilde{K}^{(j)}]$ we have

$$\begin{aligned} \mathcal{E}_I[\tilde{K}^{(j)}](X, \Psi) &= \mu \cdot (K^{(j)}(X, \Psi) - K^{(j)}(X, 0)) + \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} (\mathcal{O}_{\bar{\alpha}}^{(j)}(X, \Psi) - \mathcal{O}_{\bar{\alpha}}^{(j)}(X, 0)) \\ &\quad -\lambda_1 \lambda_2 \sum_{\substack{X_1 \cup X_2 = X \\ X_1 \cap X_2 \neq \emptyset}} \{(\mathcal{O}_1^{(j)}(X_1, \Psi) - \mathcal{O}_1^{(j)}(X_1, 0)) \mathcal{O}_2^{(j)}(X_2, 0) \\ &\quad +(1 \longleftrightarrow 2)\} + \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2), \end{aligned} \tag{5.3.6}$$

$$\|\mathcal{E}_I[\tilde{K}^{(j)}]\|_{G^{(j)},\Gamma_{\eta^2},H} \leq C_{5.2} \|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H}. \tag{5.3.7}$$

Proof. (2.4) and Proposition 4.2 imply that $\mathcal{E}_I[\tilde{K}^{(j)}] = \mathcal{E}[\tilde{K}^{(j)}, \tilde{R}]$, $\tilde{R}(X) := e^{-\omega_I[\tilde{K}^{(j)}](X)} - 1$. (5.3.4) shows that

$$\tilde{R}(X) = -(r.h.s.(5.3.4)) + \lambda_1 \lambda_2 \mathcal{O}_1^{(j)}(X, 0) \mathcal{O}_2^{(j)}(X, 0) + \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2),$$

which, together with Corollary 4.5, yields (5.3.6).

As regards (5.3.7), we proceed as follows: Because of (5.3.5), and because

$$\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} \leq \frac{4}{5} UB^{(j)} \leq 1$$

(cf. (5.2.4)) we can estimate, for any $g \geq 1,\ \mathbf{h} \geq 0$,

$$
\begin{aligned}
\|\tilde{R}\|_{g,\Gamma_\eta,\mathbf{h}} &= \sup_{\Delta} \sum_{X \supset \Delta} \Gamma_\eta(X) | \sum_{k \geq 1} \frac{(-1)^k}{k!} (\omega_I[\tilde{K}^{(j)}](X))^k | \\
&\leq \sup_{\Delta} \sum_{X \supset \Delta} \sum_{k \geq 1} (\Gamma_\eta(X) |\, \omega_I[\tilde{K}^{(j)}](X)|)^k \frac{1}{k!} \\
&\leq \sum_{k \geq 1} \frac{1}{k!} (\|\omega_I[\tilde{K}^{(j)}]\|_{g,\Gamma_\eta,\mathbf{h}})^k \\
&\leq C_{5.1} \cdot e^{C_{5.1}} \cdot \|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H}. \qquad (5.3.8)
\end{aligned}
$$

Hence, (5.2.3,5.2.4) tells us that $\|\tilde{R}\|_{G^{(j)},\Gamma_\eta,H} \leq C_{4.1}$; since (5.2.3,5.2.4) also implies that $\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma_\eta,H} \leq C_{4.1}$, we can apply Proposition 4.4 and (5.3.8) to get

$$
\begin{aligned}
\|\mathcal{E}_I[\tilde{K}^{(j)}]\|_{G^{(j)},\Gamma_{\eta 2},H} &\equiv \|\mathcal{E}[\tilde{K}^{(j)}, \tilde{R}]\|_{G^{(j)} 1^r,\Gamma_{\eta 2},H} \\
&\leq C_{4.2}\{1 + C_{5.1} \cdot e^{C_{5.1}}\} \cdot \|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H}.
\end{aligned}
$$

□

5.4 Extraction II ($\mathcal{E}_{II}$)

5.4.1 Existence

Combining Lemma 5.3.1 with Corollary 3.4.2 and Lemma 3.4.5 (and the fact that $\mathcal{E}_I[\tilde{K}^{(j)}](\cdot, \Psi = 0) = 0$) and (3.4.3), we arrive at

Lemma 5.4.1. Let μ, λ_α be as in (5.3.1); under this condition, the polymer activities $\Omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ and $\mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ exist and are regular, local, and even, and they are analytic in μ, λ_α. Furthermore

(a)
$$\Omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]](\cdot, 0) = \mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]](\cdot, 0) = 0. \qquad (5.4.1)$$

(b) $\Omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]] = \Omega_{II}[\mathcal{E}_I[\mu K^{(j)}]]$ and $\tau_0 \mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]] = \mathcal{E}_{II}[\mathcal{E}_I[\mu \cdot K^{(j)}]]$; these polymer activities are of I-type, if $\mu \in \mathbf{R}$.

(c) $\tau_{\bar{\alpha}} \mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ are $\mathcal{O}$-type.

(d)
$$\mathcal{E}^{\Box + \mathcal{E}_I[\tilde{K}^{(j)}]} \doteq e^{\Omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]} \mathcal{E}^{\Box + \mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]}. \qquad (5.4.2)$$

By Definition 3.4.1, if $X \in \mathcal{S}$ then

$$\omega_{II}[\mathcal{E}_I[K^{(j)}]](X, \Psi) = \int_X d^d z \Psi_\mu(z)\Psi_\nu(z) \cdot \omega_{\mu\nu}^{(j)}(X) \tag{5.4.3}$$

$$\omega_{\mu\nu}^{(j)}(X) := \frac{1}{|X|}\int [D(\mathbf{n} = (2, \mathbf{0}))\mathcal{E}_I[K^{(j)}]](X, \Psi = 0; (x, \mu), (y, \nu)). \tag{5.4.4}$$

Therefore

$$\Omega_{II}[\mathcal{E}_I[K^{(j)}]](\Lambda^{(j)}, \Psi^\phi) = -\sum_{\Delta \subset \Lambda^{(j)}} \frac{1}{2} \cdot \delta\sigma_{\mu\nu}^{(j)}(\Delta) \cdot \int_\Delta d^d z \partial_\mu \phi(z) \partial_\nu \phi(z),$$

where

$$-\delta\sigma_{\mu\nu}^{(j)}(\Delta) := 2 \sum_{Y \supset \Delta} \omega_{\mu\nu}^{(j)}(Y).$$

Invariance of $\mathcal{E}_I[K^{(j)}]$ (cf. Lemma 5.3.1) implies that

$$\delta\sigma_{\mu\nu}^{(j)}(\Delta) = R_{\mu'\mu} R_{\nu'\nu} \delta\sigma_{\mu'\nu'}^{(j)}(\Delta_E)$$

for all lattice symmetries E (cf. Section 1.4), and hence

$$\delta\sigma_{\mu\nu}^{(j)}(\Delta) = \delta\sigma^{(j)} \cdot \delta_{\mu\nu}, \quad \delta\sigma^{(j)} \text{ indep. of } \Delta.$$

Collecting everything we thus have

Corollary 5.4.2.

$$\Omega_{II}[\mathcal{E}_I[K^{(j)}]](\Lambda^{(j)}, \Psi^\phi) = -\frac{1}{2} \cdot \delta\sigma^{(j)} \cdot \int_{\Lambda^{(j)}} d^d z (\partial_\mu \phi)^2(z), \tag{5.4.5}$$

with

$$\delta\sigma^{(j)} = -\frac{2}{d} \sum_{\mu,\nu} \sum_{\substack{Y \supset \Delta \\ Y \in \mathcal{S}}} \omega_{\mu\nu}^{(j)}(Y), \text{ any fixed } \Delta \subset \Lambda^{(j)}, \tag{5.4.6}$$

where $\omega_{\mu\nu}^{(j)}$ has been given in (5.4.4).

5.4.2 Bounds

Lemma 5.4.3. For μ as in (5.3.1), $\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ is analytic in μ and satisfies

$$\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]] = \mu \cdot \mathcal{L}^{(d)}[K^{(j)} - K^{(j)}(\cdot, \Psi = 0)] + \mathcal{O}(\mu^2), \tag{5.4.7}$$

and, for any $\epsilon > 0$,

$$\|\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]\|_{G_\epsilon, \Gamma_{\eta^2}, H} \le C_{5.3} \left(1 + \frac{1}{\epsilon \cdot H^2}\right) \cdot |\mu| \cdot \|K^{(j)}\|_{G^{(j)}, \Gamma, H}. \tag{5.4.8}$$

Moreover,

$$|\delta\sigma^{(j)}| \leq C_{5.4} \cdot \frac{1}{H^2} \cdot \|K^{(j)}\|_{G^{(j)},\Gamma,H}. \tag{5.4.9}$$

Proof. Analyticity in μ of $\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ has already been noticed in Lemma 5.4.1. And Eqn. (5.4.7) follows from (3.4.1), Lemma 5.3.1(a) and (5.3.6).

Apply Lemma 4.1 of [6] to $\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$. As a result, one obtains (5.4.8) (use also (5.3.7)).

(5.4.9) is an immediate consequence of (5.4.6) and of (5.3.7). □

Lemma 5.4.4. For μ, λ_α as in (5.3.1)

$$\begin{aligned}
\mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]](X,\Psi) \;=\; & \mu \cdot \mathcal{R}^{(d)}[K^{(j)}(X,\Psi) - K^{(j)}(X,0)] \\
& + \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot (\mathcal{O}_{\bar{\alpha}}^{(j)}(X,\Psi) - \mathcal{O}_{\bar{\alpha}}^{(j)}(X,0)) \\
& - \lambda_1\lambda_2 \sum_{\substack{X_1 \cup X_2 = X \\ X_1 \cap X_2 \neq \emptyset}} \Big\{ (\mathcal{O}_1^{(j)}(X_1,\Psi) \\
& - \mathcal{O}_1^{(j)}(X_1,0))\mathcal{O}_2^{(j)}(X_2,0) + (1 \longleftrightarrow 2) \Big\} \\
& + \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2).
\end{aligned} \tag{5.4.10}$$

Define $\tau(\mathcal{S})$ by (cf. (4.11))

$$\tau(\mathcal{S}) := \max_{N,j} \max_{x \in \Lambda^{(j)}} |\{X \in \mathcal{S} : X \ni x\}|, \tag{5.4.11}$$

and assume that $\epsilon > 0$, H, $UB^{(j)}$ obey

$$\frac{2 \cdot C_{5.3}}{C_{4.1}} \cdot UB^{(j)} \leq \left(1 + \frac{\tau(\mathcal{S})}{\epsilon \cdot H^2}\right)^{-1}. \tag{5.4.12}$$

Then

$$\begin{aligned}
\|\mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]\|_{G^{(j)}G_\epsilon,\Gamma_{\eta^3},H} \leq & \; C_{5.5} \cdot \|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} \\
+ |\mu| \|K^{(j)}\|_{G^{(j)},\Gamma,H} \cdot C_{5.6} \cdot & \left(1 + \frac{1}{(\kappa^{(j)} + \epsilon)^{\frac{1}{2}} H}\right)^{C_{5.6}} \cdot \left(1 + \frac{\tau(\mathcal{S})}{\epsilon H^2}\right).
\end{aligned} \tag{5.4.13}$$

Proof. By (3.4.6), $\mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ is a function of $\mathcal{E}'_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ which, according to (3.4.1)-(3.4.3) and (4.3)-(4.6), is a power series in $\mathcal{E}_I[\tilde{K}^{(j)}]$ and

$$R := \left(e^{-\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]} - 1\right).$$

We begin by analyzing R.

R is independent of λ_α, thus by (5.4.7),

$$R = -(r.h.s(5.4.7)) + \mathcal{O}(\mu^2). \tag{5.4.14}$$

As a prerequisite to bound R we need

Corollary 5.4.5. If K_1, K_2 are regular polymer activities, then, for any $g_1, g_2, \gamma_1, \gamma_2, \mathbf{h}$,

$$\|K_1K_2\|_{g_1g_2,\gamma_1\gamma_2,\mathbf{h}} \leq \|K_1\|_{g_1,\gamma_2,\mathbf{h}} \cdot \|K_2\|_{g_2,\gamma_2,\mathbf{h}}. \tag{5.4.15}$$

Proof of Corollary. Use the definition of the norm (cf. Section 1.4). □

Thanks to (5.4.15) and because $(\Gamma_{\eta^2})^{\frac{1}{k}} \leq \Gamma_{\eta^2}$, $(G_\delta)^{\frac{1}{k}} = G_{\delta/2}$, we get

$$\|R\|_{G_{\epsilon/\tau(S)},\Gamma_{\eta^2},H} \leq \sum_{k\geq 1} \frac{1}{k!}\left(\|\omega_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]\|_{G_{\epsilon/(k\tau(S))},\Gamma_{\eta^2},H}\right)^k$$

so that (5.4.8), the bound $\frac{k^k}{k!} \leq e^k$ and $|\mu| \cdot \|K^{(j)}\|_{G^{(j)},\Gamma,H} \leq \frac{UB^{(j)}}{5}$ (cf. 5.3.1)) give

$$\leq e \cdot C_{5.3}\left(1 + \frac{\tau(S)}{\epsilon H^2}\right) \cdot |\mu| \|K^{(j)}\|_{G^{(j)},\Gamma,H} \cdot \sum_{k\geq 0}\left\{\left(1 + \frac{\tau(S)}{\epsilon H^2}\right) \cdot \frac{eC_{5.3}}{5} \cdot UB^{(j)}\right\}^k,$$

which, together with the conditions (5.4.12) and $C_{4.1} \leq 1$ implying that $\{\quad\} \leq \frac{1}{2}$, yields

$$\leq 2 \cdot eC_{5.3}\left(1 + \frac{\tau(S)}{\epsilon H^2}\right) \cdot |\mu| \cdot \|K^{(j)}\|_{G^{(j)},\Gamma,H}. \tag{5.4.16}$$

Next, we take a look at $\mathcal{E}'_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]] \equiv \mathcal{E}[\mathcal{E}_I[\tilde{K}^{(j)}], R]$. Due to (5.4.14), (5.3.6), and Corollary 4.5, we find

$$\begin{aligned}
\mathcal{E}'_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]](X,\Psi) \;=\;& \mu \cdot (1 - \mathcal{L}^{(d)})[K^{(j)}(X,\Psi) - K^{(j)}(X,0)] \\
&+ \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot (\mathcal{O}_{\bar{\alpha}}^{(j)}(X,\Psi) - \mathcal{O}_{\bar{\alpha}}^{(j)}(X,0)) \\
&-\lambda_1\lambda_2 \sum_{\substack{X_1\cup X_2 = X\\ X_1\cap X_2\neq\emptyset}} \Big\{(\mathcal{O}_1^{(j)}(X_1,\Psi) \\
&- \mathcal{O}_1^{(j)}(X_1,0))\mathcal{O}_2^{(j)}(X_2,0) + (1 \longleftrightarrow 2)\Big\} \\
&+\mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2);
\end{aligned} \tag{5.4.17}$$

combined with (3.4.6) and Theorem 3.1.1, part (i), this yields (5.4.10).

