Symmetries of the Chern-Simons Theory in the Axial Gauge, Manifold with Boundary[1]

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Abstract. The field equations of the Chern-Simons theory quantized in the axial gauge are shown to be completely determined by supersymmetry Ward identities which express the invariance of the theory under the topological supersymmetry of Delduc, Gieres and Sorella together with the usual Slavnov identity without requiring any action principle.

1 Introduction

In a previous paper [1], we show that all the Green functions of the Chern-Simon theory in three dimensions quantized in the axial gauge can be completly and uniquely determined by considering the usual BRS symmetry together with the topological supersymmetry of Delduc, Gieres and Sorella [2] without having to invoque any action principle. The construction goes as follow. The choice of a linear gauge condition allows us to find the ghost equation by commuting this gauge condition with the Slavnov identity. For the antighost equation, we show that it is equivalent to the component of the supersymetry Ward identity along the direction of the gauge defining vector. These two equations couple only to the source of the Lagrange multiplier field, thus we can find a recursion relation for the Green functions which involves only ghost, antighost and Lagrange multiplier fields. For the gauge field we have no such starting point. Nevertheless, using the transverse components of the supersymmetry Ward identity, it is possible to connect all the Green

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functions involving gauge and Lagrange multiplier field with the one's of the previous set of fields.

In fact, there exists a deeper relation between this symmetry-based approach and the content of an action principle defining the theory. Here we will show that the gauge field equation is nothing but the consistency condition between the transverse component of the supersymetry Ward identity and the ghost field equation. Having this equation, one is able to find the recursion relation which solves the gauge and Lagrange multiplier field sector independently of the ghost-antighost sector. So in a certain sense, it put the two sectors at the same level.

The paper is organized as follow. In spite of the fact that we want to show that we can find the field equations without refering to any action principle, we will begin with a short review of the 3-D Chern-Simons theory quantized in the axial gauge in order to fix the convention and notation. Section 3 is devoted to the study of the symmetry of this theory and we will show, in section 4, how the field equations arise from these symmetries. Finally we will discuss in section 5 the case with boundary.

2   Chern-Simons theory in the axial gauge

The action of the Chern-Simons model in the axial gauge reads[1]

$$
\begin{aligned}
\Sigma_{\text{CS}} = \; & -\tfrac{1}{2}\int d^3x\, \epsilon^{\mu\nu\rho}\text{Tr}\left(A_\mu\partial_\nu A_\rho + \tfrac{2}{3}g A_\mu A_\nu A_\rho\right) \\
& + \int d^3x\,\text{Tr}\left(d n^\mu A_\mu + b n^\mu D_\mu c\right),
\end{aligned}
\tag{2.1}
$$

with $D_\mu\cdot = \partial_\mu\cdot + g[A_\mu,\cdot]$ for the covariant derivative. The gauge group is chosen to be simple, all fields belong to the adjoint representation and are written as Lie algebra matrices $\varphi(x) = \varphi^a(x)\tau_a$, with

$$
[\tau_a,\tau_b] = f^c_{ab}\tau_c, \qquad \text{Tr}\left(\tau_a\tau_b\right) = \delta_{ab}.
$$

The canonical dimensions and ghost numbers of the fields are given in Table 1.

	A	d	b	c
Dimension	1	2	2	0
Ghost number	0	0	-1	1

Table 1: Dimensions and ghost numbers.

The axial gauge is defined by the following *gauge condition*

$$
n^\mu\frac{\delta Z_c}{\delta J^\mu} + J_d = 0.
\tag{2.2}
$$

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[1]**Conventions:** $\mu,\nu,\cdots = 1,2,3$, $g_{\mu\nu} = \text{diag}(1,-1,-1)$, $\epsilon^{\mu\nu\rho} = \varepsilon_{\mu\nu\rho} = \varepsilon^{[\mu\nu\rho]}$, $\varepsilon_{123} = 1$.

Without loss of generality we can choose the vector n defining the axial gauge as

$$(n^\mu) = (0, 0, 1). \qquad (2.3)$$

The coordinates transverse to n will be denoted by

$$x^{\mathrm{tr}} = (x^i, \ i = 1, 2). \qquad (2.4)$$

3 Symmetries and Ward identities

The action (2.1) is invariant [1] under the BRS transformations s

$$
\begin{aligned}
sA_\mu &= -D_\mu c, \\
sb &= d, \\
sc &= gc^2, \\
sd &= 0,
\end{aligned}
\qquad (3.1)
$$

as well as under the vector supersymetry ν_ρ given by

$$
\begin{aligned}
\nu_\rho A_\mu &= \epsilon_{\rho\mu\nu} n^\nu b, \\
\nu_\rho b &= 0, \\
\nu_\rho c &= -A_\rho, \\
\nu_\rho d &= \partial_\rho b.
\end{aligned}
\qquad (3.2)
$$