(5.3.7), the standard bound $\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} \leq \frac{4}{5}UB^{(j)}$ (cf. (5.3.1)) and (5.2.4) guarantee that $\|\mathcal{E}_I[\tilde{K}^{(j)}]\|_{G^{(j)},\Gamma_{\eta 2},H} \leq C_{4.1}$. (5.4.16), (5.3.1), and (5.4.12) imply that for any $\tau \leq \tau(\mathcal{S})$, $\|R\|_{G_{\epsilon/\tau},\Gamma_{\eta 2},H} \leq C_{4.1}$. Thus we may apply Proposition 4.4 and get

$$\begin{aligned}
&\|\mathcal{E}'_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]\|_{G^{(j)}G_\epsilon,\Gamma_{\eta 3},H} \\
&\leq C_{4.2}\left\{C_{5.2}\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} + 2eC_{5.3}\cdot\left(1+\frac{\tau(\mathcal{S})}{\epsilon H^2}\right)\cdot|\mu|\cdot\|K^{(j)}\|_{G^{(j)},\Gamma,H}\right\} \\
&\leq C_{5.5}\cdot\left\{\|\tilde{K}^{(j)}\|_{G^{(j)},\Gamma,H} + \left(1+\frac{\tau(\mathcal{S})}{\epsilon H^2}\right)\cdot|\mu|\cdot\|K^{(j)}\|_{G^{(j)},\Gamma,H}\right\}. \qquad (5.4.18)
\end{aligned}$$

Abbreviate $\mathcal{E}'_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]$ by $\mathcal{E}'_{II}$. Write $\tau_0\mathcal{E}'_{II} = \tau_0\mathcal{E}'_{II}\cdot 1(X\in\mathcal{S}) + \tau_0\mathcal{E}'_{II}\cdot 1(X\notin\mathcal{S})$. Using (3.4.6) and (3.3.10) we have $\mathcal{E}_{II} = (1-\tau_0)\mathcal{E}'_{II} + \tau_0\mathcal{E}'_{II}\cdot 1(X\notin\mathcal{S}) + R^{(d)}[\tau_0\mathcal{E}'_{II}\cdot 1(X\in\mathcal{S})]$; hence, using Theorem 3.1.1(iii) with $\ell = 1$, $\epsilon\mapsto\kappa^{(j)}+\epsilon$, we obtain

$$\begin{aligned}
\|\mathcal{E}_{II}\|_{G^{(j)}G_\epsilon,\Gamma_{\eta 3},H} \leq\ & \|\mathcal{E}'_{II}\|_{G^{(j)}G_\epsilon,\Gamma_{\eta 3},H} \\
&+\|\tau_0\mathcal{E}'_{II}\|_{G^{(j)}G_\epsilon,\Gamma_{\eta 3},H}\left(1+C(d)\cdot\left(1+\frac{C_s}{(\kappa^{(j)}+\epsilon)^{\frac{1}{2}}H}\right)^{C(d)}\right).
\end{aligned}$$

Combining this with (5.4.18) we arrive at (5.4.13). □

5.5 Reblocking, Rescaling (S)

In this section we assume that $j < N$.

5.5.1 Existence

Definition 5.5.1. Let J be a regular polymer activity defined on L-scale polymers. The 1-scale regular polymer activity $S[J]$ is defined by

$$S[J](X,\Psi) := J(X_L,\Psi_L), \qquad (5.5.1)$$

where $X_L := L^dX$, $(\Psi_L)_\mu(x) := L^{-\frac{d}{2}+1-|\mu|}\Psi_\mu(L^{-1}x)$.

Corollary 5.5.2.

(a) J is local/real/even/invariant iff $S[J]$ is so.

(b) J is pinned at z iff $S[J]$ is pinned at $L^{-1}z$.

(c) For any g, γ, h,

$$\|S[J]\|_{g,\gamma,h} = \|J\|_{g_L,\gamma_L,\mathbf{h}_L}, \tag{5.5.2}$$

where γ_L has been defined in Section 4, $\mathbf{h}_L$ in (3.3.1), and g_L is given by

$$g_L(X_L, \phi_L) := g(X, \phi)$$

Definition 5.5.3. For μ, λ_α as in (5.3.1) we define the regular, local, even polymer activity $\tilde{SK}^{(j)}$ on $\Lambda^{(j+1)}$ by

$$\tilde{SK}^{(j)} := S \circ B \circ \mathcal{E}_{II} \circ \mathcal{E}_I[\tilde{K}^{(j)}], \tag{5.5.3}$$

where the reblocking operation B (resp. some of its properties) has been described in Definition 4.6 (resp. in Proposition 4.7 and in Corollary 4.9). Evidently, $\tilde{SK}^{(j)}$ is analytic in μ, λ_α and we set

$$\begin{aligned} SK^{(j)} &:= \tau_0 \tilde{SK}^{(j)}|_{\mu=1} \\ SO^{(j)}_{\bar{\alpha}} &:= \tau_{\bar{\alpha}} \tilde{SK}^{(j)}|_{\mu=1}. \end{aligned} \tag{5.5.4}$$

In addition,

$$\begin{aligned} \tilde{\Omega}^{(j)} &:= (\Omega_I + \Omega_{II} \circ \mathcal{E}_I)[\tilde{K}^{(j)}] \\ \Omega^{(j)} &:= \tau_0 \tilde{\Omega}^{(j)}|_{\mu=1}, \quad \Omega^{(j)}_{\bar{\alpha}} := \tau_{\bar{\alpha}} \tilde{\Omega}^{(j)}|_{\mu=1}. \end{aligned} \tag{5.5.5}$$

Corollary 5.5.4. $SK^{(j)}$ is I-type; $SO^{(j)}_{\bar{\alpha}}$ are $\mathcal{O}$-type, pinned at $L^{-(j+1)}x_{\bar{\alpha}}$.

Recall the notation set up in (5.1.7) and (5.1.8).

Proposition 5.5.5. Let $C^{(j)}$ be a covariance on $\mathcal{H}_s(\Lambda^{(j)})/\{\text{constants}\}$. Define $SC^{(j)}$ on $\mathcal{H}_s(\Lambda^{(j+1)})/\{\text{constants}\}$ by

$$\begin{aligned} &\int d\mu_{SC^{(j)}}(\phi) S[F](\Lambda^{(j+1)}, \Psi^\phi) \\ &\quad := \frac{\int d\mu_{C^{(j)}}(\phi) e^{\Omega_{II} \circ \mathcal{E}_I[K^{(j)}](\Lambda^{(j)}, \Psi^\phi)} F(\Lambda^{(j)}, \Psi^\phi)}{\int d\mu_{C^{(j)}}(\phi) e^{\Omega_{II} \circ \mathcal{E}_I[K^{(j)}](\Lambda^{(j)}, \Psi^\phi)}}. \end{aligned} \tag{5.5.6}$$

Assume that $Z_{\Lambda^{(j)};K^{(j)},C^{(j)}} \neq 0$. Then we have

$$\begin{aligned} \langle \mathcal{O}^{(j)}_1; \mathcal{O}^{(j)}_2 | \mathcal{O}^{(j)}_{12} \rangle_{\Lambda^{(j)};K^{(j)},C^{(j)}} &= \langle S\mathcal{O}^{(j)}_1; S\mathcal{O}^{(j)}_2 | S\mathcal{O}^{(j)}_{12} \rangle_{\Lambda^{(j+1)};SK^{(j)},SC^{(j)}} \\ &\quad + \Omega^{(j)}_{12}(\Lambda^{(j)}). \end{aligned} \tag{5.5.7}$$

Proof. Because, by hypothesis, $Z_{\Lambda^{(j)};K^{(j)},C^{(j)}} \neq 0$ it is immediate that, for $|\lambda_\alpha|$ small enough, $\int d\mu_{C^{(j)}}(\phi)\mathcal{E}^{\Box+\tilde{K}^{(j)}}(\Lambda^{(j)},\Psi^\phi)$ at $\mu := 1$ is nonzero and at least $C^{(2)}$ in λ_α. Consequently,

$$\langle \mathcal{O}_1^{(j)};\mathcal{O}_2^{(j)}|\mathcal{O}_{12}^{(j)}\rangle_{\Lambda^{(j)};K^{(j)},C^{(j)}} = \tau_{12}\log\left(\int d\mu_{C^{(j)}}(\phi)\mathcal{E}^{\Box+\tilde{K}^{(j)}}(\Lambda^{(j)},\Psi^\phi)\right).$$

Now recall that $\mathcal{E}^{\Box+\tilde{K}^{(j)}} \doteq e^{\tilde{\Omega}^{(j)}}\mathcal{E}^{\Box+\mathcal{E}_{II}\circ\mathcal{E}_I[\tilde{K}^{(j)}]}$ (cf. (5.5.5), (2.4), and (3.4.3), (3.4.7)), and that $\mathcal{E}^{\Box+\mathcal{E}_{II}\circ\mathcal{E}_I[\tilde{K}^{(j)}]} = \mathcal{E}^{\Box_L+B\circ\mathcal{E}_{II}\circ\mathcal{E}_I[\tilde{K}^{(j)}]}$ (see (4.17)). Using also that $\tilde{\Omega}^{(j)} \equiv \Omega_I + \Omega_{II}\circ\mathcal{E}_I$, where Ω_I is Ψ-independent and $\Omega_{II}\circ\mathcal{E}_I$ is λ_α-independent, and (5.5.6), (5.5.3,5.5.5) we finally get (5.5.7). □

Corollary 5.5.6. Assume the covariance $C^{(j)}$ on $\mathcal{H}_s(\Lambda^{(j)})/\{\text{constants}\}$ is given by

$$C^{(j)}(x,y) \equiv C^{(j)}(x-y) := |\Lambda^{(j)}|^{-1}\sum_{\substack{p\in(\Lambda^{(j)})^* \\ p\neq 0}} e^{ip(x-y)}\frac{\chi(p^2)}{p^2\cdot\sigma^{(j)}}, \tag{5.5.8}$$

where $\chi \geq 0$ is as specified in Section 1.4, and $\sigma^{(j)} > 0$. Then,

$$SC^{(j)}(x-y) = |\Lambda^{(j+1)}|^{-1}\sum_{\substack{p\in(\Lambda^{(j+1)})^* \\ p\neq 0}} e^{ip(x-y)}\frac{\chi(p^2/L^2)}{p^2\cdot(\sigma^{(j)}+\delta\sigma^{(j)}\cdot\chi(p^2/L^2))}. \tag{5.5.9}$$

Proof. Use (5.5.6) and Corollary 5.4.2. □

5.5.2 Bounds

Lemma 5.5.7. For μ, λ_α as in (5.3.1)

$$\begin{aligned}
\tilde{SK}^{(j)} = {} & \mu\cdot S\circ B^{(1)}[R^{(d)}[K^{(j)}(\cdot,\Psi) - K^{(j)}(\cdot,0)]] \\
& + \sum_{\bar{\alpha}}\lambda_{\bar{\alpha}}\cdot S\circ B^{(1)}[\mathcal{O}_{\bar{\alpha}}^{(j)}(\cdot,\Psi) - \mathcal{O}_{\bar{\alpha}}^{(j)}(\cdot,0)] \\
& + \lambda_1\lambda_2\left\{-S\circ B^{(1)}\left[\sum_{\substack{X_1\cap X_2=\cdot \\ X_1\cap X_2\neq\emptyset}}(\mathcal{O}_1^{(j)}(X_1,\Psi)\right.\right. \\
& \left. - \mathcal{O}_1^{(j)}(X_1,0))\mathcal{O}_2^{(j)}(X_2,0) + (1\longleftrightarrow 2)\right]
\end{aligned}$$

$$+ S \circ B^{(2)}[(\mathcal{O}_1^{(j)}(\cdot,\Psi) - \mathcal{O}_1^{(j)}(\cdot,0)),(\mathcal{O}_2^{(j)}(\cdot,\Psi) - \mathcal{O}_2^{(j)}(\cdot,0))]\Big\}$$

$$+\mathcal{O}(\mu^2,\mu\lambda_\alpha,\lambda_\alpha^2). \tag{5.5.10}$$

If $\epsilon > 0$, H, $UB^{(j)}$ satisfy e.g., the conditions (5.4.12) and

$$\frac{4^{C_{5.6}}}{C_{4.3}(L)} \cdot UB^{(j)} \leq \left(1 + \frac{\tau(S)}{\epsilon H^2}\right)^{-1}, \tag{5.5.11}$$

and if $\kappa^{(j)}$, H obey

$$\kappa^{(j)} \cdot H^2 \geq 1, \tag{5.5.12}$$

then there exists $C_{5.7}(L)$ such that

$$\|\tilde{SK}^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq C_{5.7}(L), \tag{5.5.13}$$

where the large field regulator $(G^{(j)}G_\epsilon)_{L^{-1}}$ is defined in accordance with (5.5.2), i.e., such that

$$((G^{(j)}G_\epsilon)_{L^{-1}})_L = G^{(j)}G_\epsilon. \tag{5.5.14}$$

Remarks. (1) It is easy to check that (cf. (5.1.1), (5.1.2))

$$(G^{(j)}G_\epsilon)_{L^{-1}}(X,\phi)$$
$$= \exp\left((\kappa^{(j)}+\epsilon)\left\{\sum_{|\mu|=1}^{s} L^{2-2|\mu|}\|\partial^\mu\phi\|_X^2 + \frac{1}{L\cdot c}\|\partial\phi\|_{\partial X}^2\right\}\right). \tag{5.5.15}$$

(2) The regulators $(G^{(j)}G_\epsilon)_{L^{-1}} \equiv (G_{\kappa^{(j)}+\epsilon})_{L^{-1}}$ and $\Gamma_{\eta^{-1}}$ are weaker than $G_{\kappa^{(j)}+\epsilon}$ and Γ. The fact that $\tilde{SK}^{(j)}$ can be usefully bounded *w.r.t.* the stronger norm $\|(\,\cdot\,)\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H}$ will be employed later on to control the result of the fluctuation integral *w.r.t.* the weaker, but sought-for, norm $\|(\,\cdot\,)\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H}$.

(3) We assume *w.l.g.* that $C_{5.7}(L) \geq 1$.

Proof. Use (5.5.3), (4.23), and (5.4.10) to check (5.5.10). The bound (5.5.13) emerges in the following way: By (5.5.3), (5.5.2), and (5.5.14), and because $L \geq 4$ implies $\mathbf{2H}_L \leq H$, we have

$$\|\tilde{SK}^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq \|B[\mathcal{E}_{II}[\mathcal{E}_I[\tilde{K}^{(j)}]]]\|_{G^{(j)}G_\epsilon,(\Gamma_{\eta^{-1}})_L,H}.$$

Because of (5.4.13), (5.3.1), (5.2.3,5.2.4), and (5.5.11), (5.5.12), we conclude that (r.h.s. (5.4.13) $\leq C_{4.3}(L)$. Hence we may apply Proposition 4.7(i), with $\eta \mapsto \eta^4$ and $\gamma \mapsto \Gamma_{\eta^{-1}}$ (note that $A_{\Gamma_{\eta^{-1}}} \geq (\eta^4)^{2^d}$ due to (5.2.2)), to continue with

$$\leq C_{4.4} \cdot L^d \cdot C_{4.3}(L). \qquad \square$$

Lemma 5.5.8. For μ, λ_α as in (5.3.1), write

$$\tilde{SK}^{(j)} = \mu \cdot \tilde{SK}^{(j)}_\mu + \sum_{\bar\alpha} \lambda_{\bar\alpha} \cdot \tilde{SK}^{(j)}_{\bar\alpha} + \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2).$$

If $\kappa^{(j)}$, H, L obey

$$\kappa^{(j)} \cdot H^2 \geq L^{d+2P-2}, \tag{5.5.16}$$

then (recall that $J \in \mathbf{N}_0$ has been defined by $L^J \leq |x_1 - x_2| < L^{J+1}$)

$$\|\tilde{SK}^{(j)}_\mu\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq C_{5.8} \cdot L^{-\frac{1}{2}} \cdot \|K^{(j)}\|_{G^{(j)},\Gamma,H}; \tag{5.5.17}$$

$$\|\tilde{SK}^{(j)}_\alpha\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq C_{5.8} \cdot L^{-d} \cdot \|\mathcal{O}^{(j)}_\alpha\|_{G^{(j)},\Gamma,H}; \tag{5.5.18}$$

$$\begin{aligned}\|\tilde{SK}^{(j)}_{12}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} &\leq C_{5.8} \cdot \left\{ \|\mathcal{O}^{(j)}_{12}\|_{G^{(j)},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}^{(j)}_\alpha\|_{G^{(j)},\Gamma,H} \right\} \\ &\quad \cdot \begin{cases} L^{-2d-\frac{1}{2}}, & \text{if } 0 \leq j \leq J-2 \\ L^{-2d}, & \text{if } j = J-1 \\ L^{-d}, & \text{if } J \leq j \leq N-1. \end{cases}\end{aligned} \tag{5.5.19}$$

[In the last equation a stronger decay can be obtained by choosing A larger in Γ.]

Proof. (1) We begin with the proof of (5.5.17). The explicit formula for $\tilde{SK}^{(j)}_\mu$ can be looked up in (5.5.10). Write $\Delta K^{(j)} := K^{(j)}(\cdot, \Psi) - K^{(j)}(\cdot, 0)$ and $\Delta K^{(j)}_\mathcal{S} := \Delta K^{(j)} \cdot 1(X \in \mathcal{S})$, $\Delta K^{(j)}_\mathcal{L} := \Delta K^{(j)} - \Delta K^{(j)}_\mathcal{S}$. With this notation, with (5.5.2) and the linearity of $R^{(d)}$, $B^{(1)}$ we have

$$\begin{aligned}&\|\tilde{SK}^{(j)}_\mu\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \\ &\quad \leq \sum_{\beta=\mathcal{S},\mathcal{L}} \|B^{(1)}[R^{(d)}[\Delta K^{(j)}_\beta]]\|_{G^{(j)}G_\epsilon,(\Gamma_{\eta^{-1}})_L,2\mathbf{H}_L}.\end{aligned} \tag{5.5.20}$$

(1a) $\beta = \mathcal{L}$ term in (5.5.20): Due to (3.3.10) we have $R^{(d)}[\Delta K^{(j)}_\mathcal{L}] = \Delta K^{(j)}_\mathcal{L}$; now, apply (4.19) (with $\gamma \mapsto \Gamma_{\eta^{-1}}$), and recall that $A_{\Gamma_{\eta^{-1}}} \geq L^{d+\frac{1}{2}}$ (cf. (5.2.2)), that $G^{(j)}G_\epsilon \geq G^{(j)}$ and that $2\mathbf{H}_L \leq H$ to get

$$\|B^{(1)}[R^{(d)}[\Delta K^{(j)}_\mathcal{L}]]\|_{G^{(j)}G_\epsilon,(\Gamma_{\eta^{-1}})_L,2\mathbf{H}_L} \leq \eta^{2^{d+1}} \cdot L^{-\frac{1}{2}} \cdot \|\Delta K^{(j)}_\mathcal{L}\|_{G^{(j)},\Gamma,H}.$$

We continue with $\|\Delta K^{(j)}_\mathcal{L}\|_{g,\gamma,h} \leq \|\Delta K^{(j)}\|_{g,\gamma,h}$, and with $\|\Delta K^{(j)}\|_{g',\gamma,h} \leq 2\|K^{(j)}\|_{g',\gamma,h}$ for any $g' \geq 1$ with $g'(\cdot, \phi = 0) = 1$.

(1b) $\beta = \mathcal{S}$ term in (5.5.20): Apply (4.19) and $G^{(j)}G_\epsilon \geq G^{(j)}$, $2\mathbf{H}_L \leq \mathbf{H}_\ell$ with $\ell := L/4$, to obtain

$$\|B^{(1)}[R^{(d)}[\Delta K^{(j)}_\mathcal{L}]]\|_{G^{(j)}G_\epsilon,(\Gamma_{\eta^{-1}})_L,2\mathbf{H}_L} \leq \eta^{2^{d+1}} \cdot L^d \cdot \|R^{(d)}[\Delta K^{(j)}_\mathcal{S}]\|_{G^{(j)},\Gamma,\mathbf{H}_\ell}.$$

Use Theorem 3.1.1, part (iii), and (5.5.16) and $\|\Delta K_{\mathcal{S}}^{(j)}\|_{G^{(j)},\Gamma,H} \leq 2\|K^{(j)}\|_{G^{(j)},\Gamma,H}$ to arrive at (5.5.17).

(2) Proof of (5.5.18): We use the same notation as before. Then, according to (5.5.10), (5.5.2), and (4.20),

$$\|\tilde{SK}_{\alpha}^{(j)}\|_{(G^{(j)}G_{\epsilon})_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq \eta^{2^{d+1}} \cdot \|\Delta\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,\mathbf{H}_{\ell}}.$$

Apply Lemma 4.3 of [6] to proceed

$$\leq \eta^{2^{d+1}} \cdot C \cdot \left(1 + \frac{\ell^{d+2P-2}}{K^{(j)} \cdot H^2}\right)^d \|\Delta\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,\mathbf{H}_{\ell;\geq d}},$$

and use (5.5.16) to obtain (5.5.18).

(3) Proof of (5.5.19): Notation as before. Use (5.5.10), (5.5.2) to see that

$$\begin{aligned} \|\tilde{SK}_{12}^{(j)}\|_{(G^{(j)}G_{\epsilon})_{L^{-1}},\Gamma_{\eta^{-1}},2H} &\leq \|B^{(1)}[\Delta\mathcal{O}_{12}^{(j)}]\|_{G^{(j)},(\Gamma_{\eta^{-1}})_L,\mathbf{H}_{\ell}} \\ &+ \left\{\left\|B^{(1)}\left[\sum_{\substack{X_1\cup X_2=\cdot \\ X_1\cap X_2\neq\phi}} \Delta\mathcal{O}_1^{(j)}(X_1)\cdot\mathcal{O}_2^{(j)}(X_2,0)\right]\right\|_{G^{(j)},(\Gamma_{\eta^{-1}})_L,\mathbf{H}_{\ell}} + (1\longleftrightarrow 2)\right\} \\ &+ \|B^{(2)}[\Delta\mathcal{O}_1^{(j)},\Delta\mathcal{O}_2^{(j)}]\|_{G^{(j)},(\Gamma_{\eta^{-1}})_L,\mathbf{H}_{\ell}}. \end{aligned} \tag{5.5.21}$$

(3a) $0 \leq j \leq J-2$: Then $|x_1^{(j)} - x_2^{(j)}| \geq L^{J-j} \geq L^2 \geq 4L > 2^d$; hence $\Delta\mathcal{O}_{12}^{(j)}(X \in \mathcal{S}) = 0$, $\sum_{\substack{X_1\cup X_2=X\in\mathcal{S} \\ X_1\cap X_2\neq\phi}} \Delta\mathcal{O}_1^{(j)}(X_1)\cdot\mathcal{O}_2^{(j)}(X_2,0) = 0$, and upon application of (4.20), (4.21) we get

$$\begin{aligned} \leq\ & \eta^{(2^{d+2})}\cdot(A_{\Gamma_{\eta^{-1}}})^{-1}\Big\{\|\Delta\mathcal{O}_{12}^{(j)}\|_{G^{(j)},\Gamma,\mathbf{H}_{\ell}} \\ &+\{\|\Delta\mathcal{O}_1^{(j)}\|_{G^{(j)},\Gamma,\mathbf{H}_{\ell}} + (1\longleftrightarrow 2)\} + \prod_{\alpha=1}^{2}\|\Delta\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,\mathbf{H}_{\ell}}\Big\}. \end{aligned}$$

And now we apply once more Lemma 4.3 of [6] (to generate a decay factor L^{-d}), and (5.2.2) provides us with an additional (in principle arbitrarily strong) decay factor from $(A_{\Gamma_{\eta^{-1}}})^{-1}$ so that (5.5.19) can be established.

(3b) $j = J-1$: Since $|x_1^{(j)} - x_2^{(j)}| \geq L > 2^d$ the $B^{(1)}$-terms on the r.h.s. of (5.5.21) can be bounded as before. When estimating the $B^{(2)}$-term, however, (4.21) yields only a factor 1 (instead of $(A_{\Gamma_{\eta^{-1}}})^{-1}$); this loss is compensated by applying Lemma 4.3 of [6] to both factors in $\prod_{\alpha=1,2}\|\Delta\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,\mathbf{H}_{\ell}}$.

(3c) $J \leq j \leq N-1$: Proceed as in (3a); but instead of the factor $(A_{\Gamma_{\eta^{-1}}})^{-1}$ we only get 1 because $|x_1^{(j)} - x_2^{(j)}|$ is too small. □

Without loss of generality we will assume that

$$2 \cdot C_{5.3} \leq 4^{C_{5.6}} \tag{5.5.22}$$

(cf. (5.4.8), (5.4.13) and (5.4.12), (5.5.11)). Define $C_{5.9}(L)$ by (cf. (5.5.11))

$$C_{5.9}(L) := 2 \cdot 4^{C_{5.6}} / C_{4.3}(L). \tag{5.5.23}$$

We now collect the results of Lemmas 5.5.7, 5.5.8. For the sake of clarity of our statement we will, for the first time in this section, explicitly state the hypotheses (1)-(4) of Section 5.2.

Theorem 5.5.9. Assume that $L, A, H, \kappa^{(j)}, \epsilon, UB^{(j)}$ and $\|K^{(j)}\|_{G^{(j)},\Gamma,H}$, $\|\mathcal{O}_{\bar{\alpha}}^{(j)}\|_{G^{(j)},\Gamma,H}$ satisfy the conditions[15]

(a)

$$L \geq 2^{d+1}, \tag{5.5.24}$$

(b)

$$A \geq \max\{\eta^{2^{d+2}}, L^{d+\frac{1}{2}}\},$$

(c)

$$(\|K^{(j)}\|_{G^{(j)},\Gamma,H})^{\frac{1}{4}} \leq UB^{(j)} \leq C_{5.0}(L), \tag{5.5.25}$$

(d)

$$\|\mathcal{O}_{\bar{\alpha}}^{(j)}\|_{G^{(j)},\Gamma,H} < \infty, \ \forall\, \bar{\alpha},$$

and

(e)

$$\kappa^{(j)} \cdot H^2 \geq L^{d+2P-2}, \tag{5.5.26}$$

(f)

$$C_{5.9}(L) \cdot UB^{(j)} \leq \min\left\{1, \frac{\epsilon H^2}{\tau(S)}\right\}, \tag{5.5.27}$$

where $C_{5.0}(L)$, $C_{5.9}(L)$ have been defined in (5.2.4), (5.5.23).