The BRS invariance of the theory can be expressed, formally, by the functional identity

$$\mathrm{Tr} \int d^3x \left(-J^\mu [D_\mu c] \cdot Z_{\mathrm{c}} - gJ_c[c^2] \cdot Z_{\mathrm{c}} - J_b \frac{\delta Z_{\mathrm{c}}}{\delta J_d} \right) = 0. \qquad (3.3)$$

Here $Z_{\mathrm{c}}(J^\mu, J_b, J_c, J_d)$ is the generating functional of the connected Green functions, J^μ, J_d, J_b and J_c denoting the sources of the fields A_μ, d, b and c, respectively. We have used the notation

$$[O] \cdot Z_{\mathrm{c}}(J^\mu, J_b, J_c, J_d)$$

for the generating functional of the connected Green functions with the insertion of the local field polynomial operator O. Usually, such insertions must be renormalized, their renormalization is controlled by coupling them to external fields and the identity (3.3) becomes the Slavnov identity [4]. We shall however see below that, in the axial gauge, these insertions are trivial and thus the Slavnov identity is replaced by a local gauge Ward identity.

The invariance under the supersymmetry transformations ν_μ (3.2) leads to the *supersymmetry Ward identities*

$$\text{Tr} \int d^3x \left(J^\mu \varepsilon_{\rho\mu\nu} n^\nu \frac{\delta}{\delta J_b} + J_c \frac{\delta}{\delta J^\rho} + J_d \partial_\rho \frac{\delta}{\delta J_b} \right) Z_c = 0. \tag{3.4}$$

4 Field equations recovered

Now our goal is to recover the field equations using only the functional identities (3.3), (3.4) together with the gauge condition (2.2).

4.1 Ghost-antighost sector

It is well known that the choice of a linear gauge condition leads to an equation for the ghost field. In the axial gauge, it takes the form of the *ghost equation*

$$- J_b + \left(n^\mu \partial_\mu \frac{\delta}{\delta J_c} - g \left[J_d, \frac{\delta}{\delta J_c} \right] \right) Z_c = 0. \tag{4.1}$$

Furthermore, we have already see [1] that the projection of the supersymmetry Ward identity along the gauge vector n:

$$\text{Tr} \int d^3x J_d \left(-J_c + n^\mu \partial_\mu \frac{\delta Z_c}{\delta J_b} \right) = 0.$$

leads to the local *antighost equation*

$$- J_c + \left(n^\mu \partial_\mu \frac{\delta}{\delta J_b} - g \left[J_d, \frac{\delta}{\delta J_b} \right] \right) Z_c = 0. \tag{4.2}$$

The ghost equation (4.1) and the antighost equation (4.2) express the "freedom" of the ghosts in the axial gauge [5]: they couple only to the n-component of the gauge field, *i.e.*, to the external source J_d. The effect of (4.1) is to factorize out the contributions of the ghost field c to the composite fields appearing in the BRS Ward identity (3.3). We can thus replace the latter by the local *gauge Ward identity*:

$$\begin{aligned} -\partial_\mu J^\mu + \bigg(g \left[J^\mu, \frac{\delta}{\delta J^\mu} \right] + g \left[J_d, \frac{\delta}{\delta J_d} \right] + g \left\{ J_b, \frac{\delta}{\delta J_b} \right\} \\ + g \left\{ J_c, \frac{\delta}{\delta J_c} \right\} - n^\mu \partial_\mu \frac{\delta}{\delta J_d} \bigg) Z_c = 0. \end{aligned} \tag{4.3}$$

4.2   Gauge sector

Up to now, using only symmetry principle, we got the following set of constraint (with the gauge vector $n_\mu = (0, 0, 1)$)

$$\mathcal{G}^a(x)Z_c = \left(\partial_3 \frac{\delta}{\delta J_c} - g\left[J_d, \frac{\delta}{\delta J_c}\right]\right)^a Z_c = J_b^a$$

$$\mathcal{A}^a(x)Z_c = \left(\partial_3 \frac{\delta}{\delta J_b} - g\left[J_d, \frac{\delta}{\delta J_b}\right]\right)^a Z_c = J_c^a$$

$$\mathcal{W}^a(x)Z_c = \left(\partial_3 \frac{\delta}{\delta J_d} - g\sum_\varphi \left[J_\varphi, \frac{\delta}{\delta J_\varphi}\right]\right)^a Z_c = \partial_\mu J_\mu^a$$