Then the I-type polymer activity $SK^{(j)}$ and the $\mathcal{O}$-type polymer activities $S\mathcal{O}_{\bar{\alpha}}^{(j)}$ (cf. (5.5.4)) obey the bounds

$$\|SK^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq C_{5.10} \cdot L^{-\frac{1}{2}} \cdot \|K^{(j)}\|_{G^{(j)},\Gamma,H}; \tag{5.5.28}$$

$$\|S\mathcal{O}_{\alpha}^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq C_{5.10} \cdot L^{-d} \cdot \|\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,H}; \tag{5.5.29}$$

[15] We wish to remind the reader that conditions (a)-(d) (or, rather, the somewhat weaker forms (1)-(4) in Section 5.2.) have so far always been assumed to hold, but according to our rules we didn't mention them explicitly.

$$\|S\mathcal{O}_{12}^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} \leq C_{5.10}\cdot\left\{\|\mathcal{O}_{12}^{(j)}\|_{G^{(j)},\Gamma,H}+\prod_{\alpha=1}^{2}\|\mathcal{O}_{\alpha}^{(j)}\|_{G^{(j)},\Gamma,H}\right\} \cdot\begin{cases} L^{2d-\frac{1}{2}}, & 0\leq j\leq J-2\\ L^{-2d}, & j=J-1\\ L^{-d}, & J\leq j\leq N-1.\end{cases} \tag{5.5.30}$$

Proof. First of all: Note that (5.5.24) $\Rightarrow$ (5.2.1); (5.5.25) $\Rightarrow$ (5.2.3,5.2.4) because $UB^{(j)}\leq\frac{1}{2}$; hence the hypotheses in Section 5.2 hold; (5.5.24) and (5.5.26) $\Rightarrow$ (5.5.12); (5.5.27) $\Rightarrow$ (5.4.12) and (5.5.11); hence we may apply Lemmas 5.5.7 and 5.5.8.

We will write $\|K^{(j)}\|$ instead of $\|K^{(j)}\|_{G^{(j)},\Gamma,H}$, and similarly for $\|\mathcal{O}_{\alpha}^{(j)}\|$.

Define $\delta_0:=\frac{1}{5}\,\|K^{(j)}\|^{-\frac{3}{4}}$, $\delta_\alpha:=\frac{1}{5}\frac{\|K^{(j)}\|^{\frac{1}{4}}}{\|\mathcal{O}_{\alpha}^{(j)}\|}$. Due to (5.5.25) and $UB^{(j)}\leq\frac{1}{2}$, we have $\delta_0\geq\frac{8}{5}$ (the important point is that δ_0 is strictly larger than 1, uniformly in $\|K^{(j)}\|$; the precise value of the lower bound for δ_0, here $\frac{8}{5}$ is irrelevant). Thus, the curve $C_{\delta_0}:=\{\mu\in\mathbf{C}:|\mu|=\delta_0\}$ enclosed the points 0 and 1 and is, by (5.5.25), contained in the disk (5.3.1); moreover,

$$\max_{\mu\in C_{\delta_0}}\left|\frac{\mu}{\mu-1}\right|\leq 1+\frac{1}{\delta_0-1}\quad(\leq\frac{8}{3})$$

is bounded from above, uniformly in $\|K^{(j)}\|$. Evidently, for $\lambda_1=0$ the curve $C_{\delta_2}\equiv\{\lambda_2\in\mathbf{C}:\ |\lambda_2|=\delta_2\}$ is contained in the set (5.3.1), and similarly for C_{δ_1}.

(1) Proof of (5.5.28): Because $\frac{1}{\mu-1}=\frac{1}{\mu}\left(1+\frac{1}{\mu-1}\right)$ (and thus $\frac{1}{\mu-1}=\frac{1}{\mu}+\frac{1}{\mu^2}+\frac{1}{\mu^2(\mu-1)}$), and since $\tilde{SK}^{(j)}$ obeys $\tilde{SK}^{(j)}|_{\mu=\lambda_\alpha=0}=0$ (cf. (5.5.10)) and $\tilde{SK}_{\mu}^{(j)}=\frac{\partial}{\partial\mu}\tilde{SK}^{(j)}|_{\mu=\lambda_\alpha=0}$ (cf. Lemma 5.5.8), we have the following contour integral representation for $SK^{(j)}$:

$$\begin{aligned} SK^{(j)} &\equiv (2\pi i)^{-1}\oint_{C_{\delta_0}}\frac{d\mu}{\mu-1}\tilde{SK}^{(j)}|_{\lambda_\alpha=0}\\ &= \tilde{SK}_{\alpha}^{(j)}+(2\pi i)^{-1}\oint_{C_{\delta_0}}\frac{d\mu}{\mu^3}\left(\frac{\mu}{\mu-1}\right)\tilde{SK}^{(j)}|_{\lambda_\alpha=0}. \end{aligned}\tag{5.5.31}$$

Applying Lemmas 5.5.7 and 5.5.8 we obtain

$$\begin{aligned} \|SK^{(j)}\|_{(G^{(j)}G_\epsilon)_{L^{-1}},\Gamma_{\eta^{-1}},2H} &\leq C_{5.8}\cdot L^{-\frac{1}{2}}\|K^{(j)}\|\\ &+ C_{5.7}(L)\cdot(\delta_0)^{-2}\max_{\mu\in C_{\delta_0}}\left|\frac{\mu}{|\mu-1|}\right|, \end{aligned}$$

which, together with (5.5.25) (implying that $\|K^{(j)}\|^{\frac{1}{2}}\cdot C_{5.7}(L)\leq L^{-\frac{1}{2}}$), leads to (5.5.28).

(2) Proof of (5.5.29) for $\alpha=1$: Again by $\frac{1}{\mu-1}=\frac{1}{\mu}+\frac{1}{\mu(\mu-1)}$,

$$\begin{aligned} S\mathcal{O}_{1}^{(j)} &\equiv (2\pi i)^{-2}\oint_{C_{\delta_1}}\frac{d\lambda_1}{(\lambda_1)^2}\oint_{C_{\delta_0}}\frac{d\mu}{\mu-1}\tilde{SK}^{(j)}|_{\lambda_2=0}\\ &= \tilde{SK}_{1}^{(j)}+(2\pi i)^{-2}\oint_{C_{\delta_1}}\frac{d\lambda_1}{(\lambda_1)^2}\oint_{C_{\delta_0}}\frac{d\mu}{\mu^2}\left(\frac{\mu}{\mu-1}\right)\tilde{SK}^{(j)}|_{\lambda_2=0}; \end{aligned}\tag{5.5.32}$$

and now we employ once more Lemmas 5.5.7 and 5.5.8 and (5.5.25).

(3) Proof of (5.5.30): Apart from the fact that the contours C_{δ_α} now have to be chosen more carefully, the proof is completely analogous to (1), (2); hence we content ourselves with pointing out a possible choice of circles C_{δ_α} (C_{δ_0} stays the same as before). Set

$$b_{\bar{\alpha}} := 5 \frac{\|\mathcal{O}_{\bar{\alpha}}^{(j)}\|}{K^{(j)}\|^{\frac{1}{4}}}.$$

(3a) $b_1, b_2 \geq (b_{12})^{\frac{1}{2}}$: Then we define the radii δ_1, δ_2 by $(\delta_\alpha)^{-1} := b_\alpha$.

(3b) $(b_{12})^{\frac{1}{2}} \geq b_1, b_2$: Here we set $(\delta_\alpha)^{-1} := (b_{12})^{\frac{1}{2}}$.

(3c) $b_1 > (b_{12})^{\frac{1}{2}} \geq b_2 : \Rightarrow (\delta_1)^{-1} := b_1, \ (\delta_2^{-1} := \max\left\{b_2, \frac{b_{12}}{b_1}\right\}$.

(3d) $b_2 > (b_{12})^{\frac{1}{2}} \geq b_1 :\Rightarrow (\delta_2)^{-1} := b_2, \ (\delta_1)^{-1} := \max\left\{b_1, \frac{b_{12}}{b_2}\right\}$.

This choice of δ_α is useful because it (and (5.5.25)) guarantee that

(i) λ_1, λ_2 with $|\lambda_\alpha| = \delta_\alpha$ belong to the set (5.3.1),

(ii) $\delta_1^{-1} \cdot \delta_2^{-1} \leq$ const $\cdot \|K^{(j)}\|^{-\frac{1}{2}}(\|\mathcal{O}_{12}^{(j)}\| + \prod_\alpha \|\mathcal{O}_\alpha^{(j)}\|)$. □

As a conclusion we see that the bounds (5.5.28)-(5.5.30) have been obtained by

(i) a careful estimate on the leading order contributions $\tilde{SK}_\mu^{(j)}$, $\tilde{SK}_{\bar{\alpha}}^{(j)}$ (cf. (5.5.31), (5.5.32)) which are first order, resp. zeroth order in $K^{(j)}$, as given in Lemma 5.5.8;

(ii) a rather simple estimate on the higher order remainder (represented by the contour integrals in (5.5.31), (5.5.32)) for which only a suitable choice of contours, the undetailed "nonperturbative" bound (5.5.13) and a sufficiently small $\|K^{(j)}\|_{G^{(j)},\Gamma,H}$ were needed.

6 The RG step. Part II: Fluctuation integral ($\mathcal{F}$)

6.1 Summary

For a relevant collection of notation/conventions, the reader is referred to Section 5.1. In order to avoid misunderstandings, however, one important remark has to be added: In contrast to Sections 2-5, in Sections 6 and 7 the momentum space UV cutoff function χ

(cf. Section 1.4) will enter the game. χ has been chosen once and for all and dependence of "constants" on χ will *not* be indicated explicitly (just as we did so far and continue to do with d, P, c, s, η).

Making use of the results of Section 5, that is of Proposition 5.5.5 and Theorem 5.5.9, the main goal of this chapter is to prove the

Theorem 6.1.1. Assume that L, A, H are large enough[16] and that $\kappa^{(j)} > 0$ is small enough. Let $K^{(j)}$ be an I-type polymer activity on $\Lambda^{(j)}$ whose norm $\|K^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H}$ is small enough, and let $\mathcal{O}_{\bar{\alpha}}^{(j)}$ be $\mathcal{O}$-type polymer activities on $\Lambda^{(j)}$ (pinned at $L^{-j}x_{\bar{\alpha}}$) with $\|\mathcal{O}_{\bar{\alpha}}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} < \infty$. Let j obey $0 \le j \le N-1$; and let, for $\ell = j, j+1$, $C^{(\ell)}$ be the covariance on $\mathcal{H}_s(\Lambda^{(\ell)})/\{\text{constants}\}$ whose Fourier transform, $C^{(\ell)}(p)$, is given by $C^{(\ell)}(p) = \chi(p^2)/(\sigma^{(\ell)} \cdot p^2)$, $p \neq 0$. Assume that $\sigma^{(\ell)} \ge \frac{1}{2}$ and that $|\sigma^{(j)} - \sigma^{(j+1)}|$ is small enough. Under these conditions we find:

(i) There are I-type, resp. $\mathcal{O}$-type, polymer activities $K^{(j+1)}$, resp. $\mathcal{O}_{\bar{\alpha}}^{(j+1)}$ on $\Lambda^{(j+1)}$ such that, if $Z_{\Lambda^{(j)};K^{(j)},C^{(j)}} \neq 0$,

$$\begin{aligned}\langle \mathcal{O}_1^{(j)}; \mathcal{O}_2^{(j)} | \mathcal{O}_{12}^{(j)} \rangle_{\Lambda^{(j)};K^{(j)},C^{(j)}} &= \langle \mathcal{O}_1^{(j+1)}; \mathcal{O}_2^{(j+1)} | \mathcal{O}_{12}^{(j+1)} \rangle_{\Lambda^{(j+1)};K^{(j+1)},C^{(j+1)}} \\ &\quad + \Omega_{12}^{(j)}(\Lambda^{(j)}),\end{aligned}$$

where $\Omega_{12}^{(j)}$ is a polymer activity on $\Lambda^{(j)}$ pinned at $L^{-j}x_1, L^{-j}x_2$; moreover, $Z_{\Lambda^{(j+1)};K^{(j+1)},C^{(j+1)}} \neq 0$.

(ii) Assume that $\epsilon \ge \mathcal{O}(H^{-2} \cdot (\|K^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H})^{\frac{1}{4}})$. Then, for any $\delta > 0$, if L is large enough, $L^{(j+1)}$ and $\mathcal{O}_{\bar{\alpha}}^{(j+1)}$ obey the bounds

$$\begin{aligned}\|K^{(j+1)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} &\le L^{-\frac{1}{2}+\delta} \cdot \|K^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \\ \|\mathcal{O}_{\alpha}^{(j+1)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} &\le L^{-d+\delta} \cdot \|\mathcal{O}_{\alpha}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \\ \|\mathcal{O}_{12}^{(j+1)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} &\le \left\{ \|\mathcal{O}_{12}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}_{\alpha}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \right\} \\ &\quad \cdot \begin{cases} L^{-2d-\frac{1}{4}}, & 0 \le j \le J-2 \\ L^{-2d+\delta}, & j = J-1 \\ L^{-d+\delta}, & J \le j \le N-1. \end{cases}\end{aligned}$$

The contents of Sections 6.1-6.5 are mostly independent of the previous sections, and a vague familiarity with Sections 2-5 is sufficient to understand Section 6. In Sections 6.1-6.5 we assume that $j < N$, and that $C^{(j)}, C^{(j+1)}$ are as in Theorem 6.1.1, i.e., for $\ell = j, j+1$,

$$C^{(\ell)}(x-y) := |\Lambda^{(\ell)}|^{-1} \sum_{\substack{p \in (\Lambda^{(\ell)})^* \\ p \neq 0}} e^{ip(x-y)} \frac{\chi(p^2)}{\sigma^{(\ell)} \cdot p^2}. \tag{6.1.1}$$

[16]Cf. Theorem 6.5.4 for more details.

6.2 Deforming the covariance

Definition 6.2.1. For given $\sigma^{(j)}$ and $\delta\sigma^{(j)}$, we set (cf. Corollary 5.5.6)

$$\sigma^{(j+1)} := \sigma^{(j)} + \delta\sigma^{(j)}. \tag{6.2.1}$$

For $\ell \in [1, L]$ we define the covariance $C_\ell^{(j)}$ on $\mathcal{H}_s(\Lambda^{(j+1)})/\{\text{constants}\}$ by

$$C_\ell^{(j)}(p) := \frac{\chi\left(p^2 \cdot \frac{\ell^2}{L^2}\right)}{p^2\sigma^{(j+1)} \cdot \left\{1 - \pi(\ell) \cdot \frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\left(1 - \chi\left(p^2 \cdot \frac{\ell^2}{L^2}\right)\right)\right\}}, \tag{6.2.2}$$

where π could be "any" function with $\pi(1) = 1$ and $\pi(L) = 0$; for the sake of simplicity we choose π to be linear, i.e., $\pi(\ell) := (L - \ell)/(L - 1)$.