$$\mathcal{V}_i(x)Z_c = \mathrm{Tr} \int d^3x \left(J^j \varepsilon_{ij} \frac{\delta}{\delta J_b} + J_c \frac{\delta}{\delta J^i} + J_d \partial_i \frac{\delta}{\delta J_b}\right) Z_c = 0,$$

which correspond respectively to the ghost equation (4.1), the antighost equation (4.2), the local gauge Ward identity (4.3) and the transverse component of the supersymmetry Ward identity (3.4) written as functional differential equations. These operators obey the following algebra

$$\left[\mathcal{W}^a(x), \mathcal{W}^b(y)\right] = g\delta^{(3)}(x-y)f^{abc}\mathcal{W}^c(x)$$

$$\left\{\mathcal{W}^a(x), \mathcal{G}^b(y)\right\} = g\delta^{(3)}(x-y)f^{abc}\mathcal{G}^c(x)$$

$$\left\{\mathcal{W}^a(x), \mathcal{A}^b(y)\right\} = g\delta^{(3)}(x-y)f^{abc}\mathcal{A}^c(x)$$

and

$$[\mathcal{W}^a(x), \mathcal{V}_i(y)] = \delta^{(3)}(x-y)\partial_3\partial_i \frac{\delta}{\delta J_b^a} \tag{4.4}$$

$$\{\mathcal{G}^a(x), \mathcal{V}_i(y)\} = \delta^{(3)}(x-y)\left(\partial_3 \frac{\delta}{\delta J_i} - g\left[J_d, \frac{\delta}{\delta J_i}\right]\right)^a \tag{4.5}$$

all the other brackets being zero. If we apply (4.4) and (4.5) on Z_c, it gives the following constistency conditions

$$\partial_i \left(\partial_3 \frac{\delta}{\delta J_c} - g\left[J_d, \frac{\delta}{\delta J_c}\right]\right)^a Z_c = \partial_i J_b^a \tag{4.6}$$

$$\left(\partial_3 \frac{\delta}{\delta J_i} - g\left[J_d, \frac{\delta}{\delta J-i}\right]\right)^a Z_c = \varepsilon_{ij}J_j^a - \partial_i J_d^a. \tag{4.7}$$

(4.6) is nothing new, it is just the derivative of the antighost equation, but (4.7) corresponds exactly to the *gauge field equation*.

We still have to deal with the dynamics of the Lagrange multiplier field d. Here we cannot obtain its equation of motion by using the functional identities given above. Nevertheless, gauge invariance implies that Green's functions involving only the d field are all zero. This can be check from the Ward identity (4.3) taken at $J_\varphi = 0 \ \forall \varphi \neq d$. Having noticed this, one is able to solve the full theory perturbatively from (4.1), (4.2) and (4.7) and thus, to get the same result as in [1] without invoking any action principle.

5 Manifold with boundary

The case where the theory is defined on a manifold with bounbary is important from the physical point of view. It is only in this case that topological field theory may pocess local observables, which then lie on the boundary [7, 8]. In a previous paper [6], we have already studied this case using (2.1) and shown that these observable are the two dimensionnal conserved currents generating the Kac-Moody algebra [9] of the Wess-Zumino-Witten model [10]. Here we will show that the alternative construction given above still holds in the presence of boundary effects.

Let us now introduce as boundary the plane $\mathcal{B}$ of equation $x^3 = 0$. The effect of $\mathcal{B}$ manifests itself as a breaking in the Ward identity – involving the locality and decoupling conditions discussed in [6] – of the form $\delta(x^3)\Delta$ where Δ is some polynomial in the fields. Their form is constrained by dimension and helicity arguments. For the latter, it is convenient to choose the light-cone coordinates for the transverse directions. At this point, one is faced to the same problem of multiplying distributions at the same point as in [6]. A way to fix this ambiguity is to take

$$\varphi_\pm(x^{\mathrm{tr}}) = \lim_{x^3 \to \pm 0} \frac{\delta Z_c}{\delta J_\varphi(x)}$$

for the insertion of the field $\varphi(x)$ on the right $(+)$ or on the left $(-)$ side of the boundary.

The functional identity which generalizes the supersymmetry Ward identity (3.4) for the case with boundary is

$$\mathrm{Tr}\int d^3x \left(J^\mu \varepsilon_{\rho\mu\nu} n^\nu \frac{\delta}{\delta J_b} + J_c \frac{\delta}{\delta J^\rho} + J_d \partial_\rho \frac{\delta}{\delta J_b} \right) Z_c = \\ = \mathrm{Tr}\int d^2z \left(\kappa_{\rho\pm} \frac{\delta Z_c}{\delta J_\pm^\rho} \frac{\delta Z_c}{\delta J_{b_\pm}} \right) \tag{5.1}$$

where $\kappa_\rho = (\kappa, \bar{\kappa}, \xi)$ are three a priori independent parameters of the breaking.