Corollary 6.2.2. $C_1^{(j)} \equiv SC^{(j)}$, $C_L^{(j)} \equiv C^{(j+1)}$. If $\sigma^{(j)} > 0$ and $\sigma^{(j+1)} > 0$, then $C_\ell^{(j)} \geq 0$, $\forall \ell$.

Proof. The first two identities are obvious (cf. (6.1.1) and Corollary 5.5.6). To prove $C_\ell^{(j)} \geq 0$: Use $0 \leq \chi \leq 1$ (cf. Section 1.4); if $\delta\sigma^{(j)} \leq 0$, then $\delta\sigma^{(j)}/\sigma^{(j+1)} \leq 0$; if $\delta\sigma^{(j)} \geq 0$, then $\sigma^{(j+1)} \equiv \sigma^{(j)} + \delta\sigma^{(j)} > \delta\sigma^{(j)}$, thus $\left(\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right) < 1$. □

Definition 6.2.3. For $1 \leq \ell \leq \ell' \leq L$ we put

$$C_{\ell',\ell}^{(j)} := C_\ell^{(j)} - C_{\ell'}^{(j)}. \tag{6.2.3}$$

Corollary 6.2.4. If $L \geq 2$, $\sigma^{(j+1)} > 0$ and $\left|\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right| \leq \frac{1}{4}$, then

$$C_{\ell',\ell}^{(j)} \geq 0. \tag{6.2.4}$$

Proof. The idea is to show that $\frac{d}{d\ell}C_\ell^{(j)} \leq 0$ which implies that $C_{\ell',\ell}^{(j)} \equiv -\int_\ell^{\ell'} d\ell'' \frac{d}{d\ell''}C_{\ell''}^{(j)} \geq 0$. Since

$$\begin{aligned} \frac{d}{d\ell}C_\ell^{(j)}(p) &= \left(\sigma^{(j+1)}\left\{1 - \pi(\ell) \cdot \frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}(1 - \chi(x))\right\}^2\right)^{-1} \\ &\cdot \frac{\ell^2}{L^2} \cdot \left(\frac{2}{\ell} \cdot \chi(x)' \cdot \left\{1 - \pi(\ell)\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right\} - \frac{1}{L-1}\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}} \cdot \frac{\chi(x) \cdot (1 - \chi(x))}{x}\right)\Bigg|_{x = p^2\frac{\ell^2}{L^2}} \end{aligned}$$

the above hypotheses, and the condition that $\chi'(x) \leq -\frac{1}{x}\chi(x) \cdot (1 - \chi(x))$ for $x > 0$ (cf. Section 1.4), lead to the result we sought. □

Definition 6.2.5. The symbol $\mu^{(j)}_{\ell',\ell}$ is defined by

$$(\mu^{(j)}_{\ell',\ell} * F)(\Psi) := \int d\mu_{C^{(j)}_{\ell',\ell}}(\phi) F(\psi^\phi + \Psi). \tag{6.2.5}$$

Definition 6.2.6. For $\ell \in [1, L]$ we define the large field regulator $(G_\delta)_{\ell/L}$ by

$$(G_\delta)_{\ell/L}(X,\phi) := \exp\left(\delta\left\{\sum_{|\mu|=1}^{s}\left(\frac{\ell}{L}\right)^{2|\mu|-1)}\|\partial^\mu\phi\|^2_x + \frac{1}{c}\frac{\ell}{L}\|\partial\phi\|^2_{\partial X}\right\}\right). \tag{6.2.6}$$

This definition is consistent with the definition of $(G_\delta)_{L^{-1}}$ given in (5.5.14) (cf. (5.5.15).

Evidently, for $\ell = L$ we have

$$(G_\delta)_1 \equiv G_\delta. \tag{6.2.7}$$

Lemma 6.2.7. Assume that $\sigma^{(j+1)} \geq \frac{1}{2}$, $\left|\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right| \leq \frac{1}{4}$. Let $s \geq 2$, $c > 0$, $L \geq 2$. Then there are $C_{6.1}(L) \geq 0$, $\kappa_{\max}(L) > 0$ such that for all κ with $0 \leq \kappa \leq \kappa_{\max}(L)$ and all ℓ, ℓ' with $1 \leq \ell \leq \ell' \leq L$:

$$\begin{aligned}&(\mu^{(j)}_{\ell',\ell} * (G_\kappa)_{\ell/L})(X,\phi) \cdot \ell^{\kappa \cdot C_{6.1}(L)\cdot|X|}\\ &\quad\leq (G_\kappa)_{\ell'/L}(X,\phi)\cdot(\ell')^{\kappa\cdot C_{6.1}(L)\cdot|X|}, \ \forall X \subset \Lambda^{(j+1)}, \phi.\end{aligned} \tag{6.2.8}$$

Proof. (6.2.8) follows if we can prove that

$$\frac{d}{d\ell}\left((\mu^{(j)}_{\ell',\ell} * (G_\kappa)_{\ell/L})(X,\phi)\cdot \ell^{v(\kappa)\cdot|X|}\right) \geq 0 \tag{6.2.9}$$

for some suitable function $v(\kappa)$ (which, as we claim, can be chosen to be linear homogeneous in κ); and (6.2.9) in turn is true if

$$((G_\kappa)_{\ell/L}(X,\phi))^{-1}\cdot\left\{\frac{1}{2}\dot{\Delta}^{(j)}_{\ell',\ell} + \frac{\partial}{\partial\ell} + \frac{v(\kappa)\cdot|X|}{\ell}\right\}(G_\kappa)_{\ell/L}(X,\phi) \geq 0, \ \forall X, \phi, \tag{6.2.10}$$

where

$$\dot{\Delta}^{(j)}_{\ell',\ell} \equiv \int d^dx d^dy \frac{\partial}{\partial\ell} C^{(j)}_{\ell',\ell}(x-y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}.$$

The l.h.s. of (6.2.10) is a sum of ϕ-dependent and of ϕ-independent terms, and we want to achieve the inequality (6.2.10) for both types of terms separately.

(a) The ϕ-independent terms: They are given by the l.h.s. of the next inequality and can be bounded as

$$\frac{1}{2}\dot{\Delta}^{(j)}_{\ell',\ell}\log((G_\kappa)_{\ell/L})(X,\phi)+\frac{v(\kappa)|X|}{\ell}\geq -\kappa\cdot b_s(L)\cdot\left(|X|+\frac{|\partial X|}{c}\right)+\frac{v(\kappa)}{L}|X|,$$

where

$$b_s(L):=\sum_{1\leq|\mu|\leq s}\sup_{j,\ell}\left|(\partial^\mu_x)^2\frac{\partial}{\partial\ell}C^{(j)}_\ell(x=0)\right|;$$

using $|\partial X|\leq 2d|X|$ we thus can set $v(\kappa)=C_{6.1}(L)\cdot\kappa$ in order to arrive at (6.2.10).

(b) The ϕ-dependent terms: Those originating from $\frac{\partial}{\partial\ell}(G_\kappa)_{\ell/L}$ are of order κ and can be used to dominate, for small enough κ, the remaining ϕ-dependent terms (which are of order κ^2) if we employ Young's convolution inequality and if, in order to bound the terms of the form $\kappa^2\int_X d^dx\int_Y d^{d'}y\,\partial^\mu_x\phi(x)\partial^{\mu'}_y\phi(y)\partial^\mu_x\partial^{\mu'}_y\frac{\partial}{\partial\ell}C^{(j)}_\ell(x-y)$, we integrate by parts in x to produce boundary integrals. □

6.3 A general existence theorem

The result in this section, Lemma 6.3.2, on the existence of $\mathcal{F}$ will enable us to define $K^{(j+1)}$, $\mathcal{O}^{(j+1)}_{\bar{\alpha}}$ in Section 6.4, but it is very weak. Much better bounds will be obtained in Section 6.5 by estimating the flow equation obeyed by the activities $\mathcal{F}_{\ell'}\tilde{K}^{(j)}$, $1\leq\ell'\leq L$, as in [6]. Lemma 6.3.2 could also be proved using the flow equation.

Let $\tilde{K}_1$ be a regular polymer activity on Λ, analytic in $\mu,\lambda_{\bar{\alpha}}$ and obeying $\|\tilde{K}_1\|_{g_1,1,h}<\infty$ for $\mu,\lambda_{\bar{\alpha}}$ in a neighborhood U of zero. Let g_2 be a large field regulator and $\mu_{2,1}$ represent a Gaussian measure with mean 0 and covariance $C_{2,1}$ with

$$\mu_{2,1}*g_1\leq g_2 \tag{6.3.1}$$

Definition 6.3.1.

$$\|\Psi\|:=\max_{i:1\leq i\leq P}\max_{\mu_i}\|\Psi_{\mu_i}\| \tag{6.3.2}$$

where

$$\|\Psi_{\mu_i}\|:=\max_{x\in\Lambda}|\Psi_{\mu_i}(x)|. \tag{6.3.3}$$

Finally, we set

$$\mathrm{dist}(\Psi):=\inf_\phi\|\Psi-\Psi^\phi\|. \tag{6.3.4}$$

Lemma 6.3.2. There exists a unique polymer activity $\tilde{K}_2$, regular for all Ψ with $\mathrm{dist}(\Psi) < h$, such that

$$\mathcal{E}^{\square+\tilde{K}_2}(X,\Psi) = (\mu_{2,1} * \mathcal{E}^{\square+\tilde{K}_1})(X,\Psi) \tag{6.3.5}$$

for all $X \subset \Lambda$, and Ψ with $\mathrm{dist}(\Psi) < h$. Moreover:

(a) $\tilde{K}_2$ is analytic in $\mu, \lambda_{\bar{\alpha}}$ in a neighborhood U of zero.

(b) $\tilde{K}_1$ is local/even/real $\Rightarrow$ the same holds for $\tilde{K}_2$.

(c) $\tilde{K}_1$ and $\mu_{1,2}$ are invariant $\Rightarrow$ $\tilde{K}_2$ is invariant.

(d) $\tau_0\tilde{K}_1$ is I-type and $\mu_{1,2}$-invariant $\Rightarrow$ $\tau_0\tilde{K}_2$ is I-type.

(e) $\tau_{\bar{\alpha}}\tilde{K}_1$ are $\mathcal{O}$-type $\Rightarrow$ $\tau_{\bar{\alpha}}\tilde{K}_2$ are $\mathcal{O}$-type.

Sketch of proof. If we show that, for $\mathrm{dist}(\Psi) < h$, the integral $I(X,\Psi) := (\mu_{2,1} * \mathcal{E}^{\square+\tilde{K}_1})(X,\Psi)$ exists, is regular, and exhibits properties analogous to (a)-(e), then, using $I(\emptyset,\Psi) = 1$ and induction in $|X|$, the claim for $\tilde{K}_2$ follows.

Fix Ψ with $\mathrm{dist}(\Psi) < h$; we will show that $I(X,\Psi)$ exists. First, since $\mathrm{dist}(\Psi) < h$ there exists ϕ' such that $\|\Psi - \Psi^{\phi'}\| \leq h$. Write $\Psi = \Psi^{\phi'} + \Delta$ and perform the Taylor series expansion of $\mathcal{E}^{\square+\tilde{K}_1}(X, \Psi + \Psi^{\phi})$ in Δ around $\Delta = 0$. In this way, and applying the inequality $\|J_1 \circ J_2\|_{g,1,h} \leq |\Lambda|^2 \cdot \prod_{i=1}^{2} \|J_i\|_{g,1,h}$, we obtain

$$\sup_{\phi,\phi'} (g_1(X,\phi+\phi'))^{-1} |\mathcal{E}^{\square+\tilde{K}_1}(X,\Psi+\Psi^{\phi})| \leq |\Lambda| \cdot e^{|\Lambda|\cdot\|\tilde{K}_1\|_{g,1,h}}.$$

Therefore, applying (6.3.1) yields

$$\begin{aligned} |I(X,\Psi)| &\leq \int d\mu_{C_{2,1}}(\phi) g_1(X,\phi+\phi')(g_1(X,\phi+\phi'))^{-1} |\mathcal{E}^{\square+\tilde{K}_1}(X,\Psi+\Psi^{\phi})| \\ &\leq g_2(X,\phi') \cdot |\Lambda| \cdot e^{|\Lambda|\cdot\|\tilde{K}_1\|_{g,1,h}}. \end{aligned}$$

In a similar way one can prove regularity and analyticity in $\mu, \lambda_{\bar{\alpha}}$; and now the (analogues of the) properties (b)-(e) follow immediately. $\square$

6.4 Definition of $K^{(j+1)}, \mathcal{O}_{\bar{\alpha}}^{(j+1)}$

Definition 6.4.1. The regular, local, even polymer activity $\tilde{\mathcal{F}_1 K}^{(j)}$ on $\Lambda^{(j+1)}$ is defined by

$$\tilde{\mathcal{F}_1 K}^{(j)} := \mu \cdot SK^{(j)} + \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot S\mathcal{O}_{\bar{\alpha}}^{(j)}, \text{ for } \mu, \lambda_{\bar{\alpha}} \in \mathbf{C}. \tag{6.4.1}$$

As a consequence of Proposition 5.5.5, Theorem 5.5.9 and Corollaries 6.2.2, 6.2.4, and Lemmas 6.2.7 and 6.3.2, we find

Theorem 6.4.2. Let $0 \leq j \leq N-1$ and assume:

$$\begin{aligned} &\text{(a)} \quad \text{Hypotheses of Theorem 5.5.9,} \\ &\text{(b)} \quad \sigma^{(j+1)} \geq \frac{1}{2}, \quad \left|\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right| \leq \frac{1}{4}, && (6.4.2) \\ &\text{(c)} \quad \kappa^{(j)} + \epsilon \leq \kappa_{\max}(L). && (6.4.3) \end{aligned}$$

Then:

(i) For all ℓ' with $1 \leq \ell' \leq L$, there exists a unique regular, local, even polymer activity $\mathcal{F}_{\ell'}^{\sim} K^{(j)}$, regular for all Ψ with $\operatorname{dist}(\Psi) < 2H$, analytic in $\mu, \lambda_\alpha \in \mathbf{C}$, such that

$$\mathcal{E}^{\Box + \mathcal{F}_{\ell'}^{\sim} K^{(j)}} = \left(\mu_{\ell',1}^{(j)} * \mathcal{E}^{\Box + \mathcal{F}_1^{\sim} K^{(j)}}\right). \tag{6.4.4}$$

(ii) $K^{(j+1)}$ is I-type and $\mathcal{O}_{\bar\alpha}^{(j+1)}$ are $\mathcal{O}$-type (pinned at $L^{-(j+1)} x_{\bar\alpha}$), where

$$\begin{aligned} K^{(j+1)} &:= \tau_0 \mathcal{F}_L^{\sim} K^{(j)}|_{\mu=1} \\ \mathcal{O}_{\bar\alpha}^{(j+1)} &:= \tau_{\bar\alpha} \mathcal{F}_L^{\sim} K^{(j)}|_{\mu=1} \end{aligned} \tag{6.4.5}$$

(iii) Assume also $Z_{\Lambda^{(j)};K^{(j)},C^{(j)}} \neq 0$. Then $Z_{\Lambda^{(j+1)};K^{(j+1)},C^{(j+1)}} \neq 0$ (and vice versa) and

$$\begin{aligned} &\langle \mathcal{O}_1^{(j)}; \mathcal{O}_2^{(j)} | \mathcal{O}_{12}^{(j)} \rangle_{\Lambda^{(j)};K^{(j)},C^{(j)}} \\ &= \langle \mathcal{O}_1^{(j+1)}; \mathcal{O}_2^{(j+1)} | \mathcal{O}_{12}^{(j+1)} \rangle_{\Lambda^{(j+1)};K^{(j+1)},C^{(j+1)}} + \Omega_{12}^{(j)}(\Lambda^{(j)}). \end{aligned} \tag{6.4.6}$$

6.5 Bounds on $K^{(j+1)}$, $\mathcal{O}_{\bar\alpha}^{(j+1)}$

Recall the definition of the function $\theta_A : \mathbf{N}_0 \to \mathbf{R}_+$ (cf. Section 1.4), which is involved in the definition of the large set regulator Γ. It is easy to verify that for $n \in \mathbf{N}$,

$$\theta_A(n) \leq A^Q \cdot n^{Q \cdot \log_L(A)} \tag{6.5.1}$$

If $C : \Lambda \times \Lambda \to \mathbf{C}$ is sufficiently smooth we set

$$\|C\|_\gamma := \sup_{\Delta \subset \Lambda} \sum_{\Delta' \subset \Lambda} \gamma(\Delta \cup \Delta') \cdot C(\Delta, \Delta'), \tag{6.5.2}$$

where

$$C(\Delta, \Delta') := \sup_{\substack{1 \leq |\boldsymbol{\mu}^\#| \leq P \\ x^\# \in \Delta^\#}} |\partial_x^{\mu} \partial_{x'}^{\mu'} C(x, x')|. \tag{6.5.3}$$

Corollary 6.5.1. Let $\sigma^{(j+1)} \geq \frac{1}{2}$, $\left|\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right| \leq \frac{1}{4}$. There is $C_{6.2}(L,A,Q)$ such that

$$\int_1^L d\ell' \left\| \frac{\partial}{\partial \ell'} C^{(j)}_{\ell',1} \right\|_{\Gamma_{\eta-1}} \leq \frac{C_{6.2}(L,A,Q)}{16}. \tag{6.5.4}$$

Proof. Use (6.5.1). The details are left to the reader. □

Next, we note that if μ, λ_α are as in (5.3.1), and if $L^{\frac{1}{2}} \geq C_{5.10}$, then (cf. (6.4.1), (5.5.28)-(5.5.30)

$$\|\tilde{\mathcal{F}_1 K}^{(j)}\|_{(G_{\kappa^{(j)}+\epsilon})_{1/L},\Gamma_{\eta-1},2H} \leq UB^{(j)}, \tag{6.5.5}$$

because $\frac{UB^{(j)}}{5} \leq 1$ (cf. (5.2.4)). Therefore we get

Lemma 6.5.2. For μ, λ_α as in (5.3.1), and under the conditions

(a) hypotheses (a)-(c) of Theorem 6.4.2,

(b) $L \geq (C_{5.10})^2$,

(c) $H^2 \geq UB^{(j)} \cdot C_{6.2}(L,A,Q)$,

(d) $L^{C_{6.1}(L)\cdot(\kappa^{(j)}+\epsilon)} \leq \eta$,

we find that

$$\|\tilde{\mathcal{F}_L K}^{(j)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} \leq 1.$$

Proof. Use Theorem B of [6] or equivalently Theorem 7.1 of [3] in conjunction with Lemma 6.2.7, (6.2.7), (6.5.5), and (c) of Corollary 6.5.1, and the fact that $UB^{(j)} \leq 1$. □

Lemma 6.5.3. Expand

$$\tilde{\mathcal{F}_L K}^{(j)} = \mu \cdot (\tilde{\mathcal{F}_L K}^{(j)})_\mu + \sum_{\bar{\alpha}} \lambda_{\bar{\alpha}} \cdot (\tilde{\mathcal{F}_L K}^{(j)})_{\bar{\alpha}} + \mathcal{O}(\mu^2, \mu\lambda_\alpha, \lambda_\alpha^2).$$

If we impose:

(i) conditions (a) and (d) of Lemma 6.5.2,

(ii) $H^2 \geq L^{\frac{1}{2}} \cdot C_{6.2}(L, A, Q)$,

then

$$\|(\tilde{\mathcal{F}}_L K^{(j)})_\mu\|_{G_{\kappa(j)+\epsilon},\Gamma,H} \leq \|SK^{(j)}\|_{(G_{\kappa(j)+\epsilon})_{1/L},\Gamma_\eta-1,2H} \tag{6.5.6}$$

$$\|(\tilde{\mathcal{F}}_L K^{(j)})_\alpha\|_{G_{\kappa(j)+\epsilon},\Gamma,H} \leq \|S\mathcal{O}_\alpha^{(j)}\|_{(G_{\kappa(j)+\epsilon})_{1/L},\Gamma_\eta-1,2H} \tag{6.5.7}$$

$$\begin{aligned} \|(\tilde{\mathcal{F}}_L K^{(j)})_{12}\|_{G_{\kappa(j)+\epsilon},\Gamma,H} \leq\; & \|S\mathcal{O}_{12}^{(j)}\|_{(G_{\kappa(j)+\epsilon})_{1/L},\Gamma_\eta-1,2H} \\ & + L^{-\frac{1}{2}} \cdot \prod_\alpha \|S\mathcal{O}_\alpha^{(j)}\|_{(G_{\kappa(j)+\epsilon})_{1/L},\Gamma_\eta-1,2H}. \end{aligned} \tag{6.5.8}$$

Remark. The factor $L^{-\frac{1}{2}}$ in (6.5.8) is gained thanks to condition (ii). We chose this power of L in order to match it to the $\frac{1}{2}$ appearing in $L^{-2d-\frac{1}{2}}$ in (5.5.30).

Proof. (a) Due to (6.4.1), (6.4.4) we have $(\tilde{\mathcal{F}}_L K^{(j)})_\mu = (\mu_{L,1}^{(j)} * SK^{(j)})$. Therefore, making use of (6.2.8) and assumption (d), the proof of (6.5.6) runs as follows:

$$\begin{aligned} \| \; (\tilde{\mathcal{F}}_L K^{(j)})_\mu \; & \|_{G_{\kappa(j)+\epsilon},\Gamma,H} \\ \equiv \quad & \sum_{\mathbf{n}} H^{\mathbf{n}} \sup_\Delta \sum_{X \supset \Delta} \Gamma(X) \sup_\phi (G_{\kappa(j)+\epsilon}(X,\phi))^{-1} \\ & \cdot \|D(\mathbf{n})(\tilde{\mathcal{F}}_L K^{(j)})_\mu(X, \Psi^\phi) \cdot 1_{\boldsymbol{\Delta}}\| \\ \leq \quad & \sum_{\mathbf{n}} H^{\mathbf{n}} \sup_\Delta \sum_{X \supset \Delta} \Gamma(X) \sup_\phi (G_{\kappa(j)+\epsilon}(X,\phi))^{-1} \\ & \cdot \int d\mu_{L,1}^{(j)}(\phi')(G_{\kappa(j)+\epsilon})_{1/L}(X, \phi+\phi') \cdot \sup_{\phi+\phi'}((G_{\kappa(j)+\epsilon})_{1/L}(X,\phi+\phi')^{-1} \\ & \cdot \|D(\mathbf{n})SK^{(j)})(X, \Psi^\phi + \Psi^{\phi'}) \cdot 1_{\boldsymbol{\Delta}}\| \\ \leq \quad & \sum_{\mathbf{n}} H^{\mathbf{n}} \sup_\Delta \sum_{X \supset \Delta} \Gamma(X) \cdot \eta^{|X|} \cdot \|D(\mathbf{n})SK^{(j)}(X)\|_{(G_{\kappa(j)+\epsilon})_{1/L}} \\ = \quad & \|SK^{(j)}\|_{(G_{\kappa(j)+\epsilon})_{1/L},\Gamma_\eta-1,H}. \end{aligned} \tag{6.5.9}$$

(b) In precisely the same way one proves, for $1 \leq \ell' \leq L$, that $(\tilde{\mathcal{F}}_L K^{(j)})_\alpha = (\mu_{\ell',1}^{(j)} * S\mathcal{O}_\alpha^{(j)})$ and that, for $z \in \{0,1\}$,

$$\left(\frac{\partial}{\partial H}\right)^z \|(\tilde{\mathcal{F}}_{\ell'} K^{(j)})_\alpha\|_{(G_{\kappa(j)+\epsilon})_{\ell'/L},(\Gamma)_{\ell'/L},H} \leq \left(\frac{\partial}{\partial H}\right)^z \|S\mathcal{O}_\alpha^{(j)}\|_{(G_{\kappa(j)+\epsilon})_{1/L},\Gamma_\eta-1,H}, \tag{6.5.10}$$

where

$$(\Gamma)_{\ell'/L}(X) := \Gamma(X) \cdot \left(\frac{L}{\ell'}\right)^{(\kappa^{(j)}+\epsilon)\cdot C_{6.1}(L)\cdot|X|},$$

thereby establishing, among others, inequality (6.5.7).

(c) Proof of (6.5.8): As is evident from (6.4.1) and (6.4.4) we have $(\mathcal{F}_L\tilde{} K^{(j)})_{12} = (\mu_{L,1}^{(j)} * SO_{12}^{(j)}) + (\mu_{L,1}^{(j)} * (SO_1^{(j)} \circ SO_2^{(j)})) - (\mu_{L,1}^{(j)} * (SO_1^{(j)}) \circ (\mu_{L,1}^{(j)} * SO_2^{(j)})$. This formula can be rewritten as

$$(\mathcal{F}_L\tilde{} K^{(j)})_{12} = (\mu_{L,1}^{(j)} * SO_{12}^{(j)}) - \int_1^L d\ell \frac{\partial}{\partial \ell}\left\{\mu_{L,\ell}^{(j)} * ((\mu_{\ell,1}^{(j)} * SO_1^{(j)}) \circ (\mu_{\ell,1}^{(j)} * SO_2^{(j)}))\right\}. \quad (6.5.11)$$

The first term on the r.h.s. of (6.5.11) can be estimated as before. As regards the 2^{nd} term on the r.h.s. of (6.5.11): Writing $\dot{\Delta}_\ell^{(j)} := \int d^dx d^dx' \sum_{1\le|\mu^\#|\le P} \frac{\partial}{\partial \ell}\partial_x^{\mu}\partial_{x'}^{\mu'} C_\ell^{(j)}(x - x')\frac{\delta}{\delta\Psi_\mu(x)}\frac{\delta}{\delta\Psi_{\mu'}(x')}$ we have (cf. (6.2.3)) $\frac{\partial}{\partial\ell}(\mu_{\ell',\ell}^{(j)} * F)(X,\Psi) = \frac{1}{2}\dot{\Delta}_\ell^{(j)}(\mu_{\ell',\ell}^{(j)} * F)(X,\Psi)$ and $\frac{\partial}{\partial\ell}(\mu_{\ell,\ell''}^{(j)} * F)(X,\Psi) = -\dot{\Delta}_\ell^{(j)}(\mu_{\ell,\ell''}^{(j)} * F)(X,\Psi)$, hence

$$(2^{nd} \text{ term on r.h.s. of } (6.5.11)) = -\int_1^L d\ell\, \mu_{L,\ell}^{(j)} * [((\mathcal{F}_\ell\tilde{} K^{(j)})_1)_\Psi, \dot{C}_\ell^{(j)}((\mathcal{F}_\ell\tilde{} K^{(j)})_2)_\Psi], \quad (6.5.12)$$

where

$$[(J_1)_\Psi, C(J_2)_\Psi] := \int d^dx d^dx' \sum_{1\le|\mu^\#|\le P} \partial_x^{\mu}\partial_{x'}^{\mu'} C(x,x') \cdot \left(\frac{\partial}{\delta\Psi_\mu(x)}J_1\right) \circ \left(\frac{\partial}{\delta\Psi_{\mu'}(x')}J_2\right).$$

The r.h.s. of (6.5.12) can be bounded in much the same way as $(\mathcal{F}_L\tilde{} K^{(j)})_\mu$, using (6.5.2), (6.5.3), and (6.5.10). We obtain

$$\|\text{r.h.s. } (6.5.12)\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} \le \int_1^L d\ell \left\|\frac{\partial}{\partial\ell} C_{\ell,1}^{(j)}\right\|_\Gamma \cdot \prod_\alpha \frac{\partial}{\partial H}\|SO_\alpha^{(j)}\|_{(G_{\kappa^{(j)}+\epsilon})_{1/L},\Gamma_{\eta^{-1}},H}. \quad (6.5.13)$$