As for the case without boundary, the antighost equation is a direct consequence of the projection of the supersymmetry Ward identity along the gauge vector n. Thus, the antighost equation in presence of the boundary takes the following form:

$$\left(\partial_3 \frac{\delta}{\delta J_b} - g\left[J_d, \frac{\delta}{\delta J_b} \right] + \delta(x^3)\xi_\pm \frac{\delta}{\delta J_b} \right)^a Z_c = J_c^a. \tag{5.2}$$

For the ghost equation, we modify (4.1) by adding a term expressing the breaking due to the boundary. The parameter of this breaking, which is a priori arbitrary, must be fixed to $-\xi_\pm$ due to consistency with (5.2). So we get

$$\left(\partial_3 \frac{\delta}{\delta J_c} - g\left[J_d, \frac{\delta}{\delta J_c} \right] - \delta(x^3)\xi_\pm \frac{\delta}{\delta J_c} \right)^a Z_c = J_b^a \tag{5.3}$$

which is the ghost equation in presence of the boundary.

Then, the consistency between (5.3) and (5.1) gives the gauge field equation for the case with boundary:

$$\left(\partial_3 \frac{\delta}{\delta J^i} - g\left[J_d, \frac{\delta}{\delta J^i}\right] - \delta(x^3)\,(\xi - \kappa)_{\pm}\,\frac{\delta}{\delta J^i}\right) Z_c = \varepsilon_{ij}J^j - \partial_i J_d \qquad (5.4)$$

where $\kappa = \bar{\kappa}$ due to consistency between the transverse component of (5.1).

These equations together with the fact that, due to BRS invariance, all the Green functions involving only d's are zero, are sufficient to get the results obtained in [6]: the finiteness of the theory as well as the existence of a Kac-Moody algebra on the boundary. We can also check that all the field equations are invariant under the discrete *parity transformation* $z \leftrightarrow \bar{z}$, $u \rightarrow -u$, under which the fields transform as:

$$A \leftrightarrow \bar{A}\,, \qquad\qquad A_u \rightarrow -A_u\,,$$
$$d \rightarrow -d\,, \qquad\qquad b \rightarrow -c\,, \qquad\qquad c \rightarrow b\,.$$

As in [6], this invariance implies $\xi_{\pm} = -\xi_{\mp}$ and therefore, it allows us to find the relation between the behaviour of the fields on the boundary: A and $\bar{A}$ can not be simultanuously zero, but one of them does. Thus, if we choose $A(z, \bar{z}, +0) = 0$, then $\bar{A}(z, \bar{z}, +0) \neq 0$ generates the Kac-Moody algebra. For the other side, the roles of A and $\bar{A}$ are interchanged: $\bar{A}$ is zero on the boundary and A generates the Kac-Moody algebra.

6   Conclusion

We have shown that, given the field content of a topological field theory, imposing BRS invariance and topological supersymmetry, together with the choice of a linear gauge condition, is enough to get all the field equations, except that of the Lagrange multiplier, whose Green's functions are fixed by BRS symmetry. Thus, at least for the specific theory treated in this paper, this is an approach, for defining a theory, which is an alternative to the usual one based on the action principle. This approach seems to be easily extendable to all topological fields theories.

References

[1] A.Brandhuber, M.Langer, M.Schweda, S.P. Sorella, S. Emery and O.Piguet, *Symmetries of the Chern-Simons theory in the axial gauge*,

[2] F. Delduc, F. Gieres and S.P. Sorella, *Phys. Lett.* B225 (1989) 367;

[3] A.Blasi, O. Piguet, S.P. Sorella, *Nucl. Phys.* B356 (1991) 154;

[4] O. Piguet and A. Rouet, *Phys. Rep.* 76 (1981) 1.

[5] W. Kummer, *Acta Physica Austriaca, Suppl.* XV (1976) 423;

[6] S. Emery and O. Piguet, *Helv. Phys. Acta* 64 (1991) 1256;

[7] E. Witten, *Commun. Math. Phys.* 121 (1989) 351;

[8] C.M. Hull and B. Spence, *Mod. Phys. Lett.* A6 (1991) 969;

[9] V.G. Kac, *Funct. Anal. App. 1 (1967) 328)*;
     R.V. Moody, *Bull. Amer. Math. Soc. 73 (1967) 217*;

[10] E. Witten *(modèle de WZW) Commun. Math. Phys.* 92 (1984) 455;