Next, we use (6.5.4) (which is guaranteed by condition (a)) and the fact that for any J

$$\frac{\partial}{\partial H}\|J\|_{\cdot,\cdot,H} = \frac{1}{2\pi i}\oint_{|h|=2H} dh \frac{1}{(h-H)^2}\|J\|_{\cdot,\cdot,h} \le \frac{2}{H}\cdot\|J\|_{\cdot,\cdot,2H},$$

so that (6.5.13) may be continued as

$$\le \frac{C_{6.2}(L,A,Q)}{4\cdot H^2}\prod_\alpha \|SO_\alpha^{(j)}\|_{(G_{\kappa^{(j)}+\epsilon})_{1/L},\Gamma_{\eta^{-1}},2H}.$$

Taking into account the condition (ii) we thus arrive at (6.5.8). □

In Lemma 6.5.2 we established, for μ, λ_α as in (5.3.1), a nondetailed upper bound on $\|\tilde{\mathcal{F}_L K^{(j)}}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H}$, and in the previous lemma we showed that there is an easy way to obtain useful estimates on the interesting leading order parts of $\tilde{\mathcal{F}_L K^{(j)}}$. We now combine these pieces of information with the analyticity $\tilde{\mathcal{F}_L K^{(j)}}$ in μ, λ_α (cf. Theorem 6.4.2) in order to bound $K^{(j+1)}$, $\mathcal{O}_{\vec{\alpha}}^{(j+1)}$. The method is exactly the same as the one applied in the proof of Theorem 5.5.9 and hence we omit all the details here. As a result one obtains

Theorem 6.5.4. Assume that $0 \le j \le N-1$, that the covariances $C^{(j)}$, $C^{(j+1)}$ obey $C^{(\ell)}(p) = \chi(p^2)/(\sigma^{(\ell)} \cdot p^2)$ for $\ell = j, j+1$ and $p \neq 0$, and that the following conditions hold:

(a) Hypotheses of Theorem 5.5.9

(b) $\sigma^{(j+1)} \ge \frac{1}{2}$, $\left|\frac{\delta\sigma^{(j)}}{\sigma^{(j+1)}}\right| \le \frac{1}{4}$

(c) $\kappa^{(j)} + \epsilon \le \kappa_{\max}(L)$

(d) $L \ge (C_{5.10})^2$

(e) $H^2 \ge L^{\frac{1}{2}} \cdot C_{6.2}(L, A, Q)$

(f) $L^{C_{6.1}(L)\cdot(\kappa^{(j)}+\epsilon)} \le \eta$.

Then the polymer activities $K^{(j+1)}$, $\mathcal{O}_{\vec{\alpha}}^{(j+1)}$ obey

$$\|K^{(j+1)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} \le C_{6.3} \cdot L^{-\frac{1}{2}} \|K^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \tag{6.5.14}$$

$$\|\mathcal{O}_{\alpha}^{(j+1)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} \le C_{6.3} \cdot L^{-d} \|\mathcal{O}_{\alpha}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \tag{6.5.15}$$

$$\|\mathcal{O}_{12}^{(j+1)}\|_{G_{\kappa^{(j)}+\epsilon},\Gamma,H} \le C_{6.3} \cdot \left\{ \|\mathcal{O}_{12}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}_{\alpha}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \right\} \cdot \begin{cases} L^{-2d-\frac{1}{2}}, & \text{if } 0 \le j \le J-2 \\ L^{-2d}, & \text{if } j = J-1 \\ L^{-d}, & \text{if } J \le j \le N-1. \end{cases} \tag{6.5.16}$$

7 Conclusions

In this final section we will evaluate the information gathered in Theorems 6.4.2 and 6.5.4. In Section 7.1 we will show that the so-far unspecified parameters $L, A, \ldots$, can indeed be fixed in a way which is consistent with the conditions mentioned in Theorem 6.5.4, and so that the RG transformation $(K^{(j)}, \mathcal{O}_{\vec{\alpha}}^{(j)}, C^{(j)}) \to (K^{(j+1)}, \mathcal{O}_{\vec{\alpha}}^{(j+1)}, C^{(j+1)})$ can be iterated. And in Section 7.2 we will prove our upper bound on the correlation function $\langle \mathcal{O}_1^{(0)}; \mathcal{O}_2^{(0)} | \mathcal{O}_{12}^{(0)} \rangle_{\Lambda^{(0)};K^{(0)},C^{(0)}}$.

7.1 Iterating the RG transformation

Recall, first of all, that the parameters d, s, c, P, η and the UV cutoff χ have been fixed (cf. Section 1.4). These being given, we now give one possible method to choose the parameters $L, A, Q, H, \kappa^{(j)}, \epsilon$. Note that, in what follows, all the choices made in item (n) depend at most on those made in $(1), (2), \ldots, (n-1)$ (and on $d, s, c, \ldots$, naturally).

(1)

$$\text{(a) Choose } \delta \text{ with } 0 < \delta < \frac{1}{2}. \tag{7.1.1}$$

$$\text{(b) Choose } Q \geq 1. \tag{7.1.2}$$

(2) Choose $L \geq L_{\min}$, where

$$L_{\min} := \max\left\{2^{d+1}, (C_{5.10})^2, 2^{\frac{4}{(\frac{1}{2}-\delta)}}, (C_{6.3})^{\frac{1}{\delta}}\right\}. \tag{7.1.3}$$

(3) (a) Choose

$$A \geq \max\{\eta^{2^{d+2}}, L^{d+\frac{1}{2}}\}. \tag{7.1.4}$$

(b) Choose $\kappa^{(0)}$ with $0 < \kappa^{(0)} \leq \kappa^{(0)}_{\max}$, where

$$\kappa^{(0)}_{\max} := \min\left\{\frac{1}{2}\kappa_{\min}(L), \log_L(\eta)/(2 \cdot C_{6.1}(L))\right) ; \tag{7.1.5}$$

and define

$$\kappa^{(j)} := \kappa^{(0} \cdot \sum_{i=0}^{j} 2^{-i}, \tag{7.1.6}$$

and

$$\epsilon := \epsilon^{(j)} \equiv \kappa^{(0} \cdot 2^{-j-1}. \tag{7.1.7}$$

(4) Choose $H \geq H_{\min}$, where

$$H_{\min} := \max\left\{\left(\frac{L^{d+2P-2}}{\kappa^{(0)}}\right)^{\frac{1}{2}}, (L^{\frac{1}{2}} \cdot C_{6.2}(L, A, Q))^{\frac{1}{2}}\right\}. \tag{7.1.8}$$

Recall that, given $x_1, x_2 \in \mathbf{R}^d$ with $|x_1 - x_2| \geq 1$ and given L, the parameter $J \in \mathbf{N}_0$ was defined by $L^J \leq |x_1 - x_2| < L^{J+1}$. Furthermore, given L, the initial torus $\Lambda^{(0)}$ is determined by $N \in \mathbf{N}$ (as $\Lambda^{(0)} = \left[-\frac{L^N}{2}, \frac{L^N}{2}\right]^d$).

Definition 7.1.1. For $\sigma \in \mathbf{R}$ we set

$$\rho(L, H, \kappa^{(0)}, \sigma) := \min\left\{(C_{5.0}(L))^4, \left(\min\left\{1, \frac{\kappa^{(0)} H^2}{2 \cdot \tau(S)}\right\} / C_{5.9}(L)\right)^4, \right.$$
$$\left. \frac{H^2 \cdot (\sigma - \frac{1}{2})}{2 \cdot C_{5.4}}, \frac{H^2}{8 \cdot C_{5.4}}\right\}. \tag{7.1.9}$$

Clearly, $\rho(L, H, \kappa^{(0)}, \sigma) > 0$ iff $\sigma > \frac{1}{2}$.

Theorem 7.1.2. Choose the parameters $L, A, Q, H, \kappa^{(j)}, \epsilon$ as in (1)-(4) above. Fix x_1, x_2 and let $N > (J+1)$. Let $C^{(0)}$ be the χ-regularized inverse Laplacian on $\Lambda^{(0)}$ with dielectric constant $\sigma^{(0)} \geq \frac{1}{2}$.[17] Let $K^{(0)}$ be an I-type polymer activity on $\Lambda^{(0)}$ obeying

$$\|K^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \leq \rho(L, H, \kappa^{(0)}, \sigma^{(0)}),$$

and let $\mathcal{O}_{\bar{\alpha}}^{(0)}$ be $\mathcal{O}$-type polymer activities on $\Lambda^{(0)}$ with

$$\|\mathcal{O}_{\bar{\alpha}}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} < \infty.$$

Then we can iterate the RG transformation $(K^{(0)}, \mathcal{O}_{\bar{\alpha}}^{(0)} C^{(0)}) \to (K^{(1)}, \mathcal{O}_{\bar{\alpha}}^{(1)}, C^{(1)}) \to \cdots (N-1)$-times. For $0 \leq j \leq N$, the χ-regularized inverse Laplacian on $\Lambda^{(j)}, C^{(j)}$, has dielectric constant

$$\sigma^{(j)} \equiv \sigma^{(0)} + \sum_{i=0}^{j-1} \delta\sigma^{(i)} \geq \frac{1}{2},$$

where $\delta\sigma^{(i)}$ has been defined in (5.4.6), and $K^{(j)}, \mathcal{O}_{\bar{\alpha}}^{(j)}$ satisfy the bounds (recall that δ obeys $0 < \delta < \frac{1}{2}$ (cf. (7.1.1))

$$\|K^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \leq L^{-j(\frac{1}{2}-\delta)} \|K^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \tag{7.1.10}$$

$$\|\mathcal{O}_{\alpha}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \leq L^{-j(d-\delta)} \|\mathcal{O}_{\alpha}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \tag{7.1.11}$$

and

$$\|\mathcal{O}_{12}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \leq L^{-2(d-\delta)j} \left\{ L^{-(\frac{1}{2}+\delta)j} \|\mathcal{O}_{12}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}_{\alpha}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \right\}, \text{ if } 0 \leq j \leq J-1, \tag{7.1.12}$$

$$\|\mathcal{O}_{12}^{(j)}\|_{G_{\kappa^{(j)}},\Gamma,H} \leq L^{-2(d-\delta)J-(d-\delta)(j-J)} \cdot C_{7.1} \cdot \left\{ L^{-(\frac{1}{2}+\delta)(J-1)} \|\mathcal{O}_{12}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}_{\alpha}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \right\}, \text{ if } J \leq j \leq N. \tag{7.1.13}$$

Proof. The proof is carried out by induction in j, using Theorem 6.5.4 and the bound (5.4.9). It is very easy to work out the details and hence we leave this task to the reader (in particular, one sees that $C_{7.1} \leq 4$). □

[17] We need $\sigma^{(0)} \geq \frac{1}{2}$ because of condition (b) in Theorem 6.5.4. Actually, any strictly positive lower bound would have been acceptable as well (e.g., $\sigma^{(0)} \geq 10^{-20}$), but for the sake of simplicity we chose $\frac{1}{2}$ throughout Section 6.

7.2 Bounding the correlation function

Remember the definition (5.5.5) of the polymer activity $\Omega_{12}^{(j)}$.

Lemma 7.2.1. Hypotheses:

(a) Conditions as in Theorem 7.1.2, but we strengthen (7.1.4) by

(b) $A \geq \max\left\{\eta^{2^{d+2}}, L^{2d+\frac{1}{2}}\right\}$.

Then we find, for $0 \leq j \leq N$:

$$\begin{aligned} |\Omega_{12}^{(j)}(\Lambda^{(j)})| \;\leq\; & |x_1 - x_2|^{-2d} \cdot C_{7.2}(L) \cdot \Big\{ |x_1 - x_2|^{-\frac{1}{2}} \cdot \|\mathcal{O}_{12}^{(0)}\|_{G_{\kappa(0)},\Gamma,H} \\ & + \prod_{\alpha=1}^{2} \|\mathcal{O}_{\alpha}^{(0)}\|_{G_{\kappa(0)},\Gamma,H} \Big\} \cdot \begin{cases} L^{2\delta j}, & 0 \leq j \leq J-1 \\ L^{2\delta J-(d-\delta)(j-J)}, & J \leq j \leq N. \end{cases} \end{aligned} \qquad (7.2.1)$$

Proof. (1) According to (5.5.5) and (5.3.2):

$$\Omega_{12}^{(j)}(\Lambda^{(j)}) \equiv \tau_{12}\Omega_I[\tilde{K}^{(j)}](\Lambda^{(j)})|_{\mu=1} = \sum_{X\subset\Lambda^{(j)}} \tau_{12}\omega_I[\tilde{K}^{(j)}](X)|_{\mu=1}$$

is pinned at $x_1^{(j)}$ and $x_2^{(j)}$, so the sum $\sum_{X\subset\Lambda^{(j)}}$ really extends only over those X which contain both $x_1^{(j)}$ and $x_2^{(j)}$. Let $z_\Gamma := \min_{X:\mathring{X}\ni x_1^{(j)},x_2^{(j)}} \Gamma_\eta(X)$, and assume that z obeys $1 \leq z \leq z_\Gamma$. Then

$$\begin{aligned} |\Omega_{12}^{(j)}(\Lambda^{(j)})| \;&\leq\; \sum_{\substack{X\subset\Lambda^{(j)} \\ \mathring{X}\ni x_1^{(j)},x_2^{(j)}}} |\tau_{12}\omega_I[\tilde{K}^{(j)}](X)| \text{ at } \mu = 1 \\ &\leq\; z^{-1} \sum_{\substack{X\subset\Lambda^{(j)} \\ \mathring{X}\ni x_1^{(j)},x_2^{(j)}}} z_\Gamma \cdot |\tau_{12}\omega_I[\tilde{K}^{(j)}](X)| \text{ at } \mu = 1 \\ &\leq\; z^{-1}\|\tau_{12}\omega_I[\tilde{K}^{(j)}]\|_{g,\Gamma_\eta,\mathbf{h}} \text{ at } \mu = 1, \text{ for all } g \geq 1,\ \mathbf{h} \geq 0. \end{aligned} \qquad (7.2.2)$$

(2) If $j \geq J$ we put $z := 1$. Otherwise, if $j < J$, we set $z := z_\Gamma$; since $A_{\Gamma_\eta} \geq e^{2d+\frac{1}{2}}$ (and therefore $(A_{\Gamma_\eta})^{|X|} \geq |X|^{2d+\frac{1}{2}}$), and since $\theta_A(n) \geq A^{\{\log_L(n)\}}$ for $n \in \mathbf{N}$ (cf. Section 1.4), with $A \geq L^{2d+\frac{1}{2}}$ (implying that $\theta_A(n) \geq n^{2d+\frac{1}{2}}$), one concludes (exercise!) that $z_\Gamma \geq |x_1^{(j)} - x_2^{(j)}|^{2d+\frac{1}{2}}$.

(3) For μ, λ_α as in (5.3.1), $\omega_I[\tilde{K}^{(j)}]$ is analytic in μ, λ_α (cf. Lemma 5.3.3); choosing the contours C_{δ_0}, C_{δ_α} as in part (3) of the proof of Theorem 5.5.9 we have

$$\begin{aligned} \tau_{12}\omega_I[\tilde{K}^{(j)}]|_{\mu=1} &= \tau_{12}\omega_I[\tilde{K}^{(j)}]_{\mu=0} \\ &+ (2\pi i)^{-3} \oint_{C_{\delta_1}} \frac{d\lambda_1}{(\lambda_1)^2} \oint_{C_{\delta_2}} \frac{d\lambda_2}{(\lambda_2)^2} \oint_{C_{\delta_0}} \frac{d\mu}{\mu^2} \left(\frac{\mu}{\mu-1}\right) \omega_I[\tilde{K}^{(j)}]. \end{aligned} \tag{7.2.3}$$

Applying (5.3.4) to the first term on the r.h.s. of (7.2.3) we get

$$\|\tau_{12}\omega_I[\tilde{K}^{(j)}]|_{\mu=0}\|_{g,\Gamma_\eta,\mathrm{h}} \leq \|\mathcal{O}_{12}^{(j)}\|_{G^{(j)},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}_\alpha^{(j)}\|_{G^{(j)},\Gamma,H}.$$

The contour integral is estimated using (cf. (5.3.5), (5.3.1)) $\|\omega_I[\tilde{K}^{(j)}]\|_{g,\Gamma_\eta,\mathrm{h}} \leq C_{5.1}$ and (cf. (5.5.25)) $\|K^{(j)}\|_{G^{(j)},\Gamma,H} \leq 1$.

(4) Finally, we combine the results of (1)-(3) with (7.1.11), (7.1.12, 7.1.13) and with $L^{-J} \leq L \cdot |x_1 - x_2|^{-1}$ to arrive at (7.2.1). □

Lemma 7.2.2. Let C_σ be the covariance on $\mathcal{H}_s(\Lambda^{(N)})/\{\text{constants}\}$ whose Fourier transform is $C_\sigma(p) = \chi(p^2)/(\sigma \cdot p^2)$, $p \neq 0$, and $\sigma \geq \sigma' > 0$. Let $s \geq 1$, $c > 0$. Then there are $C_{7.3}(\sigma') \geq 0$ and $\kappa'_{\max}(\sigma') > 0$ such that for all κ with $0 \leq \kappa \leq \kappa'_{\max}(\sigma')$, all t, t' with $0 \leq t \leq t' \leq 1$ and all $\sigma \geq \sigma'$:

$$\left(\mu_{(t'-t)C_\sigma} * G_{\kappa\cdot e^t}\right)(\Delta, \phi) \cdot e^{t\kappa \cdot C_{7.3}(\sigma')} \leq G_{\kappa\cdot e^{t'}}(\Delta, \phi) \cdot e^{t'\kappa\cdot C_{7.3}(\sigma')}, \ \forall\, \phi. \tag{7.2.4}$$

Proof. Analogous to the one of Lemma 6.2.7. □

Definition 7.2.3.

$$\kappa'_{\max} := \kappa'_{\max}\left(\sigma' = \frac{1}{2}\right), \tag{7.2.5}$$

$$C_{7.3} := C_{7.3}\left(\sigma' = \frac{1}{2}\right). \tag{7.2.6}$$

Theorem 7.2.4. Under the conditions listed in Theorem 7.1.2, but restricting $A, \kappa^{(0)}$, $\|K^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H}$ even further by replacing

(a) (7.1.4) by $A \geq \max\left\{\eta^{2^{d+2}}, L^{2d+\frac{1}{2}}\right\}$, (7.2.7)

(b) the r.h.s. of (7.1.5) by $\min\{\text{r.h.s.}(7.1.5), \frac{1}{2}\kappa'_{\max}\}$, (7.2.8)

(c) the r.h.s. of (7.1.9) by $\min\{\text{r.h.s.}(7.1.9), \frac{1}{2}e^{-2\kappa^{(0)}\cdot C_{7.3}}\}$, (7.2.9) we find that $Z_{\Lambda^{(0)};K^{(0)},C^{(0)}} \neq 0$ and that

$$\left|\langle \mathcal{O}_1^{(0)}; \mathcal{O}_2^{(0)} | \mathcal{O}_{12}^{(0)} \rangle_{\Lambda^{(0)};K^{(0)},C^{(0)}}\right| \leq \|x_1 - x_2\|^{-2(d-\delta)} \cdot C_{7.4}(L,\delta)$$
$$\left\{ \|x_1 - x_2\|^{-\frac{1}{2}} \cdot \|\mathcal{O}_{12}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} + \prod_{\alpha=1}^{2} \|\mathcal{O}_\alpha^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \right\}. \qquad (7.2.10)$$

Remark. Instead of $\|x_1 - x_2\|^{-\frac{1}{2}}$ we could achieve $\|x_1 - x_2\|^{-r}$ for any fixed $r \geq 0$ by choosing A, H sufficently large and $\|K^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H}$ sufficiently small.

Proof. (1) By Theorem 7.1.2 we have $\sigma^{(N)} \geq \frac{1}{2}$ and, by (7.2.8) and (7.1.6), $\kappa^{(N)} \leq \kappa'_{\max}$; hence we may apply Lemma 7.2.2 to see that $\int d\mu_{C^{(N)}}(\phi) G_{\kappa^{(N)}}(\Delta,\phi) \leq e^{\kappa^{(N)} \cdot C_{7.3}} \leq e^{2\kappa^{(0)} \cdot C_{7.3}}$. As a consequence, using (7.2.9) and (7.1.10), we get

$$\begin{aligned} Z_{\Lambda^{(N)};K^{(N)},C^{(N)}} &= 1 + \int d\mu_{C^{(N)}}(\phi) K^{(N)}(\Delta, \Psi^\phi) \\ &\geq 1 - \int d\mu_{C^{(N)}}(\phi) G_{\kappa^{(N)}}(\Delta,\phi) \cdot \|K^{(N)}\|_{G_{\kappa^{(N)}},\Gamma,H} \\ &\geq \frac{1}{2}. \end{aligned} \qquad (7.2.11)$$

In particular, we see that $Z_{\Lambda^{(N)};K^{(N)},C^{(N)}} \neq 0$, and therefore, by induction in j starting at $j = N$, $Z_{\Lambda^{(j)};K^{(j)},C^{(j)}} \neq 0$ for all $0 \leq j \leq N$.

(2) Abbreviate $\langle\cdot\rangle_{\Lambda^{(N)};K^{(N)},C^{(N)}}$ by $\langle\cdot\rangle_N$. Using (5.1.8) (and the fact that $\Lambda^{(N)}$ is a single block) and (7.2.11), Lemma 7.2.2, (7.1.5) (telling us that $2\kappa^{(0)} \leq \kappa_{\max}(L)$) and (7.1.11), (7.1.12, 7.1.13), $L^{-J} \leq |x_1 - x_2|^{-1} \cdot L$, we obtain

$$\begin{aligned} &|\langle \mathcal{O}_1^{(N)};\ \mathcal{O}_1^{(N)}\ | \mathcal{O}_{12}^{(N)} \rangle_N| \\ &= \left| Z_N^{-2} \left(Z_N \int d\mu_{C^{(N)}}(\phi) \mathcal{O}_{12}^{(N)}(\Delta,\Psi^\phi) - \prod_{\alpha=1}^{2} \int d\mu_{C^{(N)}}(\phi) \mathcal{O}_\alpha^{(N)}(\Delta,\Psi^\phi) \right) \right| \\ &\leq 4 \cdot e^{4\kappa^{(0)} \cdot C_{7.3}} \left\{ \|\mathcal{O}_{12}^{(N)}\|_{G_{\kappa^{(N)}},\Gamma,H} + \prod_\alpha \|\mathcal{O}_\alpha^{(N)}\|_{G_{\kappa^{(N)}},\Gamma,H} \right\} \\ &\leq C'_{7.4}(L) \cdot |x_1 - x_2|^{-2(d-\delta)} \cdot L^{-(N-J)(d-\delta)} \\ &\quad \cdot \left\{ |x_1 - x_2|^{-\frac{1}{2}} \cdot \|\mathcal{O}_{12}^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} + \prod_\alpha \|\mathcal{O}_\alpha^{(0)}\|_{G_{\kappa^{(0)}},\Gamma,H} \right\}. \end{aligned} \qquad (7.2.12)$$

(3) By our hypotheses, the conditions of Theorem 6.4.2 are fulfilled, and $Z_j \neq 0$ (according to (1) above). Hence we may apply (6.4.6), (7.2.1), and (7.2.12) to get

$$|\langle \mathcal{O}_1^{(0)}; \mathcal{O}_2^{(0)} | \mathcal{O}_{12}^{(0)} \rangle_0| \leq \sum_{j=0}^{N-1} |\Omega_{12}^{(j)}(\Lambda^{(j)})| + |\langle \mathcal{O}_1^{(N)}; \mathcal{O}_2^{(N)} | \mathcal{O}_{12}^{(N)} \rangle_N|$$

$$\leq |x_1 - x_2|^{-2(d-\delta)} \cdot C''_{7.4}(L) \tag{7.2.13}$$
$$\cdot \left\{ \|\mathcal{O}^{(0)}_{12}\|_{G_{\kappa(0)},\Gamma,H} \cdot |x_1 - x_2|^{-\frac{1}{2}} + \prod_{\alpha=1}^{2} \|\mathcal{O}^{(0)}_{\alpha}\|_{G_{\kappa(0)},\Gamma,H} \right\}$$
$$\cdot \left[\sum_{j=0}^{J-1} |x_1 - x_2|^{-2\delta} L^{2\delta j} + \sum_{j=J}^{N-1} |x_1 - x_2|^{-2\delta} L^{2\delta J-(d-\delta)(j-J)} + L^{-(d-\delta)(N-J)} \right],$$

which, upon taking into account $|x_1 - x_2|^{-1} \leq L^{-J}$ and $|x_1 - x_2|^{-1} \leq \|x_1 - x_2\|^{-1} \cdot \sqrt{d}$ (cf. Section 1.4), immediately yields (7.2.10). □

References

[1] J. Bricmont, J. R. Fontaine, J. Lebowitz, and T. Spencer. Lattice systems with a continuous symmetry I: Perturbation theory for unbounded spins. *Commun. Math. Phys.*, 78:281–302, 1980/81.

[2] J. Bricmont, J. R. Fontaine, J. Lebowitz, and T. Spencer. Lattice systems with a continuous symmetry II: Decay of correlations. *Commun. Math. Phys.*, 78:363–371, 1980/81.

[3] D. C. Brydges. Functional integrals and their applications. Lecture Notes taken by Roberto Fernandez. Lectures Notes for a course sponsored by the "Troisième Cycle de la Physique en Suisse Romande", May 1992. Available from: Archives and Vente de Polycopies: M.D. Reymond, Université de Lausanne, Batiment des Sciences Physiques, 1015 Lausanne-Dorigny, or as a University of Virginia Preprint, or the mathematical physics bulletin board at the University of Texas at Austin.

[4] D. C. Brydges. A short course on cluster expansions. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories*, pages 131–183, Amsterdam, New York, 1984. Les Houches Summer School, North Holland.

[5] D. C. Brydges and G. Keller. Absence of Debye screening in the quantum coulomb system. To appear in J. Stat. Phys. Summer 1994.

[6] D. C. Brydges and H.T. Yau. Grad ϕ perturbations of massless Gaussian fields. *Commun. Math. Phys.*, 129:351–392, 1990.

[7] J. Dimock and T. Hurd. Construction of the two-dimensional sine-Gordon model for $\beta < 8\pi$. *Commun. Math. Phys.*, 1993. This should have appeared in CMP by now.

[8] J. Dimock and T. R. Hurd. A renormalisation group analysis of the Kosterlitz-Thouless phase. *Commun. Math. Phys.*, 137:263–287, 1991.

[9] J. Dimock and T. R. Hurd. A renormalisation group analysis of correlation functions for the dipole gas. *J. Stat. Phys.*, 66:1277–1318, 1992.

[10] J. Dimock and T. R. Hurd. A renormalization group analysis of QED. *J. Math. Phys.*, 33:814–821, 1992.

[11] J. Fröhlich and T. Spencer. On the statistical mechanics of classical Coulomb and dipole gases. *J. Stat. Phys*, 24:617–701, 1981.

[12] K. Gawedzki and A. Kupiainen. Lattice dipole gas and $(\nabla\phi)^4$ models at long distances: decay of correlations and scaling limit. *Commun. Math. Phys.*, 92:531–553, 1984.

[13] M. Kac. On the partition function of one dimensional gas. *Phys. Fluids*, 2:8–12, 1959.

[14] J. Magnen and R. Seneor. The infrared behavior of $(\nabla\phi)^4_3$. *Ann. Physics*, 152:130–202, 1984.

[15] M. O'Carroll. Wavelets and orthogonality between scales in a representation for correlation functions. the lattice dipole gas and $(\nabla\phi)^4$ models. To appear in J. Stat. Phys., 1993.

[16] Y. M. Park. Lack of screening in the continuous dipole system. *Commun. Math. Phys.*, 70:161–167, 1979.

[17] A.J.F. Siegert. Partition functions as averages of functionals of Gaussian random functions. *Physica*, 26:530–535, 1960